Model Theory in Additive Categories

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If R is a field this is enough - since the theory of vector spaces has complete elimination of quantifiers every definable set is a finite boolean combination of solution sets to systems of linear equations - but over arbitrary rings we also need *projections* of such solution sets that is, solution sets to pp (positive primitive) formulas $\phi(\overline{x})$ - those of the form $\exists \overline{y}\ \overline{x}\overline{y}B = \overline{0}$.

If $\psi(\overline{x})$ and $\phi(\overline{x})$ are pp formulas and if $\psi \rightarrow \phi$ then we say that these form a pp-pair. Then for every module M we have the quotient group $\phi(M)/\psi(M)$. Note that the condition that this group have $\leq k$ elements is expressible by a sentence of $\mathcal{L}_R$.

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Theorem

(pp-elimination of quantifiers for modules) If $\eta(\overline{x})$ is a formula in the language of R-modules then there is a finite boolean combination $\chi(\overline{x})$ of pp formulas and a sentence σ such that, modulo the theory of R-modules, $\eta(\overline{x}) \leftrightarrow \chi(\overline{x}) \wedge \sigma$.

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Corollary

Every definable subset of a module M is a finite boolean combination of cosets of pp-definable subgroups.

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A powerful general theory has been built and it has found many applications; usually the applications are specific to a certain flavour of representation theory and require considerable algebraic input.

Some components of the general theory

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2. **Generalised Weyl Algebras and quantum groups:** Typically these have 'wild' representation theory and there are (e.g. undecidability) results which reflect this (Prest, Puninski). There are also Herzog's results on the model theory of pseudo-finite-dimensional representations of $\mathrm{sl}_2(k)$.

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4. **Structure theory for modules over serial rings:** good descriptions of the model theory (Eklof, Herzog, Puninski) and the resolution of various conjectures on direct-sum decomposition of modules over such rings (Puninski).

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Suppose also that $\mathcal{C}$ is preadditive. A **definable subcategory** $\mathcal{D}$ of $\mathcal{C}$ is the full subcategory of $\mathcal{C}$ consisting of those objects which satisfy a given set of conditions of the form $|\phi(-)/\psi(-)| = 1$ where $\psi \leq \phi$ is a pp-pair.

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It turns out that the model theory of (the objects of) $\mathcal{D}$ is implicit in the structure of $\mathcal{D}$ as a category.

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For a definable subcategory $\mathcal{D}$ of Mod-R the category, $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$, of pp-imaginaries of $\mathcal{D}$ is defined as for Mod-R - it has the same objects but now the arrows are the maps which are pp-definable relative to the common theory of objects of $\mathcal{D}$.

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The geometry of a definable category

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Equivalences of 2-categories

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Then what was described above gives an anti-equivalence of $\mathbb{ABEX}$ and $\mathbb{DEF}$ (and there is an (anti-)equivalent third 2-category, with objects the locally coherent additive categories, and "geometric" morphisms between them - an additive analogue of the category of coherent toposes and geometric morphisms).

M. Prest and R. Rajani, Structure sheaves of definable additive categories, J. Pure Applied Algebra, 214 (2010), 1370-1383, and references therein.