Locally Finitely Presented Categories of Sheaves of Modules

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1 Introduction

We know what is meant by an element of a module. What should we mean by an *element* of a sheaf of modules? One answer is that it is simply a section (over an open set). If, however, one takes the algebraic view that an element should be of finitary character and hence should belong to a sum of subobjects iff it belongs to some finite subsum, then this is not a good answer: sections are, in general, of infinitary character.

One of our original motivations was to develop some model theory for sheaves of modules, at least to see under what conditions on a ringed space a reasonable model theory for sheaves of modules may be developed. Here we mean the usual model theory which is based on elements (of finitary character). Through this we were lead to the problem of determining when a sheaf of modules is finitely generated or finitely presented in the usual algebraic sense and to the problem of determining when the category of $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ is a ringed space, is locally finitely presented. This condition on a category is equivalent to its objects being determined by their elements.

So far as we could determine these questions in full generality had not hitherto been addressed. Presheaves are genuinely algebraic objects and categories of presheaves are always locally finitely presented. But the sheaf property is not an algebraic one and does not fit well with notions like "finitely presented" and "finitely generated" unless the base space has strong compactness properties (such as is usually the case for those spaces considered in algebraic geometry and analysis, where finiteness conditions on sheaves are of central importance).

Our other initial motivation was to investigate the representation of R-modules, where R is any ring, as sheaves over a certain ringed space which originally arose in the model theory of modules. This ringed space is the sheaf of locally definable scalars over the Gabriel-Zariski (=dual-Ziegler) spectrum of R (see, e.g. [16] [17]). This spectrum has bad (separation and compactness) properties compared with the spaces usually considered in algebraic geometry and analysis. But our interest in this space explains why we consider ringed spaces in full generality.

We begin by considering categories of presheaves. If the base space is nice enough then, from the representation of the category of sheaves as a localisation of the category of presheaves, we can deduce all that we need from the, algebraic, presheaf case. In the general case we have to work harder in order to obtain information about finitely generated and finitely presented sheaves, and that occupies us for much of the paper.

We prove our results in the context of presheaves and sheaves over topological spaces. We can see no obstruction to extending the results to locales or categories with a Grothendieck topology.

A direction that we have not pursued here is that of trying to extend the classical results on categories of quasicoherent sheaves or sheaves with quasicoherent cohomology.

Now we describe the contents of the paper in some more detail.

In Section 2, by way of introducing ideas which we will use later in the paper, we give a proof of the, known, result that the category, PreMod-$\mathcal{O}_X$, of presheaves over a presheaf, $\mathcal{O}_X$, of rings is locally finitely presented, with the presheaves of the type $j_0\mathcal{O}_U$ with U open forming a generating set of finitely presented objects (2.14). Here j_0 denotes extension by zero in the sense of presheaves. We also prove that if, for each open U, the ring, $\mathcal{O}_X(U)$, of sections over U is right coherent then the category PreMod-$\mathcal{O}_X$ is locally coherent (and that, under a flatness hypothesis, the converse is true) (2.18).

In Section 3 we use the fact that certain localisations preserve local finite presentation to derive the following result from the presheaf case. If X is a noetherian topological space and $\mathcal{O}_X$ is any sheaf of rings over X then the category, Mod-$\mathcal{O}_X$, of sheaves over $\mathcal{O}_X$ is locally finitely presented (3.11).

The fourth section gathers together the definitions and basic properties of various direct and inverse image functors on sheaves.

Our result from Section 3 is extended in the fifth section: we show that it is enough that X have a basis of compact open sets in order to guarantee that Mod-$\mathcal{O}_X$ is locally finitely presented for every ringed space, $\mathcal{O}_X$, over X (5.6). Over such a space a generating family of finitely presented objects is given by the $j_!\mathcal{O}_U$ where U ranges over a basis of compact open subsets of X and where $j_!$ denotes extension by zero (in the sheaf sense). We show that, over any ringed space, $j_!\mathcal{O}_U$ is finitely presented iff the open set U is compact (5.8).

In Section 6 we consider the weaker condition that Mod-$\mathcal{O}_X$ be locally finitely generated and we begin by showing that if F is a finitely generated sheaf then the support of F is compact (6.5). We also prove that if K is a locally closed subset of X then $j_!\mathcal{O}_K$ is finitely generated iff K is compact 6.7). Using this we obtain a necessary, but not sufficient, condition for Mod-$\mathcal{O}_X$ to be locally finitely generated. Namely, if Mod-$\mathcal{O}_X$ is locally finitely generated then for every $x \in X$ and every open neighbourhood, U, of x there is a compact locally closed set K with $x \in K \subseteq U$ (6.11). We show that if X is the (closed) unit interval in $\mathbb{R}$ with the usual topology and if $\mathcal{O}_X$ is the sheaf of continuous functions on X then Mod-$\mathcal{O}_X$ is locally finitely generated whereas, if we let $\mathcal{O}'_X$ be the constant sheaf then Mod-$\mathcal{O}_X$ is not locally finitely generated (6.12, 6.13). Therefore the property of local finite generation depends on the structure sheaf, not just on the space. We do not know whether the same applies for the property of local finite presentation.

We have not been able to determine when a sheaf of the form $j_!\mathcal{O}_K$, with K locally closed, is finitely presented but in the seventh section we show that, as well as being compact, it is necessary that K satisfy a kind of weak co-compactness condition (7.4). In the case that X is Hausdorff and locally compact then this condition is also sufficient (7.5). We finish the section by showing that when $\mathcal{O}_X$ is the sheaf of continuous functions on the real unit interval the category, Mod-$\mathcal{O}_X$, of sheaves of modules is not locally finitely presented (7.6).

In the final section we consider preservation of finite presentation under restriction. We show, in particular, that if X is locally compact and if $C \subseteq X$ is closed and compact then the restriction functor takes finitely presented $\mathcal{O}_X$-modules to finitely presented $\mathcal{O}_C$-modules (8.4).

2 The Category PreMod-$\mathcal{O}_X$

For background on sheaves and presheaves see, for instance, [7], [9], [22].

Let $\mathcal{O}_X$ be a presheaf of rings (all our rings will be associative with an identity $1 \neq 0$) and denote by PreMod-$\mathcal{O}_X$ the category of presheaves over X which are $\mathcal{O}_X$-pre-modules. That is, $M \in$ PreMod-$\mathcal{O}_X$ means that M is a presheaf of abelian groups such that, for all open sets $U \subseteq X$, $M(U)$ is a right R_U-module, where $R_U = \mathcal{O}_X(U)$ (we use this notation throughout the paper), and such that, for every inclusion $V \subseteq U \subseteq X$ of open subsets of X, the restriction map, $\mathrm{res}^M_{U,V} : M(U) \longrightarrow M(V)$, is a homomorphism of R_U-modules, where we regard $M(V)$ as an R_U-module *via* $\mathrm{res}^{\mathcal{O}_X}_{U,V} : R_U \longrightarrow R_V$.

Example 2.1 Preshv_k/X, *the category of presheaves of k-vectorspaces over X,, where k is a field has this form. Take $\mathcal{O}_X$ to be the constantly k presheaf: $\mathcal{O}_X(U) = k$, $\mathrm{res}^{\mathcal{O}_X}_{U,V} = \mathrm{id}_k$ for $U \subseteq X$ open.*

Theorem 2.2 *([2, Section I.3])* PreMod-$\mathcal{O}_X$ *is a Grothendieck abelian category.*

An object C of a (Grothendieck abelian) category $\mathcal{C}$ is **finitely presented** (**fp**) if the representable functor $(C,-):\mathcal{C}\longrightarrow \mathbf{Ab}$ commutes with direct limits in $\mathcal{C}$. That is, C is finitely presented if for every directed system $((D_\lambda)_\lambda,(g_{\lambda\mu}: D_\lambda \longrightarrow D_\mu)_{\lambda<\mu})$ in $\mathcal{C}$, with limit $(D,(g_{\lambda\infty}: D_\lambda \longrightarrow D)_\lambda)$, every $f\in (C,D)$ factors through some $g_{\lambda\infty}$. The category $\mathcal{C}$ is **locally finitely presented** (**lfp**) if the full subcategory, $\mathcal{C}^{\mathrm{fp}}$, of finitely presented objects is skeletally small and if every object of $\mathcal{C}$ is a direct limit of finitely presented objects. Grothendieck abelian categories which satisfy this condition are in many ways as well-behaved as categories of modules over rings. In particular, objects of $\mathcal{C}$ are determined by their "elements" (morphisms from finitely presented objects) and these "elements" have finitary character (as opposed to what one has for merely presentable categories). Such categories have a good model theory and they admit a useful embedding into a related functor category (see e.g., [8], [10], [13], [16]). We remark that, by [6], any locally finitely presented abelian category is Grothendieck. It is known that PreMod-$\mathcal{O}_X$ is locally finitely presented (see [4, p. 7] for example). Indeed it is a variety of finitary many-sorted algebras in the sense of [1, Section 3A] (see [20] for details) but we include a proof of local finite presentation which is relevant to the considerably more subtle question of when categories of sheaves are locally finitely presented.

Let $U\subseteq X$ be open. Our generic notation for the inclusion $U\longrightarrow V$ of one set in another is j, and hence $j_!, j^*$ etc. for the associated functors between categories of (pre)sheaves. Let $G\in$ PreMod-$\mathcal{O}_U$ (where by $\mathcal{O}_U$ we denote the restriction, $\mathcal{O}_X \restriction U$, of the structure presheaf $\mathcal{O}_X$ to U - this is another notation that will be used throughout the paper). Define the functor j_0G on the category of open sets of X by $j_0G(V)=\begin{cases} G(V) & \text{if } V\subseteq U\\ 0 & \text{otherwise}\end{cases}$ on objects V, and in the obvious way on inclusions $W\subseteq V$.

Proposition 2.3 *If $G\in$ PreMod-$\mathcal{O}_U$, then $j_0G\in$ PreMod-$\mathcal{O}_X$, indeed j_0 extends to a functor from* PreMod-$\mathcal{O}_U$ *to* PreMod-$\mathcal{O}_X$ *which is exact.*

Proof. Given $f:G\longrightarrow H$ define $j_0f:j_0G\longrightarrow j_0H$ by $j_0f.V=\begin{cases} f_V & \text{if } V\subseteq U,\\ 0 & \text{otherwise}\end{cases}$ (we write f_V instead of the notationally more consistent $fV: GV\longrightarrow HV$). Clearly this does define a functor. For presheaves, exactness is equivalent to exactness at each open set. Clearly j_0 preserves this and hence is an exact functor. $\square$

Remark 2.4 *If G already is a sheaf then one also has the sheaf extension of G by zero, $j_!G$, which is (see [7, p.68, p.111]) the sheafification of j_0G, but, as the next example shows, this may well be different from j_0G.*

Example 2.5 *Let $X = \{x, y\}$ be a two point set with the discrete topology. Let $U = \{x\}$ and define the presheaf G on U by $GU = \mathbb{Z}$. Then $j_0G.X = 0$ but $j_!G.X \neq 0$ since we have the open cover $X = U \cup (X \setminus U)$ and the compatible sections $1 \in j_!G.U$, $0 \in j_!G(X \setminus U)$ and hence a global section $a \in j_!G.X$ restricting to these sections (in particular $a \neq 0$).*

Remark 2.6 *(a) $F \in \mathrm{PreMod}\text{-}\mathcal{O}_U$ implies $(j_0F)_U \simeq F$, indeed the composite of j_0 then restriction to U is the identity on $\mathrm{PreMod}\text{-}\mathcal{O}_U$.*
(b) $G \in \mathrm{PreMod}\text{-}\mathcal{O}_X$ implies $j_0(G_U) \leq G$.
These observations follow directly from the definitions.

Proposition 2.7 *If $U \subseteq X$ is open, $F \in \mathrm{PreMod}\text{-}\mathcal{O}_U$, $G \in \mathrm{PreMod}\text{-}\mathcal{O}_X$ then there is a natural isomorphism $(j_0F, G) \simeq (F, G_U)$. Indeed $j_0 : \mathrm{PreMod}\text{-}\mathcal{O}_U \longrightarrow \mathrm{PreMod}\mathcal{O}_X$ is left adjoint to the restriction functor $(-)_U : \mathrm{PreMod}\text{-}\mathcal{O}_X \longrightarrow \mathrm{PreMod}\text{-}\mathcal{O}_U$. In particular the functor j_0 is full.*

Proof. Given $f : j_0F \longrightarrow G$ define $f' : F \longrightarrow G_U$ by, for $V \subseteq U$, $f'_V = f_V : FV = j_0F.V \longrightarrow GV = G_UV$. Conversely, given $f' : F \longrightarrow G_U$, define $f : j_0F \longrightarrow G$ by $f_V = \begin{cases} f'_V : FV \longrightarrow GV & \text{if } V \subseteq U \\ 0 : FV(=0) \longrightarrow GV & \text{otherwise} \end{cases}$. Clearly these processes are mutually inverse and the isomorphism is natural.

To see that j_0 is full, note that $(j_0F, j_0F') \simeq (F, (j_0F')_U) \simeq (F, F')$, for $F, F' \in \mathrm{PreMod}\text{-}\mathcal{O}_X$, by the remark 2.6(a). □

Corollary 2.8 *For $U \subseteq X$ open, $j_0\mathrm{PreMod}\text{-}\mathcal{O}_U$ is a localising subcategory of $\mathrm{PreMod}\text{-}\mathcal{O}_X$.*

Proof. By, for instance, [12, 4.6.3] it is enough to verify that $j_0\mathrm{PreMod}\text{-}\mathcal{O}_U$ is a Serre subcategory and is closed under direct sums in the larger category. These are trivial to check. □

Since we have a localising subcategory, there is a corresponding quotient category and localisation functor, which we now identify.

Corollary 2.9 *The localisation functor corresponding to the localising subcategory $j_0(\mathrm{PreMod}\text{-}\mathcal{O}_U)$ is given by $G \mapsto qG$ where $qG.V = \begin{cases} GV & \text{if } V \not\subseteq U \\ 0 & \text{if } V \subseteq U \end{cases}$.*

Proof. First note that $G \in \mathrm{PreMod}\text{-}\mathcal{O}_X$ is torsionfree iff for all $F \in \mathrm{PreMod}\text{-}\mathcal{O}_U$ we have $(j_0F, G) = 0$, equivalently, $(F, G_U) = 0$. Taking $F = G_U$ we deduce that $G_U = 0$ if G is torsionfree and, conversely, if $G_U = 0$ then G is torsionfree.

Now suppose that G is torsionfree and $G \leq H$ with H/G torsion. For all open $V \subseteq X$ we have the exact sequence $0 \longrightarrow GV \longrightarrow HV \longrightarrow (H/G)V \longrightarrow 0$. If V is not contained in U then $(H/G)V = 0$ and so $GV = HV$. In the case that $V \subseteq U$ we have $GV = 0$ and hence $HV = (H/G)V$. Therefore the sequence of presheaves splits and we obtain $H = G \oplus H'$ where $H' \simeq (H/G)$ indeed $H = G \oplus j_0(H/G)$.

Therefore every torsionfree sheaf is already injective with respect to the torsion theory and so the quotient category consists of all $G \in \text{PreMod-}\mathcal{O}_X$ with $G_U = 0$ and the localisation functor is as described (since it is just the functor which sends G to G/G_U. $\square$

The corresponding result for the category, $\text{Mod-}\mathcal{O}_X$, of sheaves of modules over a sheaf, $\mathcal{O}_X$, of rings holds true with $j_!\text{Mod-}\mathcal{O}_U$ in place of $j_0\text{PreMod-}\mathcal{O}_U$. In that case the localisation functor is just restriction to $X \setminus U$ and the quotient category is equivalent to $\text{Mod-}\mathcal{O}_{X\setminus U}$ (cf. [9, p. 107, 6.4]).

Lemma 2.10 *Let $G \in \text{PreMod-}\mathcal{O}_X$ and let $f \in (\mathcal{O}_X, G)$. Then f is determined by $f(1) \in GX$, where $1 \in \mathcal{O}_X(X) = R_X$ is the identity element of this ring.*

Proof. Let $U \subseteq X$ be open and let $1_U = \text{res}_{X,U}(1)$ (= the identity element of R_U). We have $f_U \text{res}^{\mathcal{O}_X}_{X,U} = \text{res}^G_{X,U} f_X$ since f is a morphism.

$$\begin{tikzcd} O_X(X) \arrow[r, "f_X"] \arrow[d, "\text{res}"'] & GX \arrow[d, "\text{res}"] \\ O_X(U) \arrow[r, "f_U"'] & GU \end{tikzcd}$$

So $f_U 1_U = \text{res}^G_{X,U}(f1)$. If $r \in \mathcal{O}_X(U)$ then, since f_U is R_U-linear, we have $f_U r = (f_U 1_U)r = (\text{res}^G_{X,U}(f1))r$, as claimed. $\square$

Proposition 2.11 *Let $G \in \text{PreMod-}\mathcal{O}_X$. Then there is a natural isomorphism $(\mathcal{O}_X, G) \simeq GX$.*

Proof. Lemma 2.10 gives the map "$\longrightarrow$". For the inverse map, given $a \in GX$, define $f \in (\mathcal{O}_X, G)$ as follows. Let $U \subseteq X$ be open. Define $f_U : \mathcal{O}_X(U) \longrightarrow GU$ by: if $r \in \mathcal{O}_X(U)$ set $f_U r = (\text{res}^G_{X,U} a)r$. By the proof of 2.10 this gives a well-defined map and we easily check that the processes are inverse and the isomorphism is natural. $\square$

Proposition 2.12 *$\mathcal{O}_X$ is a finitely presented element of* $\text{PreMod-}\mathcal{O}_X$.

Proof. Let $\{G_\lambda\}_\lambda$ be a directed system in $\text{PreMod-}\mathcal{O}_X$ and let $f \in (\mathcal{O}_X, G)$ where $G = \varinjlim G_\lambda$. Consider $f1 \in GX$. Since $G = \varinjlim G_\lambda$ there is λ and $a \in G_\lambda$ such that $g_{\lambda,\infty} a = f1$, where $g_{\lambda,\infty} : G_\lambda \longrightarrow G$ is the canonical map to the direct limit.

Define $f_\lambda : \mathcal{O}_X \longrightarrow G_\lambda$ by $f_\lambda 1 = a$ (see 2.11). Then $g_{\lambda,\infty} f_\lambda 1 = g_{\lambda,\infty} a = f1$ so, by 2.11, $f = g_{\lambda,\infty} f_\lambda$ and f factors through some G_λ, as required. $\Box$

Essential in the proof above is the fact that in the category of presheaves direct limits are computed at the level of sections: if G is the presheaf direct limit of a directed system $(G_\lambda)_\lambda$ of presheaves then. since this is how we define the direct limit of functors, for each open set U we have that GU is the direct limit of the abelian groups $G_\lambda U$. Even if all the G_λ are sheaves this recipe will yield a presheaf rather than a sheaf in general.

Proposition 2.13 *If $U \subseteq X$ is open then $j_0 \mathcal{O}_U$ is a finitely presented object of* PreMod-$\mathcal{O}_X$.

Proof. Say $G = \varinjlim G_\lambda$, where the $G_\lambda \in$ PreMod-$\mathcal{O}_X$ form a directed system and let $f : j_0\mathcal{O}_U \longrightarrow G$. By 2.7 $(j_0\mathcal{O}_U, G) \simeq (\mathcal{O}_U, G_U)$ with f corresponding to, say, $f' : \mathcal{O}_U \longrightarrow G_U$.

Now the functor $(-)_U$ commutes with direct limits: if $V \subseteq U$ then $G_U V = GV = \varinjlim (G_\lambda V)$ (as remarked already) $= (\varinjlim G_\lambda \upharpoonright U)V$. So $G \upharpoonright U = \varinjlim (G_\lambda \upharpoonright U)$. Therefore, by 2.12 applied in PreMod-$\mathcal{O}_U$, there is λ and $f'_\lambda : \mathcal{O}_U \longrightarrow (G_\lambda)_U$ with $f' = g'_{\lambda,\infty} f'_\lambda$ (where $g'_{\lambda,\infty} = (g_{\lambda,\infty})_U$ is the relevant colimit map).

Applying j_0 we obtain $j_0 f' = j_0 g'_{\lambda,\infty} j_0 f'_\lambda$. Composing with the canonical maps from $j_0 G \mid U$ to G and from $j_0 G_\lambda \mid U$ to G_λ (and noting that the composition of the first such map with $j_0 f'$ is just f, we obtain a factorisation of f through $g_{\lambda\infty}$, as required. $\Box$

Theorem 2.14 *(e.g. [4, p.7 Prop. 6]) The $j_0\mathcal{O}_U$, with $U \subseteq X$ open, form a generating set of finitely presented objects of* PreMod-$\mathcal{O}_X$.

Proof. Let $F \in$ PreMod-$\mathcal{O}_X$ and let $U \subseteq X$ be open. Let $a \in FU$. Then there is $f' : \mathcal{O}_U \longrightarrow F_U$ with $f'_U : \mathcal{O}_U(U) \longrightarrow F_U(U)$ taking 1 to a. Under the adjunction $(\mathcal{O}_U, F_U) \simeq (j_0\mathcal{O}_U, F)$, f' corresponds to $f = f_{U,a}$, say, and $f_U 1_U = a$ (by, e.g., the explicit description of j_0).

Then $\bigoplus_{U \subseteq X} \bigoplus_{a \in FU} f_{U,a} : \bigoplus_U \bigoplus_a j_0\mathcal{O}_U \longrightarrow F$ is, by construction, onto at each open $U \subseteq X$ and hence is an epimorphism, as required. $\Box$

Corollary 2.15 *For any space X and presheaf $\mathcal{O}_X$ of rings on X the category* PreMod-$\mathcal{O}_X$ *is a locally finitely presented abelian Grothendieck category.*

Note that it is necessary to use all open sets U: a basis of open sets will not suffice. For consider a presheaf F with $FX \neq 0$ but $FU = 0$ for all proper open subsets U of X. We need $\mathcal{O}_X = j_0\mathcal{O}_X$ to generate this presheaf.

We may also ask when the category of presheaves has stronger finiteness properties, such as being locally noetherian or artinian. Here we consider only local coherence, where a locally finitely presented abelian category is **locally coherent** if every finitely generated subobject of each finitely presented object is finitely presented. We say that a ring R is (**right**) **coherent** if the category, Mod-R, of (right) modules over R is locally coherent.

Let $F \in \text{PreMod-}\mathcal{O}_X$, let $U \subseteq X$ be open and let $a \in FU$. Define the presheaf $\langle a \rangle$ by $V \mapsto \begin{cases} \text{res}_{UV}(a).R_V & \text{if } V \subseteq U \\ 0 & \text{otherwise} \end{cases}$. Then $\langle a \rangle \in \text{PreMod-}\mathcal{O}_X$ and we have the natural inclusion $\langle a \rangle \leq F$.

An object F in an abelian category is **finitely generated** if $F = \sum_\lambda F_\lambda$ implies $F = \sum_{i=1}^n F_{\lambda_i}$ for some $\lambda_1, ..., \lambda_n$. If the category is locally finitely presented then it is equivalent that F be the image of a finitely presented object.

Lemma 2.16 *The presheaf* $F \in \text{PreMod-}\mathcal{O}_X$ *is finitely generated iff there exist open subsets* $U_1, ..., U_n$ *of* X *and sections* $a_i \in FU_i$ *such that* $F = \sum_1^n \langle a_i \rangle$.

Proof. ($\Rightarrow$) We have, on comparing at each open set U, $F = \sum_{U \subseteq X} \sum_{a \in FU} \langle a \rangle$ and so F finitely generated implies that F is a sum of finitely many of these.

($\Leftarrow$) If $F = \sum_\lambda F_\lambda$ (a directed sum) then choose, for each $i = 1, .., n$, some λ_i such that $a_i \in F_{\lambda_i} U_i$ (since $FU_i = \sum_\lambda F_\lambda U_i$ this exists). Then $\langle a_i \rangle \leq F_{\lambda_i}$ and so $F \leq \sum_1^n F_{\lambda_i}$. $\square$

The $j_0 \mathcal{O}_U$ form a generating set of finitely presented presheaves so, if F is finitely presented, then there is an exact sequence $\bigoplus_{j=1}^m j_0 \mathcal{O}_{V_j} \longrightarrow \bigoplus_{i=1}^n j_0 \mathcal{O}_{U_i} \xrightarrow{f} F \longrightarrow 0$ where the first map is given, in terms of components, as $(f_{ij})_{ij}$, with $f_{ij} : j_0 \mathcal{O}_{V_j} \longrightarrow j_0 \mathcal{O}_{U_i}$.

Note that $j_0 \mathcal{O}_U = \langle 1_{R_U} \rangle$ and so $f_{ij} : j_0 \mathcal{O}_{V_j} \longrightarrow j_0 \mathcal{O}_{U_i}$ equals 0 if V_j is not contained in U_i. Let $a_i = f(1_{U_i})$ (writing 1_U for 1_{R_U}) - so $F = \sum_1^n \langle a_i \rangle$.

For each j we have $f(\sum_1^n f_{ij} 1_{V_j}) = 0$. Set $f_{ij} 1_{V_j} = s_{ij} \in R_{V_j}$ so $f(s_{ij}) = \begin{cases} 0 & \text{if } V_j \not\subseteq U_i \\ \text{res}^F_{U_i V_j}(a_i).s_{ij} & \text{if } V_j \subseteq U_i \end{cases}$. Then the relations between $a_1, ..., a_n$ are generated by $\Sigma_{i: V_j \subseteq U_i} \text{res}^F_{U_i V_j}(a_i).s_{ij} = 0$, for $j = 1, ..., m$.

From this we get a presentation of each FU (as an R_U-module): the generators of FU are $\{\text{res}^F_{U_i U}(a_i) = a_i^U \text{ say } : U \subseteq U_i\}$ and the relations on these generators are $\{\sum_{i: V_j \subseteq U_i} a_i^U.s_{ij} : U \subseteq V_j\}$.

Lemma 2.17 *If* $U' \subseteq U$ *are open subsets of* X *and if* $U' \subseteq U_i$ *imples* $U \subseteq U_i$ *for all* i *and if* $U' \subseteq V_j$ *implies* $U \subseteq V_j$ *for all* j *then* $FU' \simeq FU \otimes_{R_U} R_{U'}$.

Proof. By assumption FU is generated by the a_i^U with $U \subseteq U_i$ and with a relation $\Sigma_{i: V_j \subseteq U_i} a_i^U s_{ij} = 0$ for each j with $U \subseteq V_j$. That is, we have a presentation of FU :

$\bigoplus_{j: U \subseteq V_j} R_U \longrightarrow \bigoplus_{i: U \subseteq U_i} R_U \longrightarrow FU \longrightarrow 0$ where the first map is given by the matrix $(t_{ij})_{ij}$ over R_U where $t_{ij} = \text{res}^{\mathcal{O}_X}_{V_j U}(s_{ij})$ $(= 0$ if $V_j \not\subseteq U_i)$.

Similarly we have a presentation of FU' :

$\bigoplus_{j:U'\subseteq V_j} R_U \longrightarrow \bigoplus_{i:U'\subseteq U_i} R_U \longrightarrow FU' \longrightarrow 0$

where the first map is given by the matrix $(t'_{ij})_{ij}$ over $R_{U'}$, where $t'_{ij} = \mathrm{res}^{\mathcal{O}_X}_{V_jU'}(s_{ij}) = \mathrm{res}^{\mathcal{O}_X}_{UU'}(t_{ij})$.

Applying $- \otimes_{R_U} R_{U'}$ to the first sequence we obtain, by right exactness of $\otimes$, and since $R_U \otimes_{R_U} R_{U'} \simeq R_{U'}$, the exact sequence

$\bigoplus_{j:U'\subseteq V_j} R_{U'} \longrightarrow \bigoplus_{i:U'\subseteq U_i} R_{U'} \longrightarrow FU \otimes_{R_U} R_{U'} \longrightarrow 0$

where the ij-component of the first map is $t_{ij} \otimes_{R_U} 1_{R_{U'}} = \mathrm{res}^{\mathcal{O}_X}_{UU'}(t_{ij}) = t'_{ij}$ and hence $FU' \simeq FU \otimes_{R_U} R_{U'}$. $\square$

Theorem 2.18 *Let $\mathcal{O}_X$ be a presheaf of rings on the space X. If, for each open subset U of X, the ring, $R_U = \mathcal{O}_X(U)$, of sections over U is a right coherent ring then the category,* PreMod-$\mathcal{O}_X$*, of presheaves over $\mathcal{O}_X$ is locally coherent.*

If, for each inclusion $V \subseteq U$ of open sets, the ring R_V is a flat left R_U-module (via the corresponding morphism of rings) then the converse also holds.

Proof. Suppose that F, with presentation as above, is a finitely presented presheaf and suppose that $G \leq F$ is finitely generated, say $g : \bigoplus_1^p j_0\mathcal{O}_{W_k} \longrightarrow G$ is epi. Let $b_k = g(1_{W_k})$, so $G = \sum_1^p \langle b_k \rangle$ (and, since $G \leq F$, we have $b_k = \Sigma_{i:U_i\supseteq W_k} a_i^{W_k}.t_{ik}$ for some $t_{ik} \in R_{W_k}$). We claim that, under the coherence hypothesis on the rings, G is finitely presented.

Let $U \subseteq X$ and consider $GU \leq FU$. Since GU is a finitely generated R_U-submodule of the finitely presented R_U-module FU and since R_U is right coherent, there is a finite generating set of relations on the generators $b_k^U = \mathrm{res}_{W_kU}(b_k)$, where k is such that $W_k \supseteq U$, of GU, say $\Sigma_k b_k^U r_{hk} = 0$ for some $r_{hk} \in R_U$, where $h = 1, ..., n_U$.

If $U' \subseteq U$ is such that $U' \subseteq U_i, V_j, W_k$ for any i, j, k implies the same for U then we observe that the module GU' is generated by the $b_k^{U'} = \mathrm{res}_{UU'}(b_k^U)$ and has generating set of relations $\Sigma_k b_k^{U'} \mathrm{res}^{\mathcal{O}_X}_{UU'} r_{hk} = 0$ for $h = 1, ..., n_U$: note that these follow from the above relations which hold in GU.

So, for each U of the form $U_{i_1} \cap ... \cap U_{i_N} \cap V_{j_1} \cap ... \cap V_{j_M} \cap W_{k_1} \cap ... \cap W_{k_P}$ take a finite generating set of relations on the b_k^U, $(k = k_1, ..., k_p)$.

Then, we claim, the set of all these, as U varies over the finitely many such sets, gives a presentation of G by generators $b_1, ..., b_p$ and these finitely many relations.

To prove that, we need show only that these give the correct module, GU, for each U - but that is by construction and the lemma above.

Suppose now that the category of presheaves is locally coherent and that we have the flatness hypothesis. To check that R_U is right coherent it is enough to show that every finitely generated submodule of a finitely generated free module is finitely presented. So let $R_U^{(I)} \longrightarrow R_U^m \longrightarrow M \longrightarrow 0$ where $M \leq R_U^n$ be an exact sequence. For any R_U-module N we have the $\mathcal{O}_U$-module $\bar{N}$ given by

$\bar{N}.V = N \otimes_{R_U} R_V$ with the obvious restriction maps. Since tensor product is right exact and since, clearly, $\bar{R_U} = \mathcal{O}_U$ we obtain an exact sequence $\mathcal{O}_U^{(I)} \longrightarrow \mathcal{O}_U^m \longrightarrow \bar{M} \longrightarrow 0$ and then, since j_0 is (right) exact (2.3) an exact sequence $j_0\mathcal{O}_U^{(I)} \longrightarrow j_0\mathcal{O}_U^m \longrightarrow j_0\bar{M} \longrightarrow 0$ $(*)$ in PreMod-$\mathcal{O}_X$. Since each R_V is a flat R_U-module, each morphism $M \otimes_{R_U} R_V \longrightarrow R_U^n \otimes_{R_U} R_V = R_V^n$ is monic and so $j_0\bar{M} \leq j_0\mathcal{O}_U^n$. Then, since $j_0\mathcal{O}_U$ is finitely presented, the index set I can, in the sequence $(*)$, be taken to be finite and, therefore, taking sections over U (an exact functor on presheaves), we see that I can be taken to be finite in the original exact sequence, as required. $\square$

3 Torsion theory on PreMod-$\mathcal{O}_X$ and localising presheaves to sheaves

By a torsion theory we will always mean a hereditary torsion theory - hence one determined by a class of injective objects (see e.g. [12] or [21]).

Proposition 3.1 *Let τ be a hereditary torsion theory on* PreMod-$\mathcal{O}_X$. *Then τ is determined by the collection of Gabriel filters:*
$\mathcal{U}_\tau(j_0\mathcal{O}_U) = \{\mathcal{I} \leq j_0\mathcal{O}_U : j_0\mathcal{O}_U/\mathcal{I} \in \mathcal{T}\}$,
where $\mathcal{T}$ denotes the class of τ-torsion objects.

Proof. The set of objects $j_0\mathcal{O}_U$, as U ranges over all open subsets of X, generates the category and is a set of finitely presented objects. Then it is easy to see (e.g. see [13]) that the corresponding Gabriel filters determine τ. $\square$

For the purposes of this paper one may take the next proposition as a definition of "finite type" (or see [21]).

Proposition 3.2 *A hereditary torsion theory τ on* PreMod-$\mathcal{O}_X$ *is of finite type iff for all open $U \subseteq X$ the filter $\mathcal{U}_\tau(j_0\mathcal{O}_U)$ has a cofinal set of finitely generated presheaves.*

Proof. This follows from 2.14 and [13, 3.11]. $\square$

For the remainder of this section let τ be the torsion theory corresponding to the sheafification functor and let $\mathcal{T}$, $\mathcal{F}$ denote the corresponding torsion and torsionfree classes respectively (this torsion theory is discussed in [12, Section 4.7] for example). Then $\mathcal{T}$ is the class of presheaves with sheafification equal to 0. We continue for a little longer to assume that $\mathcal{O}_X$ is just a presheaf, rather than a sheaf, of rings.

Lemma 3.3 *Let $F \in$* PreMod-$\mathcal{O}_X$. *Then $F \in \mathcal{T}$ iff for every open $U \subseteq X$ and for every $a \in FU$ there is an open cover $\{U_i\}_i$ of U such that, for all i, we have* $\mathrm{res}^F_{U,U_i} a = 0$.

Proof. Clearly such a presheaf is torsion (its sheafification must be 0). Conversely, if $F \in \mathcal{T}$ and if U, a are as given, then a must be identified with $0 \in FU$ by the sheafification process and this can happen only if there is a cover of U as described. $\square$

Corollary 3.4 *Let $G \le F \in$ PreMod-$\mathcal{O}_X$. Then G is τ-dense in F (that is, F/G is τ-torsion) iff for every open $U \subseteq X$ and for every $a \in FU$ there is an open cover $\{U_i\}_i$ of U such that, for all i we have $\mathrm{res}^F_{U,U_i} a \in GU_i$.*

Recall that the **stalk** of a (pre)sheaf F at a point x is $F_x = \varinjlim_{x \in U} FU$.

Lemma 3.5 *Let $F \in$ PreMod-$\mathcal{O}_X$. Then $F \in \mathcal{T}$ iff for every $x \in X$ we have $F_x = 0$.*

Proof. ($\Rightarrow$) $F \in \mathcal{T}$ implies $qF = 0$ where q is the sheafification functor (i.e. the localisation functor associated to τ) and this implies that $(qF)_x = 0$ for all $x \in X$ which, in turn, implies (since stalks are unchanged by sheafification) that $F_x = 0$ for all $x \in X$.

($\Leftarrow$) If every stalk of F is 0 then the same is true for qF and hence $qF = 0$, so $F \in \mathcal{T} = \ker(q)$. $\square$

Lemma 3.6 *Let $F \in$ PreMod-$\mathcal{O}_X$. Then $F \in \mathcal{F}$ iff F is a mono(=separated) presheaf.*

Proof. We have $F \in \mathcal{F}$ implies F embeds in qF which, clearly, implies that F is a monopresheaf. On the other hand $F \notin \mathcal{F}$ implies $\tau F \neq 0$ but then τF being not a monopresheaf implies the same for F. $\square$

Proposition 3.7 *Let $U \subseteq X$ be open. Then $j_0 \mathcal{O}_U$ has a cofinal family of finitely generated τ-dense subpresheaves iff U is compact.*

Proof. ($\Leftarrow$) Suppose that $G \le F = j_0 \mathcal{O}_U$ is τ-dense. Then there is an open cover $\{U_i\}_i$ of U, which, by compactness of U, we may take to be a finite cover $U_1, ..., U_n$, such that for each i, $\mathrm{res}^F_{U,U_i}(1) \in G(U_i)$, where $1 \in FU$. That is $G \restriction U_i = F \restriction U_i = \mathcal{O}_{U_i}$ for each i.

Let $G_i = j_0 \mathcal{O}_{U_i}$. Note that $G_i \le G$ (by the map corresponding to $\mathrm{Id} \in (\mathcal{O}_{U_i}, G \restriction U_i \simeq \mathcal{O}_{U_i}) \simeq (j_0 \mathcal{O}_{U_i} = G_i, G)$). So $G_1 + ... + G_n \le G$ is a finitely generated subobject of G and, by construction and by 3.4, is τ-dense, as required.

($\Rightarrow$) If U is not compact then choose an open cover $\{U_i\}_i$ of U which has no finite subcover and define $G \in$ PreMod-$\mathcal{O}_X$ by $GV = FV$ if $V \subseteq U_i$ for some i, $GV = 0$ otherwise. So $G \le F$ and G is τ-dense in F since $\mathrm{res}_{U,U_i}(1) \in GU_i$ for every i.

If there were $G' \le G$ with G' finitely generated and τ-dense in G then there would be, in particular, an open cover $\{V_j\}_j$ with $\mathrm{res}_{U,V_j}(1) \in G'V_j$ for every

j. Suppose, using 2.16, that the $a_l \in G'V_l$ with $l = 1, ..., m$ generate G'. Since $G' \leq G$, if $a_l \neq 0$ then $V_l \subseteq U_{i_l}$ for some i_l. But then, if V is an open set not contained in $U_{i_1} \cup ... \cup U_{i_m}$, and note that this is a proper subset of U, we must have $G'V = 0$ (since $G' = \sum_1^m \langle a_l \rangle$). So, choosing V_j not contained in this union, we see that $\mathrm{res}_{U,V_j}(1)$ (which is non-zero) is not in $G'V_j$, a contradiction. $\square$

Corollary 3.8 *The localisation "sheafification" is of finite type iff every open subset of X is compact.*

Proof. This is immediate by 3.7 and 3.2. $\square$

A subcategory of an lfp Grothendieck abelian category is **definable** (in the terminology of [5]) if it is closed under products, direct limits and pure subobjects, equivalently if it is axiomatisable (in the canonical language of the category) and is closed under products.

Corollary 3.9 *The class of monopresheaves is a definable subcategory of* PreMod-$\mathcal{O}_X$ *iff every open subset of X is compact.*

Proof. By [13, 3.3] which says that the torsionfree objects for a hereditary torsion theory form a definable class iff the torsion theory is of finite type. The proof is just as for the module case, for which see [14, 2.4]. $\square$

Let $\mathcal{O}_X$ be a sheaf of rings on the space X. We will see that if X is sufficiently nice then we can deduce properties of Mod-$\mathcal{O}_X$ from its representation as a localisation of PreMod-$\mathcal{O}_X$.

Let τ be a hereditary torsion theory of finite type on a locally finitely presented Grothendieck abelian category $\mathcal{C}$ which has a generating set $\mathcal{G}$ of finitely presented objects. We say that τ is an **elementary** torsion theory [15] if for every $F \leq G \in \mathcal{G}$ with G/F torsion and F finitely generated, we have that F is τ-finitely presented, meaning that if $G'' \in \mathcal{G}$ and $f : G'' \longrightarrow F$ is an epimorphism then $\ker(f)$ has a finitely generated τ-dense subobject. Note that it is enough to show that F contains a finitely presented τ-dense subobject F'. For, let G' be a finitely generated subobject of G'' with $fG' = F'$. Then $G' + \ker(f)$ is τ-dense in G'' and so, since τ is of finite type, it follows that there is a finitely generated τ-dense subobject of G'' lying between $G' + \ker(f)$ and G'. So, without loss of generality, G' is τ-dense in G''. Now, since F' is finitely presented, the kernel of $f \restriction G'$, that is $G' \cap \ker(f)$, is finitely generated and, since it is also τ-dense in $\ker(f)$ (because G' is τ-dense in G''), we have our required conclusion.

The term elementary is used because this is exactly the condition one needs for the localised category $\mathcal{C}_\tau$ to be definable when regarded as a subcategory of $\mathcal{C}$ ([15, 0.1]).

Proposition 3.10 *Let $\mathcal{O}_X$ be a sheaf of rings on the space X and let τ denote the sheafification torsion theory on* PreMod-$\mathcal{O}_X$*. Then τ is an elementary torsion theory iff τ is of finite type iff every open subset of X is compact.*

Proof. It remains to be shown that if every open subset is compact then τ is elementary. Let $F \leq G = j_0\mathcal{O}_U$ with G/F torsion and F finitely generated. As argued above it will be enough to show that F contains a finitely presented τ-dense subobject. Since F is τ-dense in G there is an open cover $(U_\lambda)_\lambda$ of U such that for each λ we have $\mathrm{res}_{U,U_\lambda}1 = 1_\lambda$ in FU_λ. Since U is compact, we may suppose that the index set is finite and so $F \geq F' = \sum_1^n j_0\langle 1_{U_i}\rangle$, say, where $U_1, ..., U_n$ cover U. Then F' is τ-dense in G (and hence in F). But clearly F' is finitely presented, the relations between its generators being generated by the $\mathrm{res}_{U_i,U_i\cap U_j}(1_{U_i}) = \mathrm{res}_{U_j,U_i\cap U_j}(1_{U_j})$ where $1 \leq i < j \leq n$. $\square$

Note that every open set of the space X is compact iff X is **noetherian** (i.e. every descending chain of closed subspaces is finite). Therefore we obtain the following as a corollary.

Theorem 3.11 *Let $\mathcal{O}_X$ be a sheaf of rings on a noetherian space X. Then* Mod-$\mathcal{O}_X$ *is locally finitely presented.*

Proof. If the Grothendieck abelian category $\mathcal{C}$ is locally finitely presented and if τ is an elementary torsion theory on $\mathcal{C}$ then, [16, A3.15] (by the proof of [15, 2.1]), the localisation of $\mathcal{C}$ at τ is locally finitely presented. $\square$

We will see that this condition is certainly not necessary for Mod-$\mathcal{O}_X$ to be locally finitely presented. For instance, X being locally noetherian will suffice.

As a consequence of the theorem, one has that if $\mathcal{O}_X$ is a sheaf of rings on a noetherian space X then the model theory of sheaves of $\mathcal{O}_X$-modules is just a part of the model theory of presheaves over $\mathcal{O}_X$ and so, being of essentially algebraic character, is well-covered by the existing body of work on the model theory of objects of locally finitely presented abelian categories. For modules, this is surveyed in [18], for more general categories it tends to be presented using algebraic (rather than model-theoretic) terminology, e.g. see [8], [10], [16], but also [19].

Corollary 3.12 *Let $\mathcal{O}_X$ be a sheaf of rings on the space X. Then* Mod-$\mathcal{O}_X$ *is a definable subcategory of* PreMod-$\mathcal{O}_X$ *iff every open subset of X is compact, that is, iff X is noetherian.*

4 Some general constructions and results on sheaves

First we recall some standard functors on categories of sheaves. Let $Y \subseteq X$ and let $F \in$ Mod-$\mathcal{O}_X$. Let $j : Y \longrightarrow X$ denote the inclusion. The sheaf j^*F is defined by: for any open subset U of Y we set $j^*F.U$ to be the set

$\{s : s$ is a set-theoretic section of the stalk space of F over U such that $\forall x \in U\, \exists V \subseteq X$ open with $x \in V \cap Y \subseteq U$ and $\exists t \in FV$ such that $\forall y \in V \cap Y$ we have $s_y = t_y\}$. That is, sections of j^*F locally look like sections of F. If Y is open in X then $j^*F.U = FU$ for $U \subseteq Y$ open.

One may check that j^*F is, indeed, a sheaf (exercise in [9, p.65] or [22, p.58]). It is also denoted $F\mid_Y$ or $F \upharpoonright Y$ and called the **restriction** of F to Y. If $F \in \text{Mod-}\mathcal{O}_X$ then $F\mid_Y \in \text{Mod-}\mathcal{O}_Y$ where we let $\mathcal{O}_Y$ denote $\mathcal{O}_X\mid_Y$ (cf. p. 106 of [7] where the notation j^{-1} is used for what we have denoted j^*: here the structure sheaf over a subspace is always that induced by the structure sheaf of the whole space, so the j^*/j^{-1} distinction does not arise).

Fact 4.1 *[9, p.97]* $j^* : \text{Mod-}\mathcal{O}_X \longrightarrow \text{Mod-}\mathcal{O}_Y$ *is exact and is left adjoint to the left exact functor* $j_* : \text{Mod-}\mathcal{O}_Y \longrightarrow \text{Mod-}\mathcal{O}_X$ *which is given by* $j_*G.U = G(U \cap Y)$ *for* $G \in \text{Mod-}\mathcal{O}_Y$ *and* U *open in* X *(the* **direct image** *functor):* $(j^*F, G) \simeq (F, j_*G)$.

Let $K \subseteq X$ be **locally closed** (the intersection of an open with a closed set), denote the inclusion by $j : K \longrightarrow X$, and let $G \in \text{Mod-}\mathcal{O}_K$. Define the sheaf $j_!G$ on X, the **extension** of G **by zero** by: $j_!G.U = \{s \in G(U \cap K) : \text{supp}(s)$ is closed in $U\}$. This is a sheaf and $j_!G \in \text{Mod-}\mathcal{O}_X$ (for an alternative description, see [22, p. 63/4]). Note that $j_!G \leq j_*G$. If K is open then this is the sheafification of the presheaf j_0F introduced in Section 2.

Fact 4.2 *[9, p.106/7]* $j_! : \text{Mod-}\mathcal{O}_K \longrightarrow \text{Mod-}\mathcal{O}_X$ *is an exact functor which is an equivalence between* $\text{Mod-}\mathcal{O}_K$ *and the category of* $\mathcal{O}_X$*-modules which have all stalks over* $X \setminus K$ *equal to 0.*

Now, given $F \in \text{Mod-}\mathcal{O}_X$, let F^K be the sheaf (clearly it is a sheaf) given by $F^K U = \{s \in FU : \text{supp}(s) \subseteq K\}$. So $F^K \leq F$. Set $j^!F = j^*F^K$, therefore sections of $j^!F$ are locally given by sections of F with support contained in K.

Fact 4.3 *[9, p.108/9] The functor* $j^! : \text{Mod-}\mathcal{O}_X \longrightarrow \text{Mod-}\mathcal{O}_K$ *is left exact and is right adjoint to the functor* $j_! : (j_!G, F) \simeq (G, j^!F)$.

Fact 4.4 *[9, p.109]*
If $K = C$ *is closed then* $j_* = j_!$.
If $K = U$ *is open then* $j^* = j^!$.
Always $j^!F \leq j^*F$.

Fact 4.5 *[9, p.110], [22, 3.8.11] If* $U \subseteq X$ *is open,* $C = X \setminus U$ *is its complement and* $F \in \text{Mod-}\mathcal{O}_X$ *then there is an exact sequence*
$0 \longrightarrow j_!(F\mid_U) \longrightarrow F \longrightarrow i_!(i^*F) = i_*(i^*F) \longrightarrow 0$
where $j : U \longrightarrow X$ *and* $i : C \longrightarrow X$ *are the inclusions.*

In particular we have an exact sequence
$0 \longrightarrow j_!\mathcal{O}_U \longrightarrow \mathcal{O}_X \longrightarrow i_!\mathcal{O}_C \longrightarrow 0.$

It follows that if C is closed, K is locally closed and $C \subseteq K \subseteq X$ then there is an exact sequence

$0 \longrightarrow i'_! \mathcal{O}_{K\setminus C} \longrightarrow \mathcal{O}_K \longrightarrow i_! \mathcal{O}_C \longrightarrow 0$

where $i : C \longrightarrow K$ and $i' : K \setminus C \longrightarrow K$ are the inclusions. Then, since $j_!$ is exact, where $j : K \longrightarrow X$ is the inclusion, we have

$0 \longrightarrow j_! i'_! \mathcal{O}_{K\setminus C} \longrightarrow j_! \mathcal{O}_K \longrightarrow j_! i_! \mathcal{O}_C \longrightarrow 0$

that is,

$0 \longrightarrow (ji')_! \mathcal{O}_{K\setminus C} \longrightarrow j_! \mathcal{O}_K \longrightarrow (ji)_! \mathcal{O}_C \longrightarrow 0$

where $ji' : K \setminus C \longrightarrow X$, $j : K \longrightarrow X$ and $ji : C \longrightarrow X$ are the inclusions.

Lemma 4.6 *Let $C \subseteq X$ be closed and let $U = X \setminus C$. The canonical sequence $0 \longrightarrow j_! \mathcal{O}_U \longrightarrow \mathcal{O}_X \longrightarrow i_! \mathcal{O}_C \longrightarrow 0$ is split iff C is open, iff U is closed. (Here and elsewhere letters such as j, i will denote the obvious inclusions.)*

Proof. Suppose that we have $g : i_! \mathcal{O}_C \longrightarrow \mathcal{O}_X$ splitting the surjection $f : \mathcal{O}_X \longrightarrow i_! \mathcal{O}_C$. Let $1_C \in i_! \mathcal{O}_C . X$ be the section with $(1_C)_x = \begin{cases} 1_x \in \mathcal{O}_{X,x} & \text{if } x \in C \\ 0 & \text{otherwise} \end{cases}$. (Note that this is a global section since $\mathrm{supp}(1_C)$ is closed in X.)

We have $i_! \mathcal{O}_C \longrightarrow \mathcal{O}_X \longrightarrow i_! \mathcal{O}_C$ with $1_C \mapsto s = g 1_C \mapsto 1_C$.

Because f_x is the identity on stalks over points $x \in C$ (see [9, p.97, 4.3]) we have $s_x = (1_C)_x = 1_x$ at $x \in C$.

Also for any morphism h and section t over U we have $\mathrm{supp}(h_U(t)) \subseteq \mathrm{supp}(t)$, so $\mathrm{supp}(s) \subseteq C$ and hence $s_x = 0$ for all $x \in X \setminus C$.

Consider $1_X - s \in \mathcal{O}_X X$: we have $(1_X - s) = \begin{cases} 1_x - 1_x = 0 & \text{if } x \in C \\ 1_x - 0 & \text{if } x \notin C \end{cases}$. So $\mathrm{supp}(1_x - s) = X \setminus C$ and, since the support of any section must be closed, we deduce that C is open in X.

Conversely, if C is open in X then we have an inclusion $0 \longrightarrow i_! \mathcal{O}_C \longrightarrow \mathcal{O}_X (\longrightarrow j_! \mathcal{O}_{X\setminus C} \longrightarrow 0)$ which clearly splits f, as required. □

Since, if $K \subseteq X$ is locally closed, the functor $j_!$ is exact, where $j : K \longrightarrow X$ is the inclusion, we have the more general result.

Lemma 4.7 *Let $C \subseteq K \subseteq X$ with C closed and K locally closed, $j : C \longrightarrow X$ and $j : L \longrightarrow X$ the inclusions. Then the canonical epimorphism $j_! \mathcal{O}_K \longrightarrow (ji)_! \mathcal{O}_C$ (from the exact sequence before 4.6) is split iff C is open in K.*

Proof. If C is open in K then we have a split exact sequence as in 4.6 so then apply $j_!$.

Conversely, if $j_! \mathcal{O}_K \longrightarrow (ji)_! \mathcal{O}_C = j_! i_! \mathcal{O}_C$ is split then apply $(j)^*$, noting that $(j)^* j_! = \mathrm{Id}$, and then apply 4.6. □

The dual statement follows immediately.

Lemma 4.8 *Let $U \subseteq K \subseteq X$ with U open and K locally closed, $i : U \longrightarrow X$ and $j : K \longrightarrow X$ the inclusions. Then the canonical monomorphism $i_!\mathcal{O}_U \longrightarrow j_!\mathcal{O}_K$ is split iff U is closed in K.*

Lemma 4.9 *Let $C = \bigcap_\lambda C_\lambda$ be closed sets, where the intersection is directed (i.e. $\forall \lambda, \mu \, \exists \nu \, C_\nu \subseteq C_\lambda \cap C_\mu$). Let $((j_!\mathcal{O}_{C_\lambda})_\lambda, (g_{\lambda\mu} : j_!\mathcal{O}_{C_\lambda} \longrightarrow j_!\mathcal{O}_{C_\mu})_{C_\lambda \supseteq C_\mu})$ be the corresponding directed system of epimorphisms. Then $\varinjlim_\lambda (j_!\mathcal{O}_{C_\lambda}) = j_!\mathcal{O}_C$ and each limit map $g_{\lambda\infty} : j_!\mathcal{O}_{C_\lambda} \longrightarrow j_!\mathcal{O}_C$ is an epimorphism.*

Proof. Let $G = \varinjlim j_!\mathcal{O}_{C_\lambda}$.

For each λ the inclusion $C \longrightarrow C_\lambda$ gives rise, by the canonical exact sequence, to an epimorphism $h_\lambda : j_!\mathcal{O}_{C_\lambda} \longrightarrow j_!\mathcal{O}_C$ and these are compatible, so we have a unique map $h : G \longrightarrow j_!\mathcal{O}_C$ with $hg_{\lambda\infty} = h_\lambda$ for all λ. We show that h is an isomorphism - so it is sufficient to show that h is an isomorphism at each stalk.

Let $x \in X$. Then $G_x (= \varinjlim_{x \in U} GU) = \varinjlim_{x\in U} \varinjlim_\lambda G_\lambda U$ (the presheaf and sheaf limits agree on stalks) $= \varinjlim_\lambda \varinjlim_{x \in U} G_\lambda U = \varinjlim_\lambda (G_\lambda)_x = \begin{cases} \mathcal{O}_x & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$.

So G has the same stalks as $j_!\mathcal{O}_C$ and since all maps in the system are, at the level of stalks, either the identity or zero, we can check that $h_x = \begin{cases} \mathrm{id} : \mathcal{O}_x \longrightarrow \mathcal{O}_x & \text{if } x \in C \\ 0 & \text{otherwise} \end{cases}$. So h_x is an isomorphism for each x and hence $G \simeq j_!\mathcal{O}_C$. Furthermore, the above yields that each $g_{\lambda\infty}$ is stalkwise an epimorphism and hence an epimorphism. $\square$

5 The category Mod-$\mathcal{O}_X$: local finite presentation

We would like to determine when a category, Mod-$\mathcal{O}_X$, of sheaves is locally finitely presented. We should point out that every category Mod-$\mathcal{O}_X$ is locally presentable because it is a Grothendieck topos (e.g. [3, 3.4.2, 3.4.16]). It has been pointed out to us by F. Borceux that, by insisting that our rings have $1 \neq 0$, we are dealing with a topos-theoretically unnatural class of ringed spaces. Nevertheless, in our motivating examples, the ringed spaces which arise do have $1 \neq 0$ and, indeed, Borceux observed that without this assumption we would not have the possibility of having a strong connection between the structure of the space X and the category of sheaves of $\mathcal{O}_X$-modules for arbitrary ringed spaces $\mathcal{O}_X$ on X.

Remark 5.1 *(cf. [2, p. 260]) The sheaves $j_!\mathcal{O}_U$, with $U \subseteq X$, open together generate* Mod-$\mathcal{O}_X$.

Proof. Since the localisation of a set of generators gives a set of generators for the quotient category (for localisation is exact), this follows from 2.14 on applying the sheafification localisation functor. $\square$

Proposition 5.2 *If $\mathcal{U}$ is a basis of open sets for X then the $j_!\mathcal{O}_U$ for $U \in \mathcal{U}$ together generate* Mod-$\mathcal{O}_X$.

Proof. (Note that this does not follow *via* localisation of PreMod-$\mathcal{O}_X$ since the corresponding result for that category is false.)

Let $F \in$ Mod-$\mathcal{O}_X$, let $x \in X$ and take $a \in F_x$. Since $F_x = \varinjlim_{x \in U} FU$ there is U open and $b = b_{U,a} \in FU$ such that the canonical map from FU to F_x takes b to a. Without loss of generality $U \in \mathcal{U}$. Define $f' : \mathcal{O}_U \longrightarrow F \upharpoonright U$ by $1 \in \mathcal{O}_U U \mapsto b_{U,a} \in FU$ (by 2.11 this defines a presheaf morphism and this is enough).

Since $j_!$ is left adjoint to $(-)_U$ there is, corresponding to $f' \in (\mathcal{O}_U, F \upharpoonright U)$, a morphism $f_{x,U,a} \in (j_!\mathcal{O}_U, F)$ with $(f_{x,U,a})_U : 1_U \mapsto b_{U,a}$.

Consider the image of 1_U in $(j_!\mathcal{O}_U)_x$, that is, $1_{\mathcal{O}_{X,x}}$. Then $(f_{x,U,a})_x : (j_!\mathcal{O}_U)_x \longrightarrow F_x$ - the stalkwise at x map induced by $f_{x,U,a}$ - maps $1_{\mathcal{O}_{X,x}}$ to $a \in F_x$.

Hence $\bigoplus_{x\in X} \bigoplus_{a \in F_x} f_{x,U,a} : \bigoplus_x \bigoplus_a j_!\mathcal{O}_{U=U(x,a)} \longrightarrow F$ is an epimorphism on stalks and hence is an epimorphism in the category of sheaves.
$\square$

Proposition 5.3 *Suppose $\varinjlim_\lambda G_\lambda = G$ in* Mod-$\mathcal{O}_X$ *and suppose that $U \subseteq X$ is compact open. Then the canonical map $g : \varinjlim_\lambda (G_\lambda U) \longrightarrow GU$ is an isomorphism.*

Proof. The sheaf G is the sheafification of the presheaf direct limit, G', of the G_λ and this is given by $G'V = \varinjlim(G_\lambda V)$ for $V \subseteq X$ open. So the proposition asserts that G' and G agree at all compact open sets. Let $g : G' \longrightarrow G$ be the sheafification map. We show that g_U is an isomorphism.

Let $s \in G'U$ and suppose that $g_U s = 0$. Then there must be an open cover $\{U_i\}_i$ of U such that, for all i, we have $\mathrm{res}^{G'}_{U,U_i} s = 0$. Since U is compact we may take the cover to be finite: $U_1, ..., U_n$ say. Then, by definition of the restriction maps in the limit, for each i there are λ_i and $a_i \in G_{\lambda_i} U$ such that $(g'_{\lambda_i,\infty})_U a_i = s$ (where $g'_{\lambda_i,\infty} : G_{\lambda_i} \longrightarrow G'$ is the canonical map) and $(g'_{\lambda_i,\infty})_{U_i} \mathrm{res}^{G_{\lambda_i}}_{U,U_i} a_i = 0$. Since there are just finitely many U_i we may take λ with $\lambda \geq \lambda_1, ..., \lambda_n$ and also such that $(g_{\lambda_i,\lambda})_U a_i = (g_{\lambda_j,\lambda})_U a_j = b$, say, for all i, j (since the a_i all map to the same element in the limit) and, furthermore, such that $(g_{\lambda_i,\lambda})_{U_i} \mathrm{res}^{G_{\lambda_i}}_{U,U_i}(a_i) = 0$ for all i (since each $\mathrm{res}^{G_{\lambda_i}}_{U,U_i} a_i$ maps to 0 in the presheaf limit). But then $\mathrm{res}^{G_\lambda}_{U,U_i}(b) = (g_{\lambda_i,\lambda})_{U_i} \mathrm{res}^{G_{\lambda_i}}_{U,U_i}(a_i) = 0$ for each i and hence, since G_λ is a sheaf, $b = 0$. Therefore $s = (g'_{\lambda,\infty})_U(b) = 0$ as required.

To see that g_U is onto, take any $t \in GU$. For each $x \in U$ there is an open neighbourhood U_x of x and $t_x \in G'U_x$ such that $g_{U_x} t_x = \mathrm{res}^G_{UU_x}(t)$. Since U is compact we may take finitely many open sets $U_1, ..., U_n$ say, with corresponding $t_i \in G'U_i$, which cover U.

For each i there is λ_i and $s_i \in G_{\lambda_i} U_i$ with $g'_{\lambda_i \infty} s_i = t_i$. Since there are only finitely many λ_i we may take $\lambda \geq \lambda_1, ..., \lambda_n$ and we may suppose that $s_i \in G_\lambda U_i$ for each i.

We have, furthermore, that for each pair, i, j, of indices, $\mathrm{res}^{G'}_{U_i, U_i \cap U_j}(t_i) = \mathrm{res}^{G'}_{U_j, U_i \cap U_j}(t_j)$ that is, $g'_{\lambda\infty} \mathrm{res}^{G_\lambda}_{U_i, U_i \cap U_j}(s_i) = g'_{\lambda\infty} \mathrm{res}^{G_\lambda}_{U_j, U_i \cap U_j}(s_j)$ and hence there is $\mu_{ij} \geq \lambda$ such that $g'_{\lambda\mu_{ij}} \mathrm{res}^{G_\lambda}_{U_i, U_i \cap U_j}(s_i) = g'_{\lambda\mu_{ij}} \mathrm{res}^{G_\lambda}_{U_j, U_i \cap U_j}(s_j)$. So, choosing $\mu \geq \mu_{ij}$ for each i, j and setting $s'_i = g_{\lambda\mu}(s_i)$ we may suppose that for all i, j we have $\mathrm{res}^{G_\mu}_{U_i, U_i \cap U_j}(s'_i) = \mathrm{res}^{G_\mu}_{U_j, U_i \cap U_j}(s'_j)$. Since G_μ is a sheaf, there is $s \in G_\mu U$ such that $\mathrm{res}^{G_\mu}_{U, U_i}(s) = s_i$ for each $i = 1, ..., n$. Then $g'_{\mu\infty} s \in G'U$ with (since G is separated) $g g'_{\mu\infty} s = t$, as required. $\square$

Corollary 5.4 *Suppose that every open subset of X is compact. Let $(G_\lambda)_\lambda$ be a directed system in* $\mathrm{Mod}\text{-}\mathcal{O}_X$*. Then the direct limit $\varinjlim G_\lambda$, computed in* $\mathrm{PreMod}\text{-}\mathcal{O}_X$ *is a sheaf and hence equals $\varinjlim G_\lambda$, computed in* $\mathrm{Mod}\text{-}\mathcal{O}_X$*:*

$q(\varinjlim{}_\lambda (iG_\lambda) = \varinjlim{}_\lambda G_\lambda$

where i is the inclusion of $\mathrm{Mod}\text{-}\mathcal{O}_X$ *in* $\mathrm{PreMod}\text{-}\mathcal{O}_X$ *and q is the sheafification functor.*

From the adjunction $(j_! \mathcal{O}_U, F) \simeq (\mathcal{O}_U, F \upharpoonright U) \simeq \Gamma(F \upharpoonright U) \simeq FU$ for $U \subseteq X$ open we deduce that $\Gamma_U(-) \simeq (j_! \mathcal{O}_U, -)$ as functors in $(\mathrm{Mod}\text{-}\mathcal{O}_X, \mathbf{Ab})$. Here Γ denotes the global section functor, $F \mapsto FX$, and Γ_U is the functor $F \mapsto FU$.

Corollary 5.5 *If $U \subseteq X$ is compact open then $j_! \mathcal{O}_U$ is a finitely presented object of* $\mathrm{Mod}\text{-}\mathcal{O}_X$*.*

Proof. Let $(G_\lambda)_\lambda$ be a directed system in $\mathrm{Mod}\text{-}\mathcal{O}_X$ with direct limit G. Then, as just noted, $(j_! \mathcal{O}_U, G) \simeq GU$ and $\varinjlim{}_\lambda (j_! \mathcal{O}_U, G_\lambda) \simeq \varinjlim (G_\lambda U)$ - and these coincide by 5.3, as required. $\square$

Theorem 5.6 *[20] If X has a basis of compact open sets then* $\mathrm{Mod}\text{-}\mathcal{O}_X$ *is locally finitely presented, with the $j_! \mathcal{O}_U$ for $U \subseteq X$ compact open (or with the U from any basis of such sets) as a generating set of finitely presented objects of* $\mathrm{Mod}\text{-}\mathcal{O}_X$*.*

Corollary 5.7 *If X is locally noetherian and $\mathcal{O}_X$ is any sheaf of rings on X then* $\mathrm{Mod}\text{-}\mathcal{O}_X$ *is locally finitely presented.*

We also have the converse to 5.5.

Proposition 5.8 *If U is open then $j_!\mathcal{O}_U$ is finitely presented in* $\text{Mod-}\mathcal{O}_X$ *iff U is compact.*

Proof. If U is compact then we have 5.5 so suppose that U is not compact - say $\{U_i\}_i$ is an open cover with no finite subcover. We may suppose that $\{U_i\}_i$ is closed under finite union. Set $G_i = j_!\mathcal{O}_{U_i}$ and $G = \varinjlim j_!\mathcal{O}_{U_i}$. Then $G = j_!\mathcal{O}_U$ since from the canonical inclusions $j_!\mathcal{O}_{U_i} \longrightarrow j_!\mathcal{O}_U$ we obtain a map $G = \varinjlim j_!\mathcal{O}_{U_i} \longrightarrow j_!\mathcal{O}_U$ which locally, and hence stalkwise, is an isomorphism and which is, therefore, an isomorphism (cf. 4.9).

So the identity map of G would factor through some $j_!\mathcal{O}_{U_i}$ if $G = j_!\mathcal{O}_U$ were finitely presented - but since $U_i \neq U$ there can be no such factorisation (recall that if $x \notin U_i$ then $(j_!\mathcal{O}_{U_i})_x = 0$). $\square$

Example 5.9 *If $F \in$* $\text{Mod-}\mathcal{O}_X$ *is finitely presented and $U \subseteq X$ is open then $F\mid_U$ might not even be finitely generated. Let $X = [0,1]$ and take $U = (0,1)$. Take $\mathcal{O}_X$ to be the sheafification of the constant presheaf k. By 5.5 $\mathcal{O}_X$ is a finitely presented sheaf but, by 5.8, $\mathcal{O}_U$ is not, because U is not compact. Indeed, $\mathcal{O}_U$ is not even finitely generated, as one sees by writing $\mathcal{O}_U$ as the sum over $n \geq 1$ of the sheaves $j_!O_{(\frac{1}{n},1-\frac{1}{n})}$.*

6 The category $\text{Mod-}\mathcal{O}_X$: local finite generation

Now we determine some properties of finitely presented and finitely generated sheaves.

Suppose that $F \in \text{Mod-}\mathcal{O}_X$ and $s \in FX$. Define a subpresheaf $\langle s\rangle^0$ of F by setting: $\langle s\rangle^0 X = sR_X$, $\langle s\rangle^0 U = \text{res}^F_{X,U}(s).R_U$ (recall that $R_U = \mathcal{O}_X U$) and with the restriction maps coming from F. This is a separated presheaf, but is not necessarily a sheaf.

Let $\langle s\rangle$ denote the sheafification of $\langle s\rangle^0$ - so this is a subsheaf of F and, up to isomorphism, it is defined by the exact sequence $0 \longrightarrow \text{ann}_{\mathcal{O}_X} s \longrightarrow \mathcal{O}_X \longrightarrow \langle s\rangle \longrightarrow 0$.

More generally, if $U \subseteq X$ is open and $s \in FU$ then we define the subsheaf of F **generated by** s to be $j_!\langle s\rangle$, where $j : U \longrightarrow X$ is the inclusion and $\langle s\rangle \leq F\mid_U$ is defined as above.

Recall that there is an inclusion $j_!F\mid_U \longrightarrow F$ and so, since $j_!$ is (left) exact, $j_!\langle s\rangle$ is indeed a subsheaf of F. Although we call it the sheaf generated by s it need not be a finitely generated sheaf - unless U is compact s might not be a "finitary element" of F. Here, though, is some justification for the terminology.

Lemma 6.1 *Let $F \in$* $\text{Mod-}\mathcal{O}_X$*, let $U \subseteq X$ be open and let $s \in FU$. Suppose that $G \leq F$ is a subsheaf such that $s \in GU$. Then $j_!\langle s\rangle \leq G$.*

Proof. It is immediate from the definition that $\langle s\rangle^0$ is a subpresheaf of G_U and hence that $\langle s\rangle \leq G_U$. Therefore $j_!\langle s\rangle \leq j_!(G_U) \leq G$. $\square$

Lemma 6.2 *Let $F \in \text{Mod-}\mathcal{O}_X$. Then $F = \sum\{j_!\langle s\rangle : U \subseteq X$ is open and $s \in FU\}$ (we write $j_!$ for $(j_U)_!$ where $j_U : U \longrightarrow X$ is the inclusion).*

Proof. By the remarks above, F contains the right hand side. Conversely, given $U \subseteq X$ open and $s' \in FU$ we have $s' \in j_!\langle s'\rangle.U$ (since $s' \in \langle s'\rangle^0 U$) so $s' \in (\sum_U \sum_{s\in FU} j_!\langle s\rangle).U$ as required. $\square$

Note that infinite sums in Mod-$\mathcal{O}_X$ are obtained by first forming the presheaf sum (that is, algebraic sum of U-sections at each open $U \subseteq X$) and then sheafifying. We say that a sheaf F is **finitely generated** if whenever $F = \sum_\lambda F_\lambda$ with the F_λ subsheaves of F, then there are $\lambda_1, ..., \lambda_n$ such that $F = F_{\lambda_1} + ... + F_{\lambda_n}$. It is immediate that any finitely presented sheaf is finitely generated.

Lemma 6.3 *If $F \in \text{Mod-}\mathcal{O}_X$, $s \in FX$ and $(V_\lambda)_\lambda$ is an open cover of X then $\langle s\rangle = \sum_\lambda j_!\langle \text{res}^F_{X,V_\lambda} s\rangle$.*

Proof. Arguing as above, the right hand side, G say, is a subpresheaf of $\langle s\rangle$. We have, for each λ, a section in GV_λ which agrees with s on V_λ and hence, since $\langle s\rangle$ is a sheaf, we deduce that $s \in GX$. Therefore, by 6.1, $\langle s\rangle \leq G$, as required. $\square$

If $s \in FU$ then the **support** of s is $\text{supp}(s) = \{x \in X : s_x \neq 0\}$ and this is always a closed subset of U. Also set $\text{supp}(F) = \{x \in X : F_x \neq 0\}$.

Lemma 6.4 *Suppose that $F \in \text{Mod-}\mathcal{O}_X$ is finitely generated. Then there are open subsets $U_1, ..., U_n$ of X and $s_i \in F(U_i)$ such that $\text{supp}(F) = \bigcup_1^n \text{supp}(s_i)$. In particular, $\text{supp}(F)$ is a locally closed subset of X.*

If $F = \sum_i j_!\langle s_i\rangle$ where $s_i \in F(U_i)$ then we may take the U_i and s_i from this representation.

Proof. Take a representation of F as given (we know there is such by 6.2). Since F is finitely generated we have $F = \sum_1^n j_!\langle s_i\rangle$, say. Certainly $\text{supp}(F) \supseteq \bigcup_1^n \text{supp}(s_i)$.

Conversely, if $x \in \text{supp}(F)$ then there is an open set V containing x and $t \in FV$ such that $t_x \neq 0$. We have $FV = \sum_1^n j_!\langle s_i\rangle.V$. If $a \in j_!\langle s_i\rangle.V$ then $\text{supp}(a) \subseteq \text{supp}(s_i)$ (using the definition of $\langle s\rangle^0$). But then $t \in \sum_1^n j_!\langle s_i\rangle.V$ implies $x \in \text{supp}(s_i)$ for some i, as required.

Finally, $\text{supp}(F)$ is locally closed since it is a finite union of closed subsets of open sets. $\square$

Proposition 6.5 *Let $F \in \text{Mod-}\mathcal{O}_X$ be finitely generated. Then $\text{supp}(F)$ is compact.*

Proof. We know that $F = \sum_1^n j_!\langle s_i\rangle$ for some $s_i \in F(U_i)$ for some open U_i and then, by the lemma above, $\text{supp}(F) = \bigcup_1^n \text{supp}(s_i)$.

Suppose that $\text{supp}(F)$ is not compact. Then there are open subsets V_λ of X such that no finite number of these cover $\text{supp}(F)$. For each $i = 1, ..., n$ the $V_\lambda \cap U_i$ cover U_i and so, by 6.3, $F = \sum_{i=1}^n \sum_\lambda (j_{V_\lambda \cap U_i, X})_! \langle \text{res}^F_{U_i, V_\lambda \cap U_i}(s_i) \rangle$ (ignoring those where $V_\lambda \cap U_i = \emptyset$).

Since F is finitely generated there is a finite subsum $F = \sum_{k=1}^m (j_{V_{\lambda_k} \cap U_{i_k}, X})_! \langle \text{res}^F_{U_{i_k}, V_{\lambda_k} \cap U_{i_k}}(s_{i_k}) \rangle$. By 6.4 above we have $\text{supp}(F) = \bigcup_1^m \text{supp}(s_{i_k}) \cap V_{\lambda_k} \subseteq \bigcup_1^m V_{\lambda_k}$ - contradiction, as required. $\square$

Corollary 6.6 *Let $F \in \text{Mod-}\mathcal{O}_X$ be finitely presented. Let $s \in FX$. Then $\text{supp}(s)$ is compact. In particular, if F is finitely presented and $FX \neq 0$ then X contains a compact closed set.*

Proof. We have $\text{supp}(s) \subseteq \text{supp}(F)$ which is compact by 6.5 above. Therefore, $\text{supp}(s)$, being a closed subset of a compact set, is compact, as required. $\square$

Proposition 6.7 *Let $K \subseteq X$ be compact and locally closed. Then $j_! \mathcal{O}_K$ is finitely generated iff K is compact and locally closed.*

Proof. If $j_! \mathcal{O}_K$ is finitely generated then, by 6.4 and 6.5 K is locally closed and compact.

For the converse, suppose that K is locally closed and compact. We have $K = U_0 \cap C$ for some open U_0 and closed C. Let $U \subseteq U_0$ be open. Then $j_! \mathcal{O}_K . U = \{s \in \mathcal{O}_K(K \cap U) : \text{supp}(s) \text{ is closed in } U\}$ is generated by $1_{K \cap U}$ since $\text{supp}(1_{K \cap U}) = K \cap U = C \cap U$ is closed in U. Hence $j_! \mathcal{O}_K = \langle 1_K \rangle$.

Now suppose that $j_! \mathcal{O}_K = \sum_\lambda F_\lambda$ for some subsheaves F_λ. We may suppose that the sum is directed. Let $x \in K$. Since $\sum_\lambda F_\lambda$ is the sheafification of the presheaf sum of the F_λ there is an open set U_x containing x (wlog $U_x \subseteq U_0$) such that $1_{U_x \cap K} (\in j_! \mathcal{O}_K . U_x)$ belongs to the presheaf sum of the $F_\lambda U_x$ and hence (since the sum is directed) belongs to $F_\lambda U_x$ for some λ.

As x varies over K we get a cover $(U = U_x)_{x \in K}$ and so, by compactness, some finite subset, $U_{x_1}, ..., U_{x_n}$, covers K. Set $U_1 = U_{x_1} \cup ... \cup U_{x_n} \subseteq U_0$. The sum is directed so we may choose λ such that $1_{K \cap U_{x_i}} \in F_\lambda(U_{x_i})$ for $i = 1, ..., n$ and hence such that $1_K = 1_{K \cap U_1} \in F_\lambda(U_1)$ (since F_λ is a sheaf). Therefore $j_! \mathcal{O}_K = \langle 1_K \rangle \leq F_\lambda$, as required.
$\square$

Example 6.8 *If F is finitely presented and $x \in X$ it might not be the case that there is a local presentation of the form $\mathcal{O}_U^m \longrightarrow \mathcal{O}_U^n \longrightarrow F_U \longrightarrow 0$ for an open neighbourhood U of x, even if X has a basis of compact open sets. This is in contrast to way that the terms finitely generated and finitely presented are sometimes used in the algebraic-geometric situation: that is, the use of these terms in that situation, though appropriate for the kinds of sheaves considered* *there, should not be carried over verbatim to modules over arbitrary sheaves of rings.*

Take $X = \{a, b\}$ with open sets $X, \{b\}, \emptyset$ and let $\mathcal{O}_X$ be the constant sheaf k. Then $F = j_!\mathcal{O}_{\{a\}}$ (that is, the skyscraper sheaf at a), given by $X \mapsto k$, $\{b\} \mapsto 0$, is finitely presented by 5.5. The only open set containing a is X itself but any exact sequence $\mathcal{O}_X^m \longrightarrow \mathcal{O}_X^n \longrightarrow G \longrightarrow 0$ is such that $G_a \simeq G_b$ and so F cannot have such a local presentation.

Proposition 6.9 *Let $U \subseteq X$. Suppose that there is a finitely generated sheaf $F \in \text{Mod-}\mathcal{O}_X$ such that there is a non-zero homomorphism $f : F \longrightarrow j_!\mathcal{O}_U$. Then U contains a compact locally closed set. If $x \in X$ is such that the morphism of stalks $f_x : F_x \longrightarrow O_{X,x}$ is non-zero then this compact locally closed set may be taken to contain x.*

Proof. Say $F = \sum_1^n j_!\langle s_i \rangle$ with $s_i \in F(U_i)$. Let $F' = \text{im}(f)$. Being an image of a finitely generated object, F' is finitely generated. Since F' is a non-zero subfunctor of $j_!\mathcal{O}_U$, we have $\emptyset \neq \text{supp}(F') \subseteq \text{supp}(j_!\mathcal{O}_U) \subseteq U$ and, by 6.5 above, $\text{supp}(F')$ is compact. The set $\text{supp}(F')$ is also locally closed by 6.4. Finally, if $f_x \neq 0$ then $x \in \text{supp}(F')$. $\square$

A Grothendieck abelian category $\mathcal{C}$ is **locally finitely generated** if every object is an epimorphic image of a direct sum of finitely generated objects.

Proposition 6.10 *Suppose that $\text{Mod-}\mathcal{O}_X$ is locally finitely generated. Then for every $x \in X$ and open set U containing x there is a compact locally closed set K with $x \in K \subseteq U$.*

Proof. Let $x \in X$ and let U be an open neighbourhood of x. By assumption there is a finitely presented $F \in \text{Mod-}\mathcal{O}_X$ and a morphism $f : F \longrightarrow j_!\mathcal{O}_U$ such that $f_x \neq 0$. By the above result there is a compact locally closed subset of X which is contained in U and which contains x, as required. $\square$

We now strengthen this result. Say that X is **locally compact** if for every $x \in X$ and for every open set U containing x there is an open set V containing x and a compact locally closed set K with $x \in V \subseteq K \subseteq U$.

Theorem 6.11 *Suppose that $\text{Mod-}\mathcal{O}_X$ is locally finitely generated. Then X is locally compact.*

Proof. Given x and U, consider $j_!\mathcal{O}_U$. By assumption there is an epimorphism from a direct sum of finitely generated sheaves to $j_!\mathcal{O}_U$ and this must be surjective on stalks. Hence there is $F \in \text{Mod-}\mathcal{O}_X$ finitely generated and $f : F \longrightarrow j_!\mathcal{O}_U$ such that $f_x : F_x \longrightarrow (j_!\mathcal{O}_U)_x = \mathcal{O}_x$ has $1_x \in \mathcal{O}_x$ in its image (and hence which is surjective at x). Therefore there is an open set V' with $x \in V' \subseteq U$ and sections $s' \in FV'$ and $t' \in j_!\mathcal{O}_U.V'$ with $f_{V'}s' = t'$ and $t' \mapsto 1_x$ under the canonical map $\mathcal{O}_U V' \longrightarrow \mathcal{O}_x$. Since $(t' - 1_{V'})_x = 0$ there is an open set

V with $x \in V \subseteq V'$ and with $\mathrm{res}_{V'V} t' = \mathrm{res}_{V'V}(1_{V'}) = 1_V$. So there is an open set V with $x \in V \subseteq U$ and a section $(\mathrm{res}^F_{V'V}(s') =)\ s$ with $f_V s = 1_V \in \mathcal{O}_V V$. So we have $V \supseteq \mathrm{supp}(s) \supseteq \mathrm{supp}(f_V s) = V$, that is $\mathrm{supp}(s) = \mathrm{supp}(f_V s) = V$.

Now, F finitely generated implies that fF is a finitely generated subsheaf of $j_! \mathcal{O}_U$ and so, by 6.7, $\mathrm{supp}(fF) \subseteq \mathrm{supp} j_! \mathcal{O}_U = U$ is a compact locally closed subset of X which contains x. $\square$

Whether or not Mod-$\mathcal{O}_X$ is locally finitely generated depends on $\mathcal{O}_X$, not just on X.

Example 6.12 *Let X be the closed interval $[0,1]$ and let $\mathcal{O}_X$ be the sheaf of continuous functions from X to $\mathbb{R}$. We show that the $j_! \mathcal{O}_K$ with $K = [e,f], 0 \leq e < f \leq 1$ are generating (these are finitely generated by 6.7). Since, by 5.2, the $i_! \mathcal{O}_U$ with $U = (c,d)$ are generating it is enough to show that, given such an open $U, x \in U$ and $g \in \mathcal{O}_{X,x} = (i_! \mathcal{O}_U)_x$, there is such a set K and $\phi : j_! \mathcal{O}_K \longrightarrow i_! \mathcal{O}_U$ with ϕ_x having g in its image.*

Choose a continuous function $h : [0,1] \longrightarrow \mathbb{R}$ such that $\mathrm{supp}(h)$, $= K$ *say, is a closed subinterval of U and such that $h \restriction V = 1$ (the constant function) on some non-empty open set $V \subseteq K$ with $x \in V$. Then choose a function on $[0,1]$ which has germ g at x and replace the first choice of h by this function times h.*

Since this function h is a global section with support contained in K it induces a morphism $\phi : j_! \mathcal{O}_K \longrightarrow i_! \mathcal{O}_U$ (under the adjunction $(j_! \mathcal{O}_K, i_! \mathcal{O}_U) \simeq (\mathcal{O}_K, j^! i_! \mathcal{O}_U)$) and $\phi_x h_x = g$, as required.

The same argument applies if we take $\mathcal{O}_X$ to be the sheaf of smooth functions from $[0,1]$ to $\mathbb{R}$ (see e.g. [9, p.158] for the construction of a smooth function like h above).

Example 6.13 *Let $X = [0,1]$ and let $\mathcal{O}_X$ be the "constant" sheaf with values in, say, a field. Then* Mod-$\mathcal{O}_X$ *is not locally finitely generated.*

For suppose that we have a finitely generated $\mathcal{O}_X$-module, F, and a non-zero morphism $F \longrightarrow i_! \mathcal{O}_U$ where U is any non-empty proper open subset of X. By 6.5 $\mathrm{supp}(F) = K$ *is compact, hence closed. Without loss of generality, F is cyclic (factor out all but one, suitably chosen, element in a generating set). Then there is an epimorphism $j_! \mathcal{O}_K \longrightarrow F$ and hence a non-zero morphism $j_! \mathcal{O}_K \longrightarrow i_! \mathcal{O}_U$. Therefore $i_! \mathcal{O}_U$ has a non-zero section s in an open neighbourhood of K (see the description, 7.1, of the functor $(j_! \mathcal{O}_K, -)$ at the beginning of the next section). This open neighbourhood is a union of disjoint open intervals and, on each interval, each section is constant so, if non-zero, has support the whole of that interval. Therefore* $\mathrm{supp}(s)$ *is a non-empty union of disjoint open intervals and so cannot be closed - contradiction, as required.*

7 More on finitely presented and finitely generated sheaves

Recall that $j_!\mathcal{O}_K$ is finitely presented if the functor $(j_!\mathcal{O}_K, -)$ commutes with direct limits. We give an alternative description of this functor: by 4.3 we have $(j_!\mathcal{O}_K, F) \simeq (\mathcal{O}_K, j^!F)$. Hence $(j_!\mathcal{O}_K, F) \simeq j^!F(K) = j^*F^K(K)$.

Proposition 7.1 *Let K be locally closed. Then $(j_!\mathcal{O}_K, F) \simeq j^*F^K K = \varinjlim_{U\supseteq K}(F^K U)$ where the direct limit is taken over all open $U \subseteq X$ containing K.*

Proof. We remark that the second equality is proved in [9, p. 149] for general $F \in \text{Mod-}\mathcal{O}_X$ in place of F^K but with some conditions on the space X. We follow the proof there but at the point where Iversen uses an assumption that X is Hausdorff we use the particular features of F^K instead.

First note that, by definition of j^*, we have $s \in j^*F^K K$ iff for each $x \in K$ there is an open neighbourhood V_x of x in X and a section $t \in F^K V_x$ such that s and t agree on $K \cap V_x$. Since, by definition of F^K, we have $\text{supp}(t), \text{supp}(s) \subseteq K$, in fact such s and t agree on V_x.

For each open set $U \subseteq X$ with $K \subseteq U$ there is a natural map $F^K U \longrightarrow j^*F^K K$ given by restriction and hence we have a canonical map $h : \varinjlim_{U\supseteq K}(F^K U) \longrightarrow j^*F^K K$.

Suppose that $a \in \varinjlim_{U\supseteq K}(F^K U)$ with $ha = 0$. Take $U \supseteq K$ and $t \in F^K U$ such that $r_{U,\infty}t = a$, where $r_{U,\infty} : F^K U \longrightarrow \varinjlim_{V\supseteq K}(F^K V)$ is the canonical map. Note that $hr_{U,\infty}$ is restriction to K so, since $hr_{U,\infty}t = 0$, for each $x \in K$ there is an open neighbourhood U_x of x in X such that $\text{res}^{F^K}_{U,U_x} t = 0$, that is, $t_x = 0$. Since $\text{supp}(t) \subseteq K$, we have $t = 0$ and hence $a = r_{U\infty}t = 0$, as required.

To see that h is onto, let $s \in j^*F^K K$. For each $x \in K$ choose an open neighbourhood V_x of x in X and $t_x \in F^K V_x$ such that s and t_x agree on $K \cap V_x$ and hence on V_x. Note that if $y \in K$ then t_x and t_y agree not just on $(K \cap V_x) \cap (K \cap V_y)$ but on $V_x \cap V_y$ and hence there is $t \in F^K(V_x \cup V_y)$ which agrees with t_x on V_x and with t_y on V_y - hence which agrees with s on $(V_x \cup V_y) \cap K$.

Therefore, if $V = \bigcup\{V_x : x \in K\}$ then the compatible sections t_x yield a section $t \in F^K V$ which agrees with s on K. That is, $hr_{V\infty}t = s$, as required. $\square$

So, loosely, $(j_!\mathcal{O}_K, -)$ is the functor "sections with support in K" or, more precisely, "germs of sections with support in K".

Proposition 7.2 *Let $K \subseteq X$ be compact and locally closed, say $K = U \cap C$ with U open and C closed. Suppose that for every downwards-directed system $(K_\lambda)_\lambda$ of closed subsets of U with $\bigcap_\lambda K_\lambda = K$ there exists U' open and λ with $U' \cap K_\lambda = K$. Let $((G_\lambda)_\lambda, (g_{\lambda\mu})_{\lambda\leq\mu})$ be a directed system in* $\text{Mod-}\mathcal{O}_X$ *and set $G = \varinjlim_\lambda G_\lambda$ with limit maps $g_{\lambda\infty} : G_\lambda \longrightarrow G$. Then $G^K K = (\varinjlim_\lambda(G^K_\lambda))K$.*

Furthermore if g' is the canonical map from the presheaf limit, $\mathrm{p}\varinjlim_\lambda(G_\lambda^K)$ *to* G^K *then* g'_K *is monic.*

Proof. Let V be an open set containing K. Note that the $G_\lambda^K V$ form a directed system since, if $f : G_1 \longrightarrow G_2$ is a morphism of sheaves and $s \in G_1 V$, then $\mathrm{supp}(fs) \subseteq \mathrm{supp}(s)$. For the same reason we have $(g_{\lambda\infty})_V . G_\lambda^K V \leq G^K V$ for all V and all λ and hence there is induced, for each V, a canonical map $g'_V : \varinjlim_\lambda (G_\lambda^K V) \longrightarrow G^K V$ and these fit together to give a map $g' : G' = \mathrm{p}\varinjlim_\lambda(G_\lambda^K) \longrightarrow G^K$ where $\mathrm{p}\varinjlim$ denotes the presheaf direct limit.

We have a commutative diagram as shown where $G' \longrightarrow G'' = \mathrm{p}\varinjlim_\lambda G_\lambda$ is the natural map induced by the inclusions $G_\lambda^K \longrightarrow G_\lambda$.

$$\begin{tikzcd}
G'' = \mathrm{p}\varinjlim_\lambda G_\lambda \arrow[r, "g''"] & G \\
G' = \mathrm{p}\varinjlim_\lambda G_\lambda^K \arrow[r, "g'"'] \arrow[u] & G^K \arrow[u]
\end{tikzcd}$$

Since direct limit is left exact the left-hand map is an inclusion and so g' is just the restriction of $g'' : G'' = \mathrm{p}\varinjlim_\lambda(G_\lambda) \longrightarrow G$, that is, is sheafification.

We show that g'_K is monic. By 7.1 every element of $G'K$ is represented by a section $s \in G'V$ for some open $V \supseteq K$ (clearly $\mathrm{supp}(s) \subseteq K$ for every section, s, of G'). Suppose that $g'_V s = 0$. Then there is an open cover $(V_i)_i$ of V such that $\mathrm{res}^{G'}_{VV_i} s = 0$ for each i. Since K is compact there are $V_1, ..., V_n$, say, which cover K. Replace the original choice of V by $V_1 \cup ... \cup V_n$.

For $i = 1, ..., n$ choose λ_i and $a_i \in G^K_{\lambda_i}(V)$ such that $(g'_{\lambda_i\infty})_V a_i = s$ and such that $(g'_{\lambda_i\infty})_{V_i} \mathrm{res}^{G^K_{\lambda_i}}_{VV_i} a_i = 0$ (because $\mathrm{res}^{G'}_{VV_i} s = 0$ there are such λ_i and a_i). Here $g'_{\lambda,\infty}$ denotes the limit presheaf map $G_\lambda^K \longrightarrow G'$.

Since $a_1, ..., a_n$ all map to s in the limit and since a_i restricts to 0 on V_i in the limit and since there are just finitely many of these, there is $\lambda \geq \lambda_1, ..., \lambda_n$ such that for all i, j we have $(g_{\lambda_i\lambda})_V a_i = (g_{\lambda_j\lambda})_V a_j = b$ say and such that $\mathrm{res}^{G^K_\lambda}_{VV_i}(b) = (g_{\lambda_i\lambda})_{V_i} \mathrm{res}^{G^K_{\lambda_i}}_{VV_i}(a_i) = 0$. So, since G^K_λ is a sheaf, $b = 0$ and hence $s = (g'_{\lambda\infty})_V b = 0$. It follows that g'_K is indeed monic.

Now we show that if we have a section of $G^K K$, represented, using 7.1, by, say, $t \in G^K V$ where V is an open neighbourhood of K (without loss of generality, $V \subseteq U$) then there is some section of $\varinjlim_\lambda(G_\lambda^K)$ over some open neighbourhood V'' of K contained in V which maps to $\mathrm{res}^G_{V,V''}(t)$.

Set $L = \mathrm{supp}(t)$ - a closed, hence compact, subset of K.

Since $t \in GV$ and $G = \varinjlim G_\lambda$ there is, for each $x \in L$, an open neighbourhood $V_x (\subseteq V)$ of x and $t_x \in \mathrm{p}\varinjlim G_\lambda . V_x$ such that $(g'')_{V_x} t_x = \mathrm{res}^G_{V,V_x}(t)$.

Finitely many of these V_x suffice to cover L, say $V_1, ..., V_n$ (writing V_1 for V_{x_1} etc.). For each $i = 1, ..., n$ there is λ_i and $s_i \in G_{\lambda_i} V_i$ such that $(g''_{\lambda_i\infty})_{V_i} s_i = t_i$, where $g''_{\lambda\infty} : G_{\lambda_i} \longrightarrow \mathrm{p}\varinjlim G_\lambda$ is the map to the presheaf limit. We will show that we may take the s_i to have support contained in K.

Write $s_i^\lambda = (g_{\lambda_i,\lambda})_{V_i} s_i \in G_\lambda V_i$ for each $\lambda \geq \lambda_i$ and set $L_\lambda = \bigcup_i^n \mathrm{supp}(s_i^\lambda)$ for each $\lambda \geq \lambda_1, ..., \lambda_n$.

We claim that L_λ is a closed subset of $V' = V_1 \cup ... \cup V_n$. Let $y \in V' \setminus L_\lambda$. If $y \in V_i$ then, since $(s_i)_y = 0$ there is an open neighbourhood of y contained in $V_i \setminus \mathrm{supp}(s_i^\lambda)$. Taking the intersection of these neighbourhoods over those i such that $y \in V_i$, we obtain an open neighbourhood of y which is disjoint from L_λ, as required.

So the L_λ form a downwards-directed system of closed subsets of V' with, note, intersection L. Hence the $K \cup L_\lambda$ form a downwards-directed system of closed subsets of V' with intersection K and so, by hypothesis, there is an open set $V'' \subseteq V'$ and λ with $V'' \cap (K \cup L_\lambda) = K$. Replacing each V_i by $V'' \cap V_i$ and each s_i by $\mathrm{res}^{G_\lambda}_{V_i, V'' \cap V_i}((g_{\lambda_i,\lambda})_{V_i} s_i)$, we may assume now that $\mathrm{supp}(s_i) \subseteq K$ and hence that $s_i \in G_\lambda^K V_i$.

It remains to note that the s_i are locally eventually compatible and hence that, together, they correspond to an element of $(\varinjlim{}_\lambda (G_\lambda^K))K$. Given $x \in K$, if $x \in V_i \cap V_j$ then, since $\mathrm{res}^G_{V_i, V_i \cap V_j}((g' g''_{\lambda_i\infty})_{V_i} s_i) = \mathrm{res}^G_{V, V_i \cap V_j}(t) = \mathrm{res}^G_{V_j, V_i \cap V_j}((g' g''_{\lambda_j\infty})_{V_j} s_j)$, the restrictions in $\mathrm{p}\varinjlim G_\lambda^K$ of $(g''_{\lambda_i\infty})_{V_i} s_i$ and $(g''_{\lambda_j\infty})_{V_j} s_j$ agree in some neighbourhood of x. Hence the s_i glue together to form a section $s \in (\varinjlim G_\lambda^K)V''$ representing t, as required. $\square$

Of course, we would like the stronger result that, in the above situation, $G^K K$ is actually equal to the limit $\varinjlim(G_\lambda^K K)$ of the $G_\lambda^K K$, for then every morphism from $j_! \mathcal{O}_K$ to G would lift through the direct system, and we would deduce that $j_! \mathcal{O}_K$ is finitely presented.

In more detail, we have $(j_! \mathcal{O}_K, G) \simeq j^* G^K K = \varinjlim{}_{U \supseteq K}(G^K U)$ by 7.1, the correspondence being, to a morphism $f : j_! \mathcal{O}_K \longrightarrow G$ we assign the adjoint morphism $f' : \mathcal{O}_K \longrightarrow j^* G^K$ and to this we assign the image, $f' 1_K$ of $1_K \in \mathcal{O}_K K$ - this image will be represented by a section of G, over some open neighbourhood V of K, with support contained in K. So, to show that every morphism from $j_! \mathcal{O}_K$ to G lifts through some G_λ it would be enough to show that every K-germ of a section of G^K lifts to a K-germ of a section of G_λ^K for some λ.

Proposition 7.3 *Suppose that X is Hausdorff and locally compact (in the sense defined before 6.11). Let K be as in 7.2: then the morphism g'_K as there is an isomorphism. Hence $j_! \mathcal{O}_K$ is finitely presented.*

Proof. We continue with the notation of the proof of 7.2. Since j^* is a left adjoint it preserves colimits and hence $G \mid_K = j^* G = \varinjlim(G_\lambda \mid_K)$. By 5.5 $\mathcal{O}_K$ is a finitely presented object of Mod-$\mathcal{O}_K$ and so the morphism from $\mathcal{O}_K$ to $G \mid_K$ corresponding to the section $t \mid_K \in G^K K$ (in the notation of the proof of 7.2) lifts through some global section u_1 of $G_\lambda \mid_K$ for some λ.

Now we use [9, III.2.2] which has the Hausdorff hypothesis and gives that u_1 is represented by some section, u say, of $G_\lambda V_0$ for some open neighbourhood V_0 of K in X. We may assume that $V'' \subseteq V_0$ (V'' as in the proof of 7.2). This means that, in the notation at the end of the proof of 7.2, each $(g''_{\lambda_i\infty})_{V_i} s_i$ agrees with $\mathrm{res}^{G''}_{V_0,V_i}(g''_{\lambda,\infty})_{V_i}(u)$ already in the presheaf limit and hence there is $\mu \geq \lambda_1, ..., \lambda_n$ such that $\mathrm{res}^{G^K_\mu}_{V_i, V_i \cap V_j}((g_{\lambda_i\mu})_{V_i} s_i) = \mathrm{res}^{G^K_\mu}_{V_j, V_i\cap V_j}((g_{\lambda_j \mu})_{V_j} s_j)$ and so the $(g_{\lambda_i\mu})_{V_i} s_i$ glue together to form a section $s \in G^K_\mu$ with $(g'' g''_{\mu\infty} {}_{V''} s = t\,|_{V''}$, as required. $\square$

We show that the condition of 7.2 is necessary for $j_! \mathcal{O}_K$ to be finitely presented.

Proposition 7.4 *Let $K \subseteq X$ be locally closed. If $j_!\mathcal{O}_K$ is finitely presented then K is compact and also for every open set $U \supseteq K$ and for every downwards-directed set $(K_\lambda)_\lambda$ of closed subsets of U with $\bigcap_\lambda K_\lambda = K$, there is an open set V with $U \supseteq V \supseteq K$ and there is λ with $K_\lambda \cap U = K$ (for example, if K is cocompact in some open V with $U \supseteq V \supseteq K$ then we will have this).*

Proof. We know, by 6.7, that K must be compact since $K = \mathrm{supp}(j_!\mathcal{O}_K)$ and $j_!\mathcal{O}_K$ is finitely generated.

Suppose that we have an open set $U \supseteq K$ and downwards-directed system $(K_\lambda)_\lambda$ of closed subsets of U with intersection K. We have, by 4.9 $\varinjlim j_!\mathcal{O}_{K_\lambda} = j_!\mathcal{O}_K$. If $j_!\mathcal{O}_K$ is finitely presented then $\mathrm{Id}_{j_!\mathcal{O}_K}$ lifts through some $j_!\mathcal{O}_{K_\lambda}$, that is, the canonical epimorphism $j_!\mathcal{O}_{K_\lambda} \longrightarrow j_!\mathcal{O}_K$ splits and hence, by 4.7, K is open in K_λ. That is, there is V (wlog $U \supseteq V$) with $K_\lambda \cap V = K$, as required. $\square$

Theorem 7.5 *Suppose that X is Hausdorff locally compact. Let K be locally closed. Then $j_!\mathcal{O}_K$ is finitely presented iff K is compact and is "weakly cocompact" in the sense that for every downwards-directed system $(K_\lambda)_\lambda$ of closed subsets of an open set U with $\bigcap_\lambda K_\lambda = K$ there exists U' ($\subseteq U$) open and λ with $U' \cap K_\lambda = K$.*

Proof. This is immediate from 7.2, 7.3 and 7.4. $\square$

This leaves open the following questions:
When, in the general case, is a sheaf of the form $j_!\mathcal{O}_K$ finitely presented?
When is Mod-$\mathcal{O}_X$ locally finitely presented?
Does the answer to the latter question depend only on X (and not on $\mathcal{O}_X$)?
If Mod-$\mathcal{O}_X$ is locally finitely presented is it generated by finitely presented sheaves of the form $j_!\mathcal{O}_K$ with K locally closed?

We finish by showing that Mod-$\mathcal{O}_X$, where $\mathcal{O}_X$ is the sheaf of continuous real-valued functions on $[0, 1]$, is not locally finitely presented (recall, 6.12, that it is locally finitely generated). In fact, only certain properties of this ringed space are needed so we begin by assuming just that X is a space such that

$(*)$ every compact subset of X is closed

(for instance, if X is Hausdorff then we have this).

Suppose also that

$(**)$ every proper closed subset C of X is a directed intersection $C = \bigcap_\lambda C_\lambda$ of compact closed subsets C_λ such that C is not an open subset of any C_λ.

Certainly this is so for $X = [0,1]$ or, more generally, for X any locally closed subset of $\mathbb{R}^n$. [Proof for $[0,1]$: For each n set $U_n = \bigcup\{B_{\frac{1}{n+1}}(x) : x \in X \setminus C$ is such that $B_{\frac{1}{n}}(x) \cap C = \emptyset\}$ ($B_\epsilon(x)$ denotes the open ball of radius ϵ centred at x). Then $U_n \cap C = \emptyset$ and C is strictly contained in $C_n = U_n^{\mathrm{c}}$. Also $\bigcap_n C_n = C$ since, if $x \in X \setminus C$ then for some n we have $B_{\frac{1}{n}}(x) \cap C = \emptyset$. Now, if there were n and an open set V such that $C = C_n \cap V$ then $\bigcap_{m \geq n} C_m \setminus C$ would be non-empty (since each $C_m \setminus C$ would be closed, since X is compact), contained in the closure of C, yet disjoint from C - contradiction.]

Then, for such a space X and any ringed space $\mathcal{O}_X$ on X, we have, by 7.4, that if K is locally closed and $j_!\mathcal{O}_K$ is finitely presented in Mod-$\mathcal{O}_X$ then $K = X$ (and $\mathcal{O}_X$ will be finitely presented iff X is compact, by 5.5).

Now, suppose further that

$(***)$ every stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ is a (not necessarily commutative) local ring and that every section which is, at each point, in the radical, is in fact zero.

Suppose that F is a cyclic sheaf. Then, we claim, $F \simeq j_!\mathcal{O}_K$ for some locally closed $K \subseteq X$. For say $F = j_!\langle s\rangle$ where $s \in FU$ some open $U \subseteq X$. Let $K = \mathrm{supp}(s)$. Then there is a morphism $f : j_!\mathcal{O}_K \longrightarrow F$ with $f_U : (j_!\mathcal{O}_K)U \longrightarrow FU$ taking "1_K" to s and so f is an epimorphism. Let $G = \ker(f)$, so we have the exact sequence $0 \longrightarrow G \longrightarrow j_!\mathcal{O}_K \longrightarrow F \longrightarrow 0$. At $x \in K$ this is $0 \longrightarrow G_x \longrightarrow \mathcal{O}_{X,x} \longrightarrow F_x \longrightarrow 0$ so, since $F_x \neq 0$ and $\mathcal{O}_{X,x}$ is local, we have $G_x \leq \mathrm{rad}\mathcal{O}_{X,x}$ and hence for every open V and $t \in GV$ and $x \in V$ we have $t_x \in \mathrm{rad}\mathcal{O}_{X,x}$, noting that $t_x = 0$ for $x \in V \setminus K$. So, by assumption, $t = 0$. Hence $G = 0$ and so $F \simeq j_!\mathcal{O}_K$, as required.

In particular, if F is a cyclic finitely presented sheaf, then $F \simeq \mathcal{O}_X$.

It follows that every finitely presented sheaf, F, is generated by global sections, for if $F = \sum_1^n j_!\langle s\rangle$ where $s_i \in FU_i$ then, without loss of generality, $s_1 \notin \sum_2^n j_!\langle s\rangle U_1$. So $F' = F/\sum_2^n j_!\langle s\rangle$ is finitely presented and cyclic, generated by the image of s_1. By the above, $F' \simeq \mathcal{O}_X$ so, since the image of s_1 generates F', it must have support equal to X. Hence $\mathrm{supp}(s_1) = X$ and so certainly s_1 must be a global section. The same applies to any member of a minimal generating set and so we have the claim.

It follows that every non-zero sheaf generated by finitely presented sheaves must have a non-zero global section. The final assumption we need is

$(****)$ there is a non-zero sheaf which has no non-zero global sections.

Certainly not every sheaf in Mod-$\mathcal{O}_X$, where $\mathcal{O}_X$ is the sheaf of continuous fuctions on $[0,1]$, has a global section and so we conclude that Mod-$\mathcal{O}_X$ is not locally finitely presented. Therefore we have the following for $X = [0,1]$ among other spaces.

Proposition 7.6 *Let $\mathcal{O}_X$ be the sheaf of real-valued continuous functions on the closed real unit interval. Then* Mod-$\mathcal{O}_X$ *is not locally finitely presented.*

8 Restriction of finitely presented sheaves

Proposition 8.1 *Let $j : K \longrightarrow X$ be the inclusion of a locally closed set and suppose that j_* commutes with direct limits. Then j^* takes finitely presented $\mathcal{O}_X$-modules to finitely presented $\mathcal{O}_K$-modules.*

Proof. Let $F \in$ Mod-$\mathcal{O}_X$ and suppose that $G = \varinjlim G_\lambda$ in Mod-$\mathcal{O}_K$. We have $(j^*F, G) \simeq (F, j_*G) = (F, j_* \varinjlim G_\lambda) \simeq (F, \varinjlim j_*G_\lambda)$ (by hypothesis) $\simeq \varinjlim (F, j_*G_\lambda)$ (since F is finitely presented) $\simeq \varinjlim (j^*F, G_\lambda)$, as required. $\square$

That is, if j_* commutes with direct limits then the restriction of a finitely presented sheaf is finitely presented.

Example 8.2 *Let $X = [0,1]$ and take a constant sheaf $\mathcal{O}_X$. Let $K = (0,1)$. Then, as we have seen already, the restriction of the finitely presented sheaf $\mathcal{O}_X$ to $\mathcal{O}_K$ is not even finitely generated. To see directly the failure of i_*, where $i : K \longrightarrow X$ is the inclusion, to commute with direct limits, take the directed system of sheaves $(j_n)_!\mathcal{O}_{(\frac{1}{n},1-\frac{1}{n})}$ where $j_n : (\frac{1}{n}, 1-\frac{1}{n}) \longrightarrow (0,1)$ is the inclusion, which has direct limit $\mathcal{O}_{(0,1)}$. Then one may check that $\varinjlim i_*(j_n)_!\mathcal{O}_{(\frac{1}{n},1-\frac{1}{n})} = i_!\mathcal{O}_{(0,1)}$ whereas $i_*\mathcal{O}_{(0,1)} = \mathcal{O}_{[0,1]}$.*

Theorem 8.3 *Suppose that X is locally compact and that C is closed and compact. Then j_* commutes with direct limits, where $j : C \longrightarrow X$ is the inclusion.*

Proof. Let $((G_\lambda)_\lambda, (g_{\lambda\mu})_{\lambda<\mu})$ be a directed system in Mod-$\mathcal{O}_C$, with $G = \varinjlim G_\lambda$ and limit maps $g_{\lambda\infty} : G_\lambda \longrightarrow C$. The $((j_*G_\lambda)_\lambda, (j_*g_{\lambda\mu})_{\lambda<\mu})$ form a directed system in Mod-$\mathcal{O}_X$ so let $G' = \varinjlim j_*G_\lambda$, with limit maps $g'_{\lambda\infty} : j_*G_\lambda \longrightarrow G'$. Furthermore, the $(j_*g_{\lambda\infty})_\lambda$ form a coherent system of maps so there is $h : G' \longrightarrow j_*G$ such that for all λ we have $j_*(g_{\lambda\infty}) = hg'_{\lambda\infty}$. We must show that h is an isomorphism and it is enough to show that it is an isomorphism on stalks.

First we show that h is monic. Let $x \in X$ and let $t_x \in G'_x$ be represented by $t \in G'U$ for some open $U \subseteq X$ with $x \in U$. By definition of direct limit in Mod-$\mathcal{O}_X$ there is an open $V \subseteq X$ with $x \in V \subseteq U$, and there exist λ and $s \in j_*G_\lambda V$ such that $(g'_{\lambda\infty})_V(s) = \mathrm{res}^{G'}_{U,V}(t)$.

Now suppose that $h_xt_x = 0$. We have $(j_*g_{\lambda\infty})_x s_x = h_x(g'_{\lambda\infty})_x(s_x) = h_xt_x = 0$. Hence, by definition of stalk of a map, there is an open set $W \subseteq V$ with $x \in W$ such that $(j_*g_{\lambda\infty})_W(\mathrm{res}^{j_*G_\lambda}_{V,W}(s)) = 0$. Setting $s' = \mathrm{res}^{j_*G_\lambda}_{V,W}(s)$, note that

$(g'_{\lambda\infty})_W(s') = \mathrm{res}^{G'}_{U,W}(t)$. We have, regarding s' as in $G_\lambda(W \cap C) = j_*G_\lambda.W$, and regarding $(j_*g_{\lambda\infty})_W : j_*G_\lambda.W \longrightarrow j_*G.W$ as $(g_{\lambda\infty})_{W\cap C} : G_\lambda(W \cap C) \longrightarrow G(W \cap C)$, that $(g_{\lambda\infty})_{W\cap C}(s') = 0$.

Let $L \subseteq W$ be a compact locally closed neighbourhood of x, say $x \in V_0 \subseteq L$ with V_0 open. Then $C \cap L$ is a closed, hence compact, subset of L containing $C \cap V_0$.

For each $y \in C\cap L$ there is an open set W_y with $y \in W_y \subseteq W$ and some $\mu_y \geq \lambda$ such that $g_{\lambda\mu_y}(\mathrm{res}^{G_\lambda}_{W\cap C,W_y\cap C}(s')) = 0$ (we write, for example, $g_{\lambda\mu_y}$ instead of $(g_{\lambda\mu_y})_{W_y\cap C}$ here and below in order to lighten the notation a little). Since $C\cap L$ is compact there are finitely many of these, say $W_1, ..., W_n$ which cover $C\cap L$. Set $\mu = \max(\mu_1, ..., \mu_n)$, $W' = W_1 \cup ... \cup W_n$. Then we have $\mathrm{res}^{G_\mu}_{W\cap C,W_i\cap C}(g_{\lambda\mu}s') = 0$ for $i = 1, ..., n$ and hence $\mathrm{res}^{G_\mu}_{W\cap C,W'\cap C}(g_{\lambda\mu}s') = 0$. Note that x might not be a member of W', nevertheless we have $j_*G_\mu(V_0) = G_\mu(V_0 \cap C) \subseteq G_\mu(W' \cap C) = j_*G_\mu(W')$ and so $j_*g_{\lambda\mu}(\mathrm{res}^{j_*G_\lambda}_{V,V_0}(s)) = \mathrm{res}^{j_*G_\mu}_{W,V_0}(j_*g_{\lambda\mu}(s')) = 0$ and hence $\mathrm{res}^{G'}_{U,V_0}(t) = g'_{\mu\infty}(j_*g_{\lambda\mu}(\mathrm{res}^{j_*G_\lambda}_{U,V_0}(s)) = 0$ and hence $t_x = 0$, as required.

Now we show that h is epi. Let $x \in X$ and let $t'_x \in (j_*G)_x$ be represented by $t' \in (j_*G)U = G(U \cap C)$ for some open $U \subseteq X$ with $x \in U$. Choose a compact, locally closed neighbourhood, L of x, say $x \in V \subseteq L \subseteq U$ with V open. Then $C \cap V \subseteq C \cap L$ and $C \cap L$ is compact.

For each $y \in C \cap L$ there is an open $W_y \subseteq U$ with $y \in W_y$ and $\mu_y \geq \lambda$ and $s_y \in G_{\mu_y}(W_y \cap C)$ such that $g_{\mu_y\infty}s_y = \mathrm{res}^G_{U,W_y}(t')$. By compactness of $C \cap L$, finitely many of these W_y cover $C \cap L$, say $W_1, ..., W_n$. Set $W' = W_1 \cup ... \cup W_n$ and $\mu = \max(\mu_1, ..., \mu_n)$.

Also, in choosing U and $t' \in G(U \cap C) = (\varinjlim G_\lambda)(U \cap C)$, we may, without loss of generality, assume that t' is a section of the presheaf limit: $t' \in (\mathrm{p}\varinjlim G_\lambda)(U\cap C)$. Therefore, for all i, j, setting $s'_i = g_{\mu_i\mu}s_i$, we have $g_{\mu\infty}\mathrm{res}^{G_\mu}_{W_i\cap C,W_i\cap W_j\cap C}(s'_i) = \mathrm{res}_{U\cap C,W_i\cap W_j\cap C}(t')$ (restriction in the sheaf, equally in the presheaf, limit) $= g_{\mu\infty}\mathrm{res}^{G_\mu}_{W_j\cap C,W_i\cap W_j\cap C}(s'_j)$. So, since this is equality in the presheaf limit, at some point $\mu'' \geq \mu$ all these pairs are identified, that is there is $\mu'' \geq \mu$ such that, if $s''_i = g_{\mu\mu''}s'_i$, then for all i, j we have $\mathrm{res}^{G_{\mu''}}_{W_i\cap C,W_i\cap W_j\cap C}(s''_i) = \mathrm{res}^{G_{\mu''}}_{W_j\cap C,W_i\cap W_j\cap C}(s''_j)$ and hence there is a section $s \in G_{\mu''}(W'\cap C) = (j_*G_{\mu''})(W')$ with $g_{\mu''\infty}s = \mathrm{res}^G_{U,W'}(t')$.

But $W' \supseteq C \cap L \supseteq C \cap V$ so we have $s_0 = \mathrm{res}^{G_{\mu''}}_{W'\cap C,V\cap C}(s) \in G_{\mu''}(V \cap C) = j_*G_{\mu''}(V)$ is such that $(j_*g_{\mu''\infty})_V(s_0) = \mathrm{res}^{j_*G}_{W,V}(t')$ and hence $h_V((g'_{\mu''\infty})_V(s_0)) = \mathrm{res}^{j_*G}_{W,V}(t')$ and hence $h_x((g'_{\mu''\infty})_V(s_0))_x = t'_x$ and h is epi, as required. □

Corollary 8.4 *Suppose that X is locally compact and suppose that C is closed and compact. Let F be a finitely presented $\mathcal{O}_X$-module. Then $F\mid_C$ is a finitely presented $\mathcal{O}_C$-module.*

Proof. This is immediate by 8.1 and 8.3. □

Remark 8.5 *The proof shows that, with the same hypothesis on X, one may obtain the same conclusions assuming just that C is locally closed and is such that, for every compact L we have $C \cap L$ compact.*

Corollary 8.6 *Suppose that X is locally compact and suppose that C is closed and compact. If* $\text{Mod-}\mathcal{O}_X$ *is locally finitely presented, with $\mathcal{F}$ for a generating set of finitely presented objects, then* $\text{Mod-}\mathcal{O}_C$ *is locally finitely presented, with $\{F\mid_C : F \in \mathcal{F}\}$ for a generating set of finitely presented objects.*

Proof. It remains to show that the $F \mid_C = j_*F$ are generating. So let $G \in \text{Mod-}\mathcal{O}_C$. Then there is an epimorphism $f : \bigoplus_\Lambda F_\lambda \longrightarrow j_*G$ with the F_λ in $\mathcal{F}$. We have $(\bigoplus_\Lambda F_\lambda, j_*G) \simeq (j^* \bigoplus_\Lambda F_\lambda, G)$ with f corresponding to f' say. Since f is epi, so is f'. Also, j^*, being a left adjoint, commutes with direct sums. So we have an epimorphism $f' : \bigoplus_\Lambda j^*F_\lambda \longrightarrow G$ as required. $\square$

Note the following.

Remark 8.7 *If $j : K \longrightarrow X$ is such that j_* and $j^!$ commute with direct limits then, for any finitely presented $\mathcal{O}_X$-module F we have that $j_!j^*F$ is finitely presented.*

Proof. Let $H = \varinjlim H_\lambda$ be a directed system in $\text{Mod-}\mathcal{O}_X$. Then $(j_!j^*F, H) \simeq (j^*F, j^!H) \simeq (j^*F, \varinjlim j^!H_\lambda)$ (by hypothesis) $\simeq \varinjlim (j^*F, j^!H_\lambda)$ (by hypothesis and 8.1) $\simeq \varinjlim (j_!j^*F, H_\lambda)$, as required. $\square$

Finally, we return to the case of a general space X and conclude with the following observations.

Lemma 8.8 *If F is a finitely generated sheaf then for every $x \in X$ the stalk F_x is a finitely generated $\mathcal{O}_{X,x}$-module.*

Proof. Say $F = \sum_1^n j_!\langle s_i \rangle$ with $s_i \in FU_i$. At $x \in X$ this becomes $F_x = \sum_1^n j_1\langle s_i \rangle_x = \sum\{(s_i)_x.R_x : x_i \in U_i\}$ so is finitely generated. $\square$

Proposition 8.9 *If F is a finitely presented sheaf then for every $x \in X$ the stalk F_x is a finitely presented $\mathcal{O}_{X,x}$-module.*

Proof. Let $M = \varinjlim M_\lambda$ be the limit of a directed system of $\mathcal{O}_{X,x}$-modules and suppose that we have a morphism $F_x \longrightarrow M$. Apply the functor j_* where $j : \{x\} \longrightarrow X$ is the inclusion (that is, regard these as skyscraper sheaves). We have $j_*M.U = M$ if $x \in U$ and $= 0$ otherwise, and similarly for the M_λ so we have $j_*M = \varinjlim j_*M_\lambda$. Also we have the canonical map $F \longrightarrow j_*j^*F$ whose composition with $j_*j^*F = j_*F_x \longrightarrow j_*M$ lifts to $F \longrightarrow j_*M_\lambda$ for some λ. At x this morphism gives a morphism $F_x \longrightarrow M_\lambda$ as required. $\square$

Example 8.10 *If F is a finitely presented sheaf then it does not follow that $F|_U$ is a finitely presented $\mathcal{O}_U$-module, even for $U = X$. For an example, take $X = \mathbb{R}$ equipped with the Zariski topology and define the sheaf $\mathcal{O}_X$ by $\mathcal{O}_X U = \{f : U \longrightarrow \mathbb{R} : f$ is continuous with respect to the usual topology on both copies of $\mathbb{R}\}$. Let $U = \mathbb{R} \setminus \{0\}$ and let s be the constant section with value 1 on U. Set $F = j_!\langle s \rangle = j_!\mathcal{O}_U$: this is finitely presented since U is (Zariski!) compact open. But $FX = \{t \in \mathcal{O}_U U : \mathrm{supp}(t)$ is Zariski closed in $X\} = \{t \in \mathcal{O}_U U : \mathrm{supp}(t)$ is finite $\}$ and so, clearly (consider supports), the module of global sections is not finitely generated.*

References

[1] J. Adámek and J. Rosický, Locally Presentable and Accessible Categories, London Math. Soc. Lecture Note Ser., Vol. 189, Cambridge University Press, 1994.

[2] M. Artin, A. Grothendieck and J.L. Verdier, Théorie des Topos et Cohomologie Etale des Schémas (SGA4), Lecture Notes in Mathematics, Vol. 269, Springer-Verlag, 1972.

[3] F. Borceux, Handbook of Categorical Algebra, Vol. 3, Cambridge University Press, 1994.

[4] F. Borceux and G. Van den Bossche, Algebra in a Localic Topos with Applications to Ring Theory, Lecture Notes in Mathematics, Vol. 1038, Springer-Verlag, 1983.

[5] W Crawley-Boevey, Infinite-dimensional modules in the representation theory of finite-dimensional algebras, pp. 29-54 in Algebras and Modules I, Canadian Math. Soc. Conf. Proc., Vol. 23, Amer. Math. Soc, 1998.

[6] W Crawley-Boevey, Locally finitely presented additive categories, Comm. Algebra, 22 (1994), 1641-1674.

[7] R. Hartshorne, Algebraic Geometry, Springer-Verlag, 1977.

[8] I. Herzog, The Ziegler spectrum of a locally coherent Grothendieck category, Proc. London Math. Soc., 74 (1997), 503-558.

[9] B. Iversen, Cohomology of Sheaves, Springer-Verlag, 1986.

[10] H. Krause, The Spectrum of a Module Category, Mem. Amer. Math. Soc., Vol. 149 No. 707, (2001).

[11] S. Mac Lane, Categories for the Working Mathematician, Springer-Verlag, 1971.

[12] N. Popescu, Abelian Categories with Applications to Rings and Modules, Academic Press, 1973.

[13] M. Prest, Applications of Logic to Torsion Theories in Abelian Categories, Doctoral Thesis, University of Leeds, 1978.

[14] M. Prest, Some model-theoretic aspects of torsion theories, J. Pure Applied Algebra, 12 (1978), 295-310.

[15] M. Prest, Elementary torsion theories and locally finitely presented Abelian categories, J. Pure Applied Algebra, 18 (1980), 205-212.

[16] M. Prest, The (pre)sheaf of definable scalars, preprint, University of Manchester, 1995.

[17] M. Prest, Topological and geometric aspects of the Ziegler spectrum, pp. 369-392 in H. Krause and C. M. Ringel (Eds.), "Infinite Length Modules", Birkhaüser, 2000.

[18] M. Prest, Model theory and modules, preprint, to appear in Handbook of Algebra, Elsevier.

[19] M. Prest, Model theory of additive functor categories, in preparation.

[20] A. Ralph, An Approach to the Model Theory of Sheaves, Dissertation, University of Manchester, 2000.

[21] B. Stenström, Rings of Quotients, Springer-Verlag, 1975.

[22] B. R. Tennison, Sheaf Theory, London Mathematical Society Lecture Note Ser., Vol. 20, Cambridge University Press, 1975.