MODEL-THEORETIC IMAGINARIES AND ZIEGLER SPECTRA IN GENERAL CATEGORIES

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Ravi Rajani
School of Mathematics

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Ravi Rajani
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This thesis explores connections between category theory and model theory. The work is divided into three classes of categories: abelian categories in Chapter 2, toposes in chapters 3 and 4, and "affine categories" in Chapter 5.

Chapter 2 culminates in a new result (Theorem 2.4.2) which shows that the presheaf of definable scalars of a right coherent ring R generalises the presheaf of localisations of R at hereditary torsion theories on the category of right R-modules.

The bulk of Chapter 3 is devoted to a new proof of a result of Makkai and Reyes [26] which characterises a certain category of coherent sheaves as a category of "positive imaginaries" for a first-order coherent theory (Theorem 3.3.21). This result is a topos version of a localised version of a result of Burke's in module theory (Theorem 2.3.14). We establish logical characterisations of finitely generated and finitely presented sheaves along the way, as well as various results around finite type Grothendieck topologies, definable subcategories and exact functors. These results motivate our definition of an interpretation between coherent theories as an exact functor between categories of coherent sheaves (section 3.4.1). We also relate this definition to the topos literature.

Chapter 4 contains a study of pure-injectivity and Ziegler spectra for G-sets for an (infinite) group G. We highlight some parallels and point out important differences with module theory. Despite some obstacles, there is a good candidate for a (non-additive) Ziegler spectrum in this context.

The final chapter investigates whether Burke's result in the additive context, which was succesfully generalised to the topos context in Chapter 3, has an anologue in the rather unexplored territory of "affine categories". We find a counterexample (Theorem 5.4.5) which establishes that the analogue of Burke's result is not valid.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

A central theme of this thesis is the possibility of generalising a localised version of a result of Kevin Burke's in module theory to more general contexts. The result in question establishes a dictionary between the model-theoretic language of "pp-formulas" in the context of "pp-imaginaries" and the language of categories and functors (see Proposition 2.2.7 and Theorem 2.3.14). This dictionary has proved to be very useful in the application of model theory to module theory (see [30]). Chapter 2 is devoted to explaining this result in some detail. The necessary background on hereditary torsion theory and finite type localisation in locally finitely presented abelian categories is developed. All the results in this chapter have been drawn together from various sources (e.g. [29], [35], [30], and [9]) with one exception: Theorem 2.4.2 is new. This theorem uses the localisation theory developed in the first half of the chapter to prove a result relating the presheaf of definable scalars of a right coherent ring R to the presheaf of localisations of R with respect to finite type hereditary torsion theories on the category of right R-modules.

The main content of Chapter 3 is a new proof of an old result of Makkai and Reyes [26, p. 269]. The first part of this result establishes that when $\mathbf{C}$ is any locally finitely presented category, the category of coherent set-valued functors on the full subcategory of finitely presented objects $\mathrm{fp}\,\mathbf{C}$ has a logical/syntactic characterisation in terms of positive existential formulas (Theorem 3.2.12). This category can be regarded as a category of model-theoretic "positive imaginaries" for a certain first-order theory T_0, where $\mathbf{C} = \mathrm{Mod}\, T_0$.

The second part of the Makkai-Reyes result shows that if T is a theory axiomatised by sentences of a specific form (a *coherent* theory) extending T_0, there is a corresponding *coherent topos* (a category of sheaves for a finite type Grothendieck topology) whose full subcategory of coherent objects has a syntactic characterisation as above but suitably relativised to the theory T (Theorem 3.3.21). This category is a category of "positive imaginaries" for T.

Our proof of this result is very different from Makkai and Reyes'. We are motivated primarily by the model theory of modules (in contrast to Makkai and Reyes whose book is expressed in a categorical language which many model-theorists may find tough going) and we succesfully generalise techniques from that context to the context of general locally finitely presented categories and toposes. Our approach enables us to obtain syntactic/logical characterisations of, not only coherent sheaves, but also of finitely presented and finitely generated sheaves (Theorem 3.3.18, Proposition 3.3.19). These characterisations represent categories of "higher-order" imaginaries in increasing levels of generality. These results are, as far as we are aware, not present in the literature.

Another result - which may well be folklore to topos theorists but which we were unable to find in the literature - that came naturally out of our work (and has an additive analogue), is the bijective correpondence between "definable" subcategories of a locally finitely presented category $\mathbf{C}$ and finite type Grothendieck topologies on $\mathrm{fp}\, \mathbf{C}$. This is Corollary 3.3.13. We show in Proposition 3.4.4 (which also has an additive analogue) that a definable subcategory can be recovered as the category of exact functors on the corresponding category of positive imaginaries.

We end Chapter 3 by defining an interpretation between coherent theories in categorical terms (as a certain functor between categories of coherent sheaves) and demonstrate that, using the characterisation of Theorem 3.3.21, an interpretation so defined corresponds to a syntactic translation of the kind one is used to in general model-theoretic interpretations (propositions 3.4.1 and 3.4.3). Finally, we relate our definition of interpretation to a notion from topos theory, *viz.* coherent geometric morphisms between coherent toposes (section 3.4.3).

In Chapter 4 we investigate the possibility of defining a Ziegler Spectrum for G-sets. The category of (left) G-sets for an (infinite) group G is a well behaved locally finitely presented category which is naturally a category of set functors in the same way that a module category is a category of additive functors. The notions of purity and pure-injectivity (atomic compactness) for G-sets were studied in a paper by Banaschewski [7] and the initial indication was that there were clear parallels between G-sets and modules (e.g. the existence of pure-injective hulls [7, Prop 3]). In the first part of this chapter we extend some of these parallels as well as highlight some important differences. For example, we prove a quantifier elimination result (Theorem 4.2.5) which says that any formula in the language of G-sets is equivalent to a boolean combination of quantifier-free formulas and certain sentences which look rather like the "invariant conditions" for modules. This result is probably known but, as far as we are aware, has not been written down previously. We use this theorem to show that, like the modules case, every G-set embeds elementarily in its pure-injective hull (Lemma 4.3.6).

Any pure-injective module which has an indecomposable decomposition has an indecomposable pure-injective direct summand. We show that the equivalent statement for G-sets is false (Corollary 4.3.17). Moreover, we show that definable subcategories of G-sets are not determined by the indecomposable pure-injectives they contain (Proposition 4.4.2) and that this set of G-sets is therefore not a suitable set on which to define a Ziegler topology (whose closed sets should correspond to coherent theories). All is not lost however: there is a *set* of pure-injective G-sets which can function as the underlying set of points for a Ziegler spectrum (Proposition 4.4.4). It would be interesting to obtain an explicit description of this space for certain (classes of) infinite groups. This requires specialised knowledge of the type of groups in question (e.g. locally finite groups, infinite permutation groups) and, from our discussions with specialists, it appears that determination of even the points of the Ziegler spectrum for these groups is a hard problem.

It is widely recognised among category-theorists that there are remarkable similarities between an abelian category and a topos (although we have not yet found a good account of why they are so similar) and many constructions in one have analogues in the other (e.g. localisation theory). The results of Chapter 3 strongly support this view. In Chapter 5 we investigate whether Burke's result (Proposition 2.2.7) can be generalised to a context which is neither fully additive nor fully set-based (so is neither in the domain of abelian categories nor in the domain of toposes). Specifically, we expand the language of R-modules with a finite set of constants and consider models of the theory of modules in this expanded language. These models are simply modules with a designated tuple. We can make this class of *pointed modules* into a category by stipulating that an R-linear map is a *morphism* if it transforms one designated tuple pointwise into another. The hom-sets in this category are either empty or cosets of abelian groups. They have the structure of *affine spaces* which are roughly abelian groups without a distinguished identity element (or $\emptyset$). Affine spaces form a category in their own right and one can consider functors from the category of pointed modules to the category of affine spaces which preserve the affine structure on hom-sets: these are *affine functors.* An example of an affine functor is a pp-formula in the expanded language of modules with constants. One might hope that coherent affine functors can be characterised in terms of such pp-formulas in the same vein as Proposition 2.2.7. We show in Theorem 5.4.5 that this is not the case. Again, this lends support to the general philosophy that abelian categories and toposes are very similar and well-behaved, and other types of categories are somehow not as nice.

The reader of this thesis is assumed to have a basic background in category theory (limit/colimit constructions, abelian categories, etc.) as covered in e.g. [24]. We also assume a basic background in model theory (ultraproducts, types, quantifier elimination, etc.) as covered in [21] or [16]. A solid background in module theory (including its model theory [31]) is useful but not essential.

We generally use bold face capital letters (e.g. $\mathbf{C}$, $\mathbf{D}$, ...) to denote categories and bold lower case letters (e.g. $\boldsymbol{x}$, $\boldsymbol{y}$, ...) to denote tuples. An italicised word or phrase (e.g. *coherent object*) will usually indicate that it is being defined. Equations are numbered within chapters: e.g. the third equation in chapter 2 is labelled "(2.3)". Lemmas, propositions, theorems and corollaries share the same counter which is reset at the start of each section: so the third result of section 2.2, which happens to be a theorem, is labelled "Theorem 2.2.3".

Chapter 2

Modules

2.1 Overview

This chapter mainly sets up required background on locally finitely presented abelian categories and finite type hereditary torsion theory. This material on abelian categories and modules is needed in order to understand Kevin Burke's result (Proposition 2.2.7) and its localised version (Theorem 2.3.14). These results will be successfully generalised to general locally finitely presented categories in Chapter 3.

We introduce the reader to the Ziegler and Zariski spectrum of a ring in section 2.4. (We will attempt to define a non-additive Ziegler spectrum in a specific example in Chapter 4.) The chapter ends with a new result about the presheaf of definable scalars on the Zariski spectrum of a right coherent ring (Theorem 2.4.2). This theorem uses the finite type localisation theory developed earlier in the chapter.

2.2 Finitely generated, finitely presented and coherent objects

2.2.1 Locally finitely presented categories

In an arbitrary category, an object is said to be *finitely presented* (f.p.) if its associated covariant Hom-functor commutes with directed colimits. This can be restated as follows. An object X is finitely presented if whenever $((Y_i)_i, (u_{ij} : Y_i \to Y_j)_{i<j})$ is a directed system with colimit $Y = \varinjlim Y_i$, any map $f : X \to Y$ factors through some object Y_i as $f = u_i g$ for some $g : X \to Y_i$, where $u_i : Y_i \to Y$ is the canonical colimit map. Moreover, this factorisation is *essentially unique* in the sense that if $f = u_i g'$ for some other map g' then g and g' are equalised by some diagram map $u_{ij} : Y_i \to Y_j$, that is $u_{ij} g = u_{ij} g'$.

An object is *finitely generated* (f.g.) if it cannot be expressed as a directed union of proper subobjects. An object X is *coherent* if it is f.g. and whenever Y is f.g. and $f : Y \to X$ is any map, then the pullback $Y \times_X Y$ is finitely generated. In an abelian category, an object X is coherent if and only if it is f.g. and for any map $f : Y \to X$ with Y f.g., we have that $\ker f$ is f.g.

In any variety of finitary algebras (category of models of an equational theory), an algebra is f.p. if and only if it can be presented by finitely many generators and finitely many equations in the usual algebraic sense (see [1, Th 3.12]).

A set of objects $\mathcal{G}$ is a *generating set* for a category if for any pair of maps $f, g : X \to Y$ such that $f \neq g$, there is an object $G \in \mathcal{G}$ and a map $h : G \to X$ such that $fh \neq gh$. Furthermore, $\mathcal{G}$ is a *strong generating set* if, in addition, whenever $X' \to X$ is a proper subobject of X, there is a $G \in \mathcal{G}$ and a map $h : G \to X$ which does not factor through the subobject. A category is *locally finitely presented* (LFP) if it is cocomplete and has a strong generating set of f.p. objects. Equivalently, a category is LFP if it is cocomplete and has a set of f.p. objects such that every object is a directed colimit of these (see [1, p. 17]).

The following result is probably known but we were unable to find a reference.

Proposition 2.2.1. *Let $\mathcal{E}$ be an LFP category in which any map which is both epic and monic is invertible. Let $\mathcal{G}$ be a fixed strong generating set of f.p. objects.*

(a) *An object $X \in \mathcal{E}$ is finitely generated if and only if there are objects $G_1, \ldots, G_n \in \mathcal{G}$ and an epimorphism*

$$\coprod_{i=1}^{n} G_i \twoheadrightarrow X$$

(b) *An object $X \in \mathcal{E}$ is finitely presented if and only if there are objects*

$G_1, \ldots, G_n, H_1, \ldots, H_m \in \mathcal{G}$ and a coequaliser diagram of the form

$$\coprod_{i=1}^{n} G_i \rightrightarrows \coprod_{i=1}^{m} H_i \longrightarrow X$$

Proof. (a)

($\Rightarrow$) Write $A_G = \mathrm{Hom}(G, X)$ for $G \in \mathcal{G}$. Define the map

$$\coprod_{G \in \mathcal{G}} G^{(A_G)} \xrightarrow{p} X$$

by $p_{G,f} = f$ (for $f \in A_G$). Then p is epic. To see this, suppose we have two maps $g \neq h : X \to Y$. Since $\mathcal{G}$ generates the category, there is a $G \in \mathcal{G}$ and a map $t : G \to X$ with $gt \neq ht$. This gives $gp_{G,t} \neq hp_{G,t}$ and so $(gp)_{G,t} \neq (hp)_{G,t}$ which implies that $gp \neq hp$.

The components of p determine maps from any finite subcoproduct of $\coprod_G G^{(A_G)}$ to X. The images of these maps determine a directed system of subobjects of X, with colimit X. Since X is f.g., $X = \mathrm{im}(\coprod_{i=1}^{n} G_i \to X)$ for some $G_1, \ldots, G_n \in \mathcal{G}$. So there is an epimorphism

$$\coprod_{i=1}^{n} G_i \twoheadrightarrow X$$

as required.

($\Leftarrow$) Now suppose there is an epimorphism $\coprod_{i=1}^{n} G_i \to X$. Since a finite colimit of f.p. objects is f.p. (see [1, p. 12]), $\coprod_i G_i$ is a f.p. object. Suppose $X = \varinjlim X_i$ as a directed union of subobjects. Let $f_i : X_i \to X$ denote the canonical inclusions. Then the epimorphism factors through some subobject X_j as $f_j g$. So the inclusion f_j must be epic hence f_j is invertible and X is f.g.

(b) ($\Rightarrow$) Suppose that X is finitely presented. Then in particular, X is f.g. and so by (a) there must be an epimorphism $p : \coprod_{i=1}^{m} H_i \to X$ where $H_i \in \mathcal{G}$. Let

$$Y \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \coprod_{i=1}^{m} H_i$$

be the kernel pair of p (see [25, IV, Thm 7.8]). So

$$Y \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \coprod_{i=1}^{m} H_i \xrightarrow{p} X$$

is a coequaliser. Let $\{M_i\}_i$ be a directed system of subobjects of Y as in (a). So each M_i is the image of map from a finite coproduct of objects from $\mathcal{G}$ into Y. Let X_i be the coequaliser of the maps

$$\begin{tikzcd} M_i \arrow[r, tail] & Y \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \coprod_{i=1}^{m} H_i \end{tikzcd}$$

Note that an inclusion $M_i \subseteq M_j$ induces an epimorphism $u_{ij} : X_i \to X_j$. The X_i with the u_{ij} form a directed system with $X = \varinjlim X_i$ and with the cone maps $u_i : X_i \to X$ given by the universal property of the coequaliser. Since X is f.p., id_X factors through some X_j. So $\mathrm{id}_X = u_j f$ for some $f : X \to X_j$. There is an epimorphism

$$\begin{tikzcd} \coprod_{i=1}^{l} G'_i \arrow[r, two heads] & M_j \end{tikzcd}$$

where $G'_i \in \mathcal{G}$ hence, since the G'_i are finitely presented, there is a coequaliser diagram of the form

$$\begin{tikzcd} \coprod_{i=1}^{l} G'_i \arrow[r, shift left] \arrow[r, shift right] & \coprod_{i=1}^{m} H_i \arrow[r] & X_j \end{tikzcd}$$

Since X_j is a finite colimit of f.p. objects, X_j is itself f.p. Hence the factorisation

$$\begin{tikzcd} X_j \arrow[rr, "u_j"] \arrow[dr, "id"'] & & X \\ & X_j \arrow[ur, "u_j"'] & \end{tikzcd}$$

of u_i as $u_j\mathrm{id}$ is essentially unique (in the sense defined at the start of the section). So, since also $u_j = u_j(fu_j)$, there is a $k \geq j$ such that $u_{jk} = u_{jk}fu_j$. We claim that $u_k : X_k \to X$ is an isomorphism. First note that $u_k(u_{jk}f) = (u_k u_{jk})f = u_j f = \mathrm{id}_X$. But also $((u_{jk}f)u_k)u_{jk} = (u_{jk}f)(u_k u_{jk}) = u_{jk}fu_j = u_{jk}$. So since u_{jk} is epic, $(u_{jk}f)u_k = \mathrm{id}_{X_k}$. So indeed u_k is invertible. Hence $X \cong X_k$ and there is a coequaliser diagram

$$\begin{tikzcd} \coprod_{i=1}^{n} G_i \arrow[r, shift left] \arrow[r, shift right] & \coprod_{i=1}^{m} H_i \arrow[r] & X \end{tikzcd}$$

as required.

($\Leftarrow$) This direction is direct from the fact that a finite colimit of f.p. objects is f.p. $\square$

Note that the assumption in the above proposition that every map which is both monic and epic is invertible is necessary. For consider the category of commutative rings. This is an LFP category. However the map $\mathbb{Z} \to \mathbb{Q}$ is both monic and epic but not invertible. Here $\mathbb{Z}$ is a finitely presented ring but $\mathbb{Q}$ is not finitely generated. So the proposition does not hold for this category.

In an abelian category, any generating set is automatically a strong generating set (see [29, prop 3.4]) and any map which is both epic and monic must be invertible. So we can easily deduce the following corollary.

Corollary 2.2.2. *Let* $\mathbf{A}$ *be an LFP abelian category and let* $\mathcal{G}$ *be a fixed generating set of finitely presented objects.*

(a) *An object* A *is finitely generated if and only if there are objects* $G_1, \ldots, G_n \in \mathcal{G}$ *and an exact sequence*

$$\bigoplus_{i=1}^{n} G_i \longrightarrow A \longrightarrow 0$$

(b) *An object* A *is finitely presented if and only if there are objects* $G_1, \ldots, G_n, H_1, \ldots, H_m \in \mathcal{G}$ *and an exact sequence*

$$\bigoplus_{j=1}^{m} H_j \longrightarrow \bigoplus_{i=1}^{n} G_i \longrightarrow A \longrightarrow 0$$

The following corollary explains the relationship between coherent and finitely presented objects.

Corollary 2.2.3. *Let* $\mathcal{E}$ *and* $\mathcal{G}$ *be as in Proposition 2.2.1. If* X *is a coherent object of* $\mathcal{E}$*, then* X *is finitely presented.*

Proof. Since X is f.g., there is an epimorphism $\coprod_{i=1}^{n} G_i \twoheadrightarrow X$ where $G_i \in \mathcal{G}$. Let $G = \coprod_i G_i$ and put $Y = G \times_X G$. Y is f.g. since X is coherent and so there is an epimorphism $\coprod_{j=1}^{m} H_j \twoheadrightarrow Y$ with $H_j \in \mathcal{G}$ for each j. This gives us a coequaliser

$$\coprod_j H_j \rightrightarrows G \longrightarrow X$$

Hence X is f.p. as required. $\square$

In an abelian category, an object is coherent if and only if it is f.p. and every f.g. subobject is f.p. However this is not the case in general (see Proposition 3.2.8).

2.2.2 Categories of additive functors on finitely presented modules

Let **Ab** be category of abelian groups. Let R be a ring and denote by **Mod**-R and R-**Mod** its categories of right and left modules respectively. All these are locally finitely presented abelian categories. They are also categories of additive functors: if we consider $\{R\}$ to be a preadditive category (i.e. a category where the hom-sets have an abelian group structure and composition of maps is biadditive) consisting of a single object R and with maps $R \to R$ given by the elements of R (the composition of first applying r and then s is the ring element sr), then **Mod**-R is the category of contravariant functors $\{R\} \to \mathbf{Ab}$ and R-**Mod** is the category of covariant functors $\{R\} \to \mathbf{Ab}$.

For any category $\mathbf{C}$, we denote by $\mathrm{fp}\,\mathbf{C}$ the full subcategory of finitely presented objects of $\mathbf{C}$. In the specific case of **Mod**-R (resp. R-**Mod**), we write **mod**-R (resp. R-**mod**) for fp(**Mod**-R) (resp. fp(R-**Mod**)). The categories **mod**-R and R-**mod** are additive (i.e. they are preadditive and have finite direct sums).

If $\mathbf{A}$ is any small preadditive category then we can form a new abelian category by taking as objects the additive functors $\mathbf{A} \to \mathbf{Ab}$ (i.e. the functors F satisfying $F(f+g) = F(f) + F(g)$ for all parallel arrows f and g) and as maps the natural transformations between them (see [29, 3.4]). We denote the corrsponding *functor category* by $(\mathbf{A}, \mathbf{Ab})$. This is a locally finitely presented abelian category where the representable functors form a set of f.p. generators ([11, p. 19]). For any two objects X, Y of a category, we will often write (X, Y) for $\mathrm{Hom}(X, Y)$, and if X and Y are functors we may write $\mathrm{Nat}(X, Y)$ for $\mathrm{Hom}(X, Y)$. The functor category $(R\text{-}\mathbf{mod}, \mathbf{Ab})$ (as well as $(\mathbf{mod}\text{-}R, \mathbf{Ab})$ of course) has some very nice properties (e.g. Proposition 2.2.5 and [2]) and has proved to be very useful in the model theory of modules and representation theory of algebras (see [30], [5], [6]). We obtain the following characterisation of f.g. and f.p. objects in $(R\text{-}\mathbf{mod}, \mathbf{Ab})$ by Corollary 2.2.2, the fact that the set of representable functors generate the category, and $\bigoplus_{i=1}^{n} \mathrm{Hom}(A_i, -) \cong \mathrm{Hom}\left(\bigoplus_{i=1}^{n} A_i, -\right)$.

Proposition 2.2.4. *Let* $F \in (R\text{-}\mathbf{mod}, \mathbf{Ab})$.

(a) *F is finitely generated if and only if there is an object $A \in R$-**mod** and an exact sequence*

$$\mathrm{Hom}(A, -) \longrightarrow F \longrightarrow 0$$

(b) *F is finitely presented if and only if there are objects $A, B \in R$-**mod** and an exact sequence*

$$\mathrm{Hom}(A, -) \longrightarrow \mathrm{Hom}(B, -) \longrightarrow F \longrightarrow 0$$

A proof of the next result can be found in [2, 2.1].

Proposition 2.2.5. *A functor $F \in (R\text{-}\mathbf{mod}, \mathbf{Ab})$ is finitely presented if and only if it is coherent.*

Our notation for the full subcategory of coherent objects in a category $\mathbf{C}$ is $\operatorname{coh}\mathbf{C}$. So the above proposition says that

$$\operatorname{coh}(R\text{-}\mathbf{mod}, \mathbf{Ab}) = \operatorname{fp}(R\text{-}\mathbf{mod}, \mathbf{Ab})$$

2.2.3 Positive primitive formulas/functors

A formula in a first order language is *positive primitive (pp)* if it is built up from atomic formulas using only conjunction and existential quantification. Let $\mathfrak{L}_R$ be the language of left R-modules (see [31]). A pp $\mathfrak{L}_R$-formula φ in n free variables, interpreted in a module M, defines an abelian subgroup of M^n, denoted $\varphi(M)$. It is easy to see that it will in fact define an $\mathrm{End}(M)$-submodule and that it is functorial (in the sense that if $f : M \to N$ is a map of left R-modules, then f^n restricted to $\varphi(M)$ defines a map $\varphi(M) \to \varphi(N)$). We will call this map $\varphi(f)$ or f_* if the context is clear. (In general, we abide by the categorical convention of writing superscripted stars for maps induced by a contravariant functor and subscripted stars for maps induced by a covariant functor.) It is clear that a pp-formula in $\mathfrak{L}_R$ defines a covariant functor R-**Mod** $\to$ **Ab**. In particular, a pp-formula can be regarded as an object in $(R\text{-}\mathbf{mod}, \mathbf{Ab})$. We use the same notation for a formula and the functor it defines and use the terms *pp-functor* and *pp-formula* interchangeably.

The following result follows from [9, Prop 3.1.7, Prop 6.1.2].

Proposition 2.2.6. *Every finitely generated subfunctor of a representable functor* $\mathrm{Hom}(X, -)$ *in* $(R\text{-}\mathbf{mod}, \mathbf{Ab})$ *is isomorphic to a pp-formula in one free variable.*

Given two pp-formulas (or *pp-functors*) φ and ψ such that $\psi \leq \varphi$ (i.e. in all modules M, we have $\psi(M) \leq \varphi(M)$) we get a functor φ/ψ which defines, at a module M, the set of cosets of $\psi(M)$ in $\varphi(M)$. It is easy to check that functors of this form commute with products and directed colimits.

Let φ and ψ be pp-formulas with n free variables such that $\psi \leq \varphi$ and let φ' and ψ' be pp-formulas in m free variables such that $\psi' \leq \varphi'$. A pp-formula ρ in $n+m$ free variables is said to define a function $\varphi/\psi \to \varphi'/\psi'$ if for every module M, the relation $\rho(M) \leq \varphi(M) \times \varphi'(M)$ and induces a function $\varphi/\psi \to \varphi'/\psi'$ on cosets. Ivo Herzog's category $(R\text{-}\mathbf{Mod})^{\mathrm{eq}+}$ (see [14]) has as objects quotients φ/ψ of pp-formulas and as morphisms pp-formulas which define functions on these quotients (where we consider two such formulas to be the same morphism if they define the same function).

We are now in a position to state Burke's result (proved in [9, Prop 3.2.5]).

Proposition 2.2.7. *The categories* $(R\text{-}\mathbf{Mod})^{\mathrm{eq}+}$ *and* $\mathrm{coh}(R\text{-}\mathbf{mod}, \mathbf{Ab})$ *are equivalent.*

This result sets up a dictionary between model theoretic and functorial language. It allows for functorial methods to be used in the model theory of modules and gives a good point of access for the application of model theory to module theory.

2.3 Finite type localisation in LFP abelian categories

2.3.1 Hereditary torsion theory

All the results stated in this section can be found in [29, Ch 4], [35, Ch VI] and [30, Ch 12], or they can easily be deduced from what is written there.

Let $\mathbf{C}$ be an LFP abelian category. So, in particular, directed colimits in $\mathbf{C}$ commute with finite limits (see [1, 1.59]), or equivalently, the directed colimit functor

$\varinjlim$ is exact. A cocomplete abelian category with a generator in which directed colimits are exact is said to be a *Grothendieck* abelian category. In an LFP abelian category one can take the direct sum of the f.p. objects (one isomorphic copy of each) as a generator for the category. Hence $\mathbf{C}$ is Grothendieck. As such, objects in $\mathbf{C}$ have injective hulls (see [35, Cor 4.3, Prop 2.5]).

An *hereditary torsion class* $\mathcal{T}$ in $\mathbf{C}$ is a subclass closed under direct sums with the property that, for any short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

X and Z are in $\mathcal{T}$ if and only if Y is in $\mathcal{T}$. It follows that such a class is closed under directed colimits. Elements of $\mathcal{T}$ are called *torsion objects*. Every object $X \in \mathbf{C}$ has a largest torsion subobject, denoted $\tau(X)$. The operation τ is actually a functor $\mathbf{C} \to \mathbf{C}$ and is called the *torsion functor*. We say that a subobject $X \leq Y$ in $\mathbf{C}$ is *dense* if Y/X is torsion. An object Y is *torsion-free* if $\mathrm{Hom}(T, Y) = 0$ for all $T \in \mathcal{T}$, or, equivalently, if $\tau(Y) = 0$. If we denote the class of torsion-free objects by $\mathcal{F}$, then we have that

$$\mathcal{T} = \{X \in \mathbf{C} : \mathrm{Hom}(X, F) = 0 \ \forall F \in \mathcal{F}\}$$

Each of the items τ, $\mathcal{T}$ and $\mathcal{F}$ determines the other two. The triple $(\tau, \mathcal{T}, \mathcal{F})$ is called an *hereditary torsion theory* although we will often refer to it simply by the name of the torsion functor τ. If there is more than one torsion theory under discussion, the notions defined above can be prefixed by the name of the torsion functor (i.e. we can write τ*-dense* and τ*-torsion* etc.).

Given a class of injective objects $\mathcal{E}$ in $\mathbf{C}$, one can define the hereditary torsion theory *cogenerated by* $\mathcal{E}$ to be the torsion theory with torsion class

$$\{C \in \mathbf{C} : \mathrm{Hom}(C, E) = 0 \ \forall E \in \mathcal{E}\}$$

This class is clearly closed under quotients, extensions and direct sums. That it is also closed under subobjects follows from that fact that $\mathcal{E}$ is a class of injective objects.

Given an hereditary torsion theory $(\tau, \mathcal{T}, \mathcal{F})$, we can form a new category, denoted $\mathbf{C}/\mathcal{T}$, whose objects are of the form $\overline{X}$ where $X \in \mathbf{C}$ and where the hom-sets are computed as follows

$$\mathrm{Hom}(\overline{X},\overline{Y}) = \varinjlim_{\substack{X'\leq X \text{ dense} \\ Y'\leq Y \text{ torsion}}} \mathrm{Hom}(X', Y/Y')$$

The set $\mathrm{Hom}(\overline{X},\overline{Y})$ has a canonical abelian group structure. Composition is defined as follows. Let $\alpha \in \mathrm{Hom}(\overline{X},\overline{Y})$ and $\beta \in \mathrm{Hom}(\overline{Y},\overline{Z})$. Suppose α is represented by $f : X' \to Y/Y'$ and β is represented by $g : Y'' \to Z/Z'$. Let $p : Y \to Y/Y'$ and $q : Z \to Z/Z'$ be the canonical projections. Define

$$X'' = f^{-1}(p(Y'')) \quad \text{ and } \quad Z'' = q^{-1}(g(Y' \cap Y''))$$

Then X'' is a dense subobject of X and Z'' is a torsion subobject of Z. Let f' be the map

$$X'' \xrightarrow{f\restriction_{X''}} p(Y'') \cong Y''/(Y'' \cap Y')$$

and let $g' : Y''/(Y'' \cap Y') \to Z/Z''$ be the canonical map induced by g. Then the composition $\beta\alpha$ is represented by the map $g'f'$ in the group $\mathrm{Hom}(\overline{X},\overline{Z})$. This composition law may be shown to be independent of representatives and is biadditive.

Clearly, if $f : X \to Y$ is a map in $\mathbf{C}$, then f represents a map $\overline{f} : \overline{X} \to \overline{Y}$. The assignments $X \mapsto \overline{X}$ and $f \mapsto \overline{f}$ define an exact covariant functor

$$P : \mathbf{C} \to \mathbf{C}/\mathcal{T}$$

Proposition 2.3.1. *Let f be a map in $\mathbf{C}$. Then*

(a) $P(f) = 0$ *iff* $\mathrm{im}\, f \in \mathcal{T}$

(b) $P(f)$ *is monic iff* $\ker f \in \mathcal{T}$

(c) $P(f)$ *is epic iff* $\mathrm{coker}\, f \in \mathcal{T}$

(d) $P(f)$ *is invertible iff* $\ker f, \mathrm{coker}\, f \in \mathcal{T}$

By the above proposition, we can think of the category $\mathbf{C}/\mathcal{T}$ as obtained from $\mathbf{C}$ by adding inverses for all maps f such that $\ker f$ and $\mathrm{coker}\, f$ are in $\mathcal{T}$ (this can be made formal). It is for this reason that we call a category of the form $\mathbf{C}/\mathcal{T}$ a *localisation* of $\mathbf{C}$.

The functor P has a fully faithful right adjoint S, called the *section functor*. The composition $L = SP : \mathbf{C} \to \mathbf{C}$ is called the *localisation functor*. The image of S in $\mathbf{C}$

is a full reflexive subcategory (i.e. a full subcategory such that the inclusion functor has a left exact left adjoint) of $\mathbf{C}$ which we shall denote by $\mathbf{C}_\tau$. So $\mathbf{C}_\tau \simeq \mathbf{C}/\mathcal{T}$ and L is a functor $\mathbf{C} \to \mathbf{C}_\tau$. We will often write the functor L as $(-)_\tau$ so that for any object $C \in \mathbf{C}$, C_τ is $LC \in \mathbf{C}_\tau$. We have that $(C_\tau)_\tau \cong C_\tau$ for all objects $C \in \mathbf{C}$.

We say that an object $X \in \mathbf{C}$ is *τ-injective* if, whenever $Y \leq Z$ is a τ-dense inclusion, any map $Y \to X$ extends to a map $Z \to X$.

Proposition 2.3.2. *The following are equivalent for an object $X \in \mathbf{C}$.*

(i) $X \in \mathbf{C}_\tau$.

(ii) $X \cong X_\tau$.

(iii) *X is τ-torsion-free and τ-injective.*

If $\mathbf{C}$ is a functor category $(\mathbf{A}, \mathbf{Ab})$ and $(\tau, \mathcal{T}, \mathcal{F})$ is a torsion theory on $\mathbf{C}$, then τ is determined by the dense subfunctors of the representable functors. That is, a functor F is τ-torsion iff and only if for any object $A \in \mathbf{A}$ and any map $f : (A, -) \to F$, $\ker f$ is dense in $(A, -)$. (This follows from the fact that the representable functors are generating.) Subfunctors of a representable functor $(A, -)$ are called *left ideals* of A. This is because a subfunctor $\mathcal{I} \leq (A, -)$ can be thought of as a collection of maps with domain A with the property that

- If $f, g \in \mathcal{I}$ with $\operatorname{cod} f = \operatorname{cod} g$ ("cod" = "codomain"), then $f + g \in \mathcal{I}$.
- If $f \in \mathcal{I}$ and g is a map with $\operatorname{dom} g = \operatorname{cod} f$, then $gf \in \mathcal{I}$.

Suppose that for each object A in $\mathbf{A}$ we have a collection $J(A)$ of left ideals of A. Then these sets of ideals determine an hereditary torsion theory on $\mathbf{C}$ (in the above sense) provided they satisfy the following conditions.

GAB1 If $\mathcal{I} \in J(A)$ and $\mathcal{I}' \supseteq \mathcal{I}$, then $\mathcal{I}' \in J(A)$.

GAB2 If $\mathcal{I}, \mathcal{I}' \in J(A)$, then $\mathcal{I} \cap \mathcal{I}' \in J(A)$.

GAB3 If $\mathcal{I} \in J(A)$ and $g : A \to B \in \mathbf{A}$, then $(\mathcal{I} : g) \in J(B)$ where $(\mathcal{I} : g)$ is the ideal $\{f : \operatorname{dom} f = B \text{ and } fg \in \mathcal{I}\}$.

GAB4 Let $\mathfrak{I} \in J(A)$ and $\mathfrak{I}'$ be a left ideal of A. If $(\mathfrak{I}' : f) \in J(\operatorname{cod} f)$ for all $f \in \mathfrak{I}$, then $\mathfrak{I}' \in J(A)$.

An operation J satisfying the above conditions is called a *Gabriel topology* on $\mathbf{A}$. It is an additive version of a *Grothedieck topology* which we will discuss in Chapter 3. In a functor category $(\mathbf{A}, \mathbf{Ab})$ with Gabriel topology J and induced torsion theory τ, the equivalent conditions of Proposition 2.3.2 are equivalent to a *sheaf* property (see 3.3.3).

Proposition 2.3.3. *The functor $F \in (\mathbf{A}, \mathbf{Ab})$ is in $(\mathbf{A}, \mathbf{Ab})_\tau$ if and only if for all $A \in \mathbf{A}$ and all $\mathfrak{I} \in J(A)$, any natural transformation $\mathfrak{I} \to F$ has a unique extension to $(A, -)$.*

2.3.2 Finite type localisation

Let $\mathbf{C}$ be an LFP abelian category and let $(\tau, \mathcal{T}, \mathcal{F})$ be an hereditary torsion theory on $\mathbf{C}$. Then τ is of *finite type* if it satisfies the equivalent conditions of the following proposition (see [30, Prop 12.1.15]).

Proposition 2.3.4. *The following are equivalent.*

(i) *$\mathcal{T}$ is the smallest hereditary torsion class containing $\mathcal{T} \cap \operatorname{fp} \mathbf{C}$.*

(ii) *If $Y \in \mathbf{C}$ is finitely generated and $X \leq Y$ is a dense subobject, then there is a dense finitely generated $X' \leq Y$ such that $X' \leq X$.*

If τ is of finite type, and X and Y are finitely generated objects of $\mathbf{C}$, then the hom-set $\operatorname{Hom}(\overline{X}, \overline{Y}) = \operatorname{Hom}(X_\tau, Y_\tau)$ can be computed as

$$\varinjlim_{\substack{X' \text{ f.g. dense } \leq X \\ Y' \text{ f.g. torsion } \leq Y}} \operatorname{Hom}(X', Y/Y')$$

Proposition 2.3.5 ([30, 12.1.13]). *Let $\mathbf{A}$ be a small preadditive category and let $\mathbf{C} = (\mathbf{A}, \mathbf{Ab})$. Let J be a Gabriel topology on $\mathbf{A}$ with associated hereditary torsion theory τ on $\mathbf{C}$. Then τ is of finite type if and only if whenever $A \in \mathbf{A}$ and $\mathfrak{I} \in J(A)$, there is a finitely generated $\mathfrak{I}' \in J(A)$ with $\mathfrak{I}' \leq \mathfrak{I}$.*

Now let $\mathbf{C}$ be the category $(R\text{-}\mathbf{mod}, \mathbf{Ab})$ and τ be a finite type hereditary torsion theory on $\mathbf{C}$. Then, by Proposition 2.2.6, for any dense subfunctor $\mathcal{I}$ of a representable $(A, -)$, there is a dense pp-functor $\mathcal{I}' \leq (A, -)$ with $\mathcal{I}' \leq \mathcal{I}$. Let $(A, -)$ and $(B, -)$ be two representable functors. Then the hom-set $\mathrm{Nat}\,((A, -)_\tau, (B, -)_\tau)$ can be written as

$$\varinjlim \mathrm{Nat}\,(\mathcal{I}, (B, -)/\mathcal{J}) \tag{2.1}$$

where (by Proposition 2.2.6) $\mathcal{I}$ ranges over dense pp-subfunctors of $(A, -)$ and $\mathcal{J}$ ranges over torsion pp-subfunctors of $(B, -)$.

The first part of the following proposition is a special case of [30, Prop 12.2.16]. The second part follows from [30, Thm 12.2.11]

Proposition 2.3.6. *Let τ be a finite type hereditary torsion theory on $(R\text{-}\mathbf{mod}, \mathbf{Ab})$.*

(a) *Then every τ-torsion functor is a directed colimit of finitely presented (or, equivalently, coherent) τ-torsion functors.*

(b) $(R\text{-}\mathbf{mod}, \mathbf{Ab})_\tau$ *is an LFP abelian category with generating set $\{(A, -)_\tau : A \in R\text{-}\mathbf{mod}\}$ of finitely presented objects.*

(c) $\mathrm{coh}((R\text{-}\mathbf{mod}, \mathbf{Ab})_\tau) = \mathrm{fp}((R\text{-}\mathbf{mod}, \mathbf{Ab})_\tau)$.

2.3.3 Definable subcategories of modules and purity

Any functor $F : \mathbf{mod}\text{-}R \to \mathbf{Ab}$ has a canonical extension to a functor $\mathbf{Mod}\text{-}R \to \mathbf{Ab}$ which commutes with directed colimits (see [4, pp. 4-5] or [22, 5.6]). The canonical extension of the functor φ/ψ is again φ/ψ (it is already a functor defined on $\mathbf{Mod}\text{-}R$). We will use the same notation for a functor in $(\mathbf{mod}\text{-}R, \mathbf{Ab})$ and its canonical extension to $\mathbf{Mod}\text{-}R$. (Of course, the same applies to functors on finitely presented *left* R-modules.)

A *definable subcategory* of $\mathbf{Mod}\text{-}R$ is a full subcategory whose class of objects takes the form

$$Z(\{F_\lambda\}_\lambda) = \{M \in \mathbf{Mod}\text{-}R : F_\lambda(M) = 0\ \forall \lambda\}$$

where each $F_\lambda \in \mathrm{coh}(\mathbf{mod}\text{-}R, \mathbf{Ab})$. By Proposition 2.2.7 (or, rather, a version of it for *right* modules) each F_λ is of the form $\varphi_\lambda/\psi_\lambda$ for pp-formulas φ_λ and ψ_λ such that $\psi_\lambda \to \varphi_\lambda$, and so $Z(\{F_\lambda\}_\lambda)$ is the category of models of the theory $\{\varphi_\lambda \to \psi_\lambda\}_\lambda$. For examplpe, if R is a domain then the class of torsion free modules is definable: it is $Z\left(\{(xr = 0)/(x = 0)\}_{r \neq 0}\right)$. Also if R is right coherent, then the class of flat left modules is definable (see [30, ex 3.4.5]).

We say that a monomorphism $f : M \to N$ of modules is *pure* if for any pp-formula $\varphi(\boldsymbol{x})$ and tuple $\boldsymbol{a}$ from M, we have that $M \models \varphi(\boldsymbol{a})$ if $N \models \varphi(f\boldsymbol{a})$. See [30, I, Ch 2] for a proof of the following.

Proposition 2.3.7. *The following are equivalent for a monomorphism f in* $\mathbf{Mod}$-R*.*

(i) *f is pure.*

(ii) *f is a directed colimit of split monomorphisms.*

(iii) *$(f \otimes -)$ is a monomorphism in the functor category* $(R\text{-}\mathbf{mod}, \mathbf{Ab})$*.*

The following proposition is part of Theorem 3.4.6 in [30].

Proposition 2.3.8. *A subcategory $\mathbf{D} \subseteq \mathbf{Mod}\text{-}R$ is definable if and only if it is closed under products, directed colimits and pure subobjects.*

A module M is *pure-injective* if whenever $f : X \to Y$ is a pure monomorphism, any map $g : X \to M$ factors through f. The following proposition follows from general model theory (see [16, Thm 10.7.1]).

Proposition 2.3.9. *The following are equivalent for a module M.*

(i) *M is pure-injective.*

(ii) *M is a direct summand of any pure extension.*

(iii) *Any pp-type of M in finitely many variables is realised in M.*

(iv) *Any atomic type of M in any number of variables is realised in M.*

We denote the class of indecomposable pure-injective right R-modules by pinj-R. Definable subcategories of modules are determined by the indecomposable pure-injective modules they contain. The following proposition is Corollary 5.1.8 in [30].

Proposition 2.3.10. *Let $\mathbf{D}$ and $\mathbf{D}'$ be definable subcategories of* $\mathbf{Mod}$-R*.*

(a) *If $\mathbf{D} \neq 0$ then $\mathbf{D} \cap \text{pinj-}R \neq 0$.*

(b) $\mathbf{D} \cap \text{pinj-}R = \mathbf{D}' \cap \text{pinj-}R$ *if and only if* $\mathbf{D} = \mathbf{D}'$.

Proposition 2.3.11 ([38, 4.8]). *Let* $F \in \text{coh}(\mathbf{mod}\text{-}R, \mathbf{Ab})$ *and* $\mathbf{D} \subseteq \mathbf{Mod}\text{-}R$ *be definable. Then* $F(M) = 0$ *for all* $M \in \mathbf{D}$ *if and only if* $F(M) = 0$ *for all* $M \in \mathbf{D} \cap \text{pinj-}R$.

The tensor functor $M \mapsto (M \otimes -)$ and $f \mapsto (f \otimes -)$ defines a fully faithful embedding

$$T : \mathbf{Mod}\text{-}R \rightarrowtail (R\text{-}\mathbf{mod}, \mathbf{Ab})$$

It has the following useful properties (see [13]).

Proposition 2.3.12.

(a) *If N is an (indecomposable) pure-injective right R-module, then $T(N)$ is an (indecomposable) injective functor.*

(b) *Every (indecomposable) injective functor $F \in (R\text{-}\mathbf{mod}, \mathbf{Ab})$ is isomorphic to a functor of the form $T(N)$ for an (indecomposable) pure-injective right R-module N.*

See [12, Sec. 1] or [17, B16] for more details.

2.3.4 Elementary duality

There is natural duality (due independently to Auslander [3] and to Gruson and Jensen [12])

$$d : (\text{coh}(R\text{-}\mathbf{mod}, \mathbf{Ab}))^{\text{op}} \longrightarrow \text{coh}(\mathbf{mod}\text{-}R, \mathbf{Ab})$$

such that $dF(A) = \text{Nat}(F, A \otimes -)$, and if $g : A \to B$ is a map in $\mathbf{mod}\text{-}R$, we get an induced map $(g \otimes -) : (A \otimes -) \to (B \otimes -)$, and hence a map

$$dF(g) : \text{Nat}(F, A \otimes -) \longrightarrow \text{Nat}(F, B \otimes -)$$

If $f : F \to G$ is a natural transformation of functors, we define $df : dG \to dF$ by

$$(df)_M(\tau) = \tau \circ f$$

Dually we can define $d : (\mathrm{coh}(\mathbf{mod}\text{-}R, \mathbf{Ab}))^{\mathrm{op}} \to \mathrm{coh}(R\text{-}\mathbf{mod}, \mathbf{Ab})$ by $dF(A) = \mathrm{Nat}(F, - \otimes A)$ and we then have that $d^2 \cong \mathrm{id}$. So the functor d is an equivalence of categories. In other words the categories $\mathrm{coh}(R\text{-}\mathbf{mod}, \mathbf{Ab})$ and $\mathrm{coh}(\mathbf{mod}\text{-}R, \mathbf{Ab})$ are dual ([12, 5.6], [3]). If F is a coherent functor, then $\mathrm{Nat}(F, -)$ commutes with directed colimits (since F is f.p.). Given that also tensor products commute with directed colimits, we get that the canonical extension of dF to all modules is what one would expect, i.e.

$$dF(M) \cong \mathrm{Nat}(F, - \otimes M) \tag{2.2}$$

for $F \in \mathrm{coh}(\mathbf{mod}\text{-}R, \mathbf{Ab})$, $M \in R\text{-}\mathbf{Mod}$, and

$$dF(M) \cong \mathrm{Nat}(F, M \otimes -)$$

for $F \in \mathrm{coh}(R\text{-}\mathbf{mod}, \mathbf{Ab})$, $M \in \mathbf{Mod}\text{-}R$.

There is also a duality at the level of pp-formulas ([31, Sec. 8.4]). Given a pp-formula φ for left (resp. right) R-modules, there is a corresponding pp-formula $D\varphi$ for right (resp. left) R-modules such that $\varphi \to \psi$ implies $D\psi \to D\varphi$. The operation D induces a duality between the lattices of pp-formulas for right and left R-modules. The two functors d and D are related by

$$d\left(\frac{\varphi}{\psi}\right) = \frac{D\psi}{D\varphi}$$

The following result (see 1.3 in [33]) will be important in Theorem 2.4.2.

Proposition 2.3.13. *Let I be an injective R-module. Then*

$$(D\varphi)(I) = \mathrm{ann}_I(\varphi(R))$$

2.3.5 Localisation at a definable subcategory

For any abelian category $\mathbf{C}$, we denote the class of indecomposable injective objects in $\mathbf{C}$ by $\mathrm{inj}\,\mathbf{C}$. Let $\mathbf{D}$ be the definable subcategory of $\mathbf{Mod}\text{-}R$ defined by the coherent functors $\{G_\lambda\}_\lambda$ in $(\mathbf{mod}\text{-}R, \mathbf{Ab})$. So

$$\mathbf{D} = Z(\{G_\lambda\}_\lambda)$$

Then, by propositions 2.3.10, 2.3.12, and by (2.2) and the fact that d is a duality, $\mathbf{D}$ is determined by the functors $E \in \mathrm{inj}(R\text{-}\mathbf{mod}, \mathbf{Ab})$ satisfying

$$\mathrm{Nat}(dG_\lambda, E) = 0 \ \forall \lambda$$

Let us call this class of indecomposable injective functors $\mathcal{E}(\mathbf{D})$. If F is a coherent functor in $(\mathbf{mod}\text{-}R, \mathbf{Ab})$, then we have, by Proposition 2.3.11, that

$$F(M) = 0 \ \forall M \in \mathbf{D} \quad \text{iff} \quad \mathrm{Nat}(dF, E) = 0 \ \forall E \in \mathcal{E}(\mathbf{D})$$

We can now define, given $\mathbf{D}$, finite type hereditary torsion theories $(\tau_r, \mathcal{T}_r, \mathcal{F}_r)$ and $(\tau_l, \mathcal{T}_l, \mathcal{F}_l)$ on $(\mathbf{mod}\text{-}R, \mathbf{Ab})$ and $(R\text{-}\mathbf{mod}, \mathbf{Ab})$ respectively. We define the f.p. functor $F \in (\mathbf{mod}\text{-}R, \mathbf{Ab})$ to be τ_r-torsion if $F(M) = 0 \ \forall M \in \mathbf{D}$ and we define the f.p. functor $F \in (R\text{-}\mathbf{mod}, \mathbf{Ab})$ to be τ_l-torsion if $\mathrm{Nat}(F, E) = 0 \ \forall E \in \mathcal{E}(\mathbf{D})$ (this gives the finite type hereditary torsion theory cogenerated by $\mathcal{E}(\mathbf{D})$). By Proposition 2.3.6, a general torsion object in one of these functor categories is a directed colimit of f.p. torsion objects. So the class $\mathcal{T}_r$ is simply

$$\{F \in (\mathbf{mod}\text{-}R, \mathbf{Ab}) : F(M) = 0 \ \forall M \in \mathbf{D}\}$$

and an inclusion $F \leq G$ is τ_r-dense if $F(M) = G(M) \ \forall M \in \mathbf{D}$. Let $J_{\mathbf{D}}$ be the Gabriel topology corresponding to τ_r. Then we have

$$\mathcal{I} \in J_{\mathbf{D}}(A) \quad \Leftrightarrow \quad \mathcal{I}(D) = \mathrm{Hom}(A, D) \ \forall D \in \mathbf{D} \tag{2.3}$$

The duality d induces a bijection between f.p. τ_r-torsion functors in $(\mathbf{mod}\text{-}R, \mathbf{Ab})$ and f.p. τ_l-torsion functors in $(R\text{-}\mathbf{mod}, \mathbf{Ab})$. In fact ([15, 2.16], [23, 2.6]), there is an equivalence of categories

$$\left(\mathrm{coh}\left((\mathbf{mod}\text{-}R, \mathbf{Ab})_{\tau_r}\right)\right)^{\mathrm{op}} \simeq \mathrm{coh}\left((R\text{-}\mathbf{mod}, \mathbf{Ab})_{\tau_l}\right)$$

2.3.6 The localised version of Burke's result

Fix a definable subcategory $\mathbf{D} \subseteq \mathbf{Mod}\text{-}R$. Let $\mathbf{D}^{\mathrm{eq}+}$ be the category with objects pairs of pp-formulas φ/ψ and with morphisms pp-formulas $\rho : \varphi/\psi \to \varphi'/\psi'$ that define functions (as opposed to just relations) when evaluated at modules in $\mathbf{D}$. Let τ be the hereditary torsion theory of finite type on $(\mathbf{mod}\text{-}R, \mathbf{Ab})$ associated to $\mathbf{D}$ (that is τ_r above). Then (see [30, 13.3.19])

Theorem 2.3.14. *There is an equivalence of categories*

$$\mathbf{D}^{\mathrm{eq}+} \simeq \mathrm{coh}\left((\mathbf{mod}\text{-}R, \mathbf{Ab})_\tau\right)$$

2.4 Ziegler and Zariski spectra

2.4.1 Definitions

Fix a ring R. The *(right) Ziegler spectrum* of R, denoted Zg_R, is the topological space whose set of points is $\mathrm{pinj}\text{-}R$, the set of isomorphism types of indecomposable pure-injective right R-modules. Given a family of coherent functors $\{F_\lambda\}_\lambda$ in $(\mathbf{mod}\text{-}R, \mathbf{Ab})$, the set

$$Z(\{F_\lambda\}_\lambda) \cap \mathrm{pinj}\text{-}R = \{N \in \mathrm{pinj}\text{-}R : F_\lambda(N) = 0 \ \forall \lambda\}$$

is a typical closed set of Zg_R. It is the case ([38, 4.9]) that sets of this form do indeed give a topology on $\mathrm{pinj}\text{-}R$. It turns out that a study of the space Zg_R yields interesting information about the representation theory of R. See [30] for a thorough discussion.

There is another natural topology on $\mathrm{pinj}\text{-}R$ (a certain dual of the Ziegler topology) which is defined by declaring sets of the form

$$[F] = \{N \in \mathrm{pinj}\text{-}R : FN = 0\}$$

to be open, where F ranges over coherent functors in $(\mathbf{mod}\text{-}R, \mathbf{Ab})$. We have

$$[F] \cap [G] = [F \oplus G]$$

so that the set $\{[F] : F \in \mathrm{coh}(\mathbf{mod}\text{-}R, \mathbf{Ab})\}$ is a basis of open sets. We denote the corresponding topological space by $\mathrm{Zar}(R\text{-}\mathbf{mod})$ and call it the *left Zariski spectrum* of R-$\mathbf{mod}$ (the reason for this terminology will become clear later).

2.4.2 The space of indecomposable injectives

There is a certain presheaf, denoted Def_R, defined on the space $\mathrm{Zar}(R\text{-}\mathbf{mod})$ which associates to a basic open set $[F]$ the ring of "pp-definable scalars" on $Z(F)$. More precisely, $\mathrm{Def}_R[F]$ extends the ring of standard scalars R by all pp-formulas in two variables which define functions on all (right) modules $M \in Z(F)$; these are the *definable scalars*. There is another topological space $\mathrm{Zar}(R^{\mathrm{op}})$, whose set of points $\mathrm{inj}(\mathbf{Mod}\text{-}R)$ is the set of all indecomposable injective (right) modules, and there is a naturally defined presheaf of rings on this space which generalises the standard Zariski spectrum of a commutative ring. When R is right coherent, the space $\mathrm{Zar}(R^{\mathrm{op}})$ embeds (as a topological space) into $\mathrm{Zar}(R\text{-}\mathbf{mod})$ (Proposition 2.4.1). We will show in Theorem 2.4.2 that for a right coherent ring R, the natural presheaf defined on $\mathrm{Zar}(R^{\mathrm{op}})$ is the restriction of the presheaf Def_R of definable scalars defined on $\mathrm{Zar}(R\text{-}\mathbf{mod})$.

Given that we have a duality d between coherent functors on f.p. left R-modules and coherent functors on f.p. right R-modules, it is clear that we can describe the basic open sets of $\mathrm{Zar}(R\text{-}\mathbf{mod})$ as $[dF]$ where F ranges over $\mathrm{coh}(R\text{-}\mathbf{mod}, \mathbf{Ab}) = \mathrm{fp}(R\text{-}\mathbf{mod}, \mathbf{Ab})$. Moreover, since the (indecomposable) pure-injective right R-modules correspond to the (indecomposable) injective functors in $(R\text{-}\mathbf{mod}, \mathbf{Ab})$ via the map $M \mapsto (M \otimes -)$, and $dF(M) = \mathrm{Nat}(F, M \otimes -)$, we can write the open set $[dF]$ as

$$\{E \in \mathrm{inj}(R\text{-}\mathbf{mod}, \mathbf{Ab}) : \mathrm{Nat}(F, E) = 0\}$$

So we have a topology on the set of indecomposable injective "left (R-$\mathbf{mod}$)-modules" with basic open sets as above. The need for the duality d will become clear in the proof of Theorem 2.4.2.

One can similarly define a topology on the space of indecomposable injective right R-modules by declaring the following sets to be open

$$[M] = \{I \in \mathrm{inj}(\mathbf{Mod}\text{-}R) : \mathrm{Hom}(M, I) = 0\}$$

where M ranges over $\mathbf{mod}\text{-}R$. (Again, such sets are closed under finite intersections since the direct sum of two f.p. modules is f.p.) We call the resulting space *the right*

Gabriel-Zariski spectrum of R and denote it $\mathrm{Zar}(R^{\mathrm{op}})$.

When R is a commutative noetherian ring, the indecomposable injective modules are precisely the injective envelopes of the modules $R/\mathfrak{p}$ where $\mathfrak{p}$ ranges over $\operatorname{Spec} R$, the set of prime ideals of R. In fact, it can be shown (see [32, Thm 2.1]) that for such rings the space $\mathrm{Zar}(R^{\mathrm{op}})$ is homeomorphic to $\operatorname{Spec} R$ with the standard Zariski topology.

The following proposition is proved in [32, Prop 2.7].

Proposition 2.4.1. *Let R be a right coherent ring. Then the space* $\mathrm{Zar}(R^{\mathrm{op}})$ *is homeomorphic to a subspace of* $\mathrm{Zar}(R\text{-}\mathbf{mod})$ *via the map* $\mathrm{inj}(\mathbf{Mod}\text{-}R) \to \mathrm{inj}(R\text{-}\mathbf{mod}, \mathbf{Ab}) = \mathrm{pinj}\text{-}R$ *defined by*

$$I \longmapsto (I \otimes -)$$

2.4.3 The presheaf of definable scalars

Given a coherent, equivalently f.p., functor F in $(R\text{-}\mathbf{mod}, \mathbf{Ab})$, let τ_F be the finite type hereditary torsion theory on $(R\text{-}\mathbf{mod}, \mathbf{Ab})$ cogenerated by the set $[dF]$ (consisting of injective functors). Let $\mathrm{Def}_R[dF]$ be the ring $\mathrm{End}((R,-)_{\tau_F})$. That is, by (2.1), $\mathrm{Def}_R[dF]$ can be written as

$$\varinjlim \mathrm{Nat}\,(\mathfrak{I}, (R,-)/\mathfrak{J})$$

where the directed colimit is taken over all pp-formulas $\mathfrak{I}, \mathfrak{J}$ for left modules in one free variable satisfying:

$$\forall N \in \mathrm{pinj}\text{-}R,\ dF(N) = 0 \Rightarrow D\mathfrak{I}(N) = 0 \ \wedge\ D\mathfrak{J}(N) = N \tag{2.4}$$

Note that if $[dF] \subseteq [dG]$, then for all $N \in \mathrm{pinj}\text{-}R$, $dF(N) = 0 \Rightarrow dG(N) = 0$, so there is an induced map $\mathrm{Def}_R[dG] \to \mathrm{Def}_R[dF]$. These maps are ring homomorphisms and behave well with respect to composition so that Def_R is actually a presheaf of rings on $\mathrm{Zar}(R\text{-}\mathbf{mod})$. This presheaf is often referred to as the *presheaf of definable scalars.* By Theorem 2.3.14 the elements of $\mathrm{Def}_R[dF]$ are represented by pp-formulas in two variables which define $\mathrm{End}_R(M)$-endomorphisms $M \to M$ for all $M \in Z(dF) \subseteq \mathbf{Mod}\text{-}R$.

Now let $\mathcal{P}$ be the presheaf on $\mathrm{Zar}(R^{\mathrm{op}})$ which associates to the basic open set $[M]$ the *localised ring* corresponding to the finite type hereditary torsion theory τ_M on **Mod**-R cogenerated by $[M]$ (a set of injective modules). The localised ring R_M corresponding to $[M]$ is defined to be $\mathrm{End}(R_{\tau_M})$. So, since τ_M is of finite type, we can write

$$R_M = \varinjlim \mathrm{Hom}(\mathfrak{a}, R/\mathfrak{b})$$

where $\mathfrak{a}$ ranges over f.g. τ_M-dense ideals of R and $\mathfrak{b}$ ranges over f.g. τ_M-torsion ideals of R. When R is commutative noetherian, $\mathcal{P}$ corresponds to the standard structure sheaf on $\mathrm{Spec}\, R$ (see [32, Prop 7.7]).

We know from Proposition 2.4.1 that when R is right coherent, $\mathrm{Zar}(R^{\mathrm{op}})$ can be regarded as a subspace of $\mathrm{Zar}(R\text{-}\mathbf{mod})$. Is it true that $\mathcal{P}$ is the restriction of the presheaf of definable scalars Def_R? In other words, if i is the inclusion of $\mathrm{Zar}(R^{\mathrm{op}})$ in $\mathrm{Zar}(R\text{-}\mathbf{mod})$, is $\mathcal{P} = i^{-1}\mathrm{Def}_R$? The following theorem answers this question affirmatively.

Theorem 2.4.2. *Let R be a right coherent ring. The presheaf $\mathcal{P}$ defined on the space $\mathrm{Zar}(R^{\mathrm{op}})$ is the restriction of the presheaf of definable scalars Def_R on $\mathrm{Zar}(R\text{-}\mathbf{mod})$.*

Proof. Let $i : \mathrm{inj}(\mathbf{Mod}\text{-}R) \to \mathrm{inj}(R\text{-}\mathbf{mod}, \mathbf{Ab})$ be the embedding of Proposition 2.4.1. So $i(I) = (I \otimes -)$. The restriction of Def_R to $\mathrm{Zar}(R^{\mathrm{op}})$ is $i^{-1}\mathrm{Def}_R$ which takes the basic open set $[M]$ to the ring

$$\varinjlim_{i[M]\subseteq[dF]} \mathrm{End}((R,-)_{\tau_F})$$

We can rewrite this directed colimit as

$$\varinjlim \mathrm{Nat}(\mathfrak{I}, (R,-)/\mathfrak{J})$$

where $\mathfrak{I}$ and $\mathfrak{J}$ range over pp-formulas in one free variable satisfying

$$\forall I \in [M],\ D\mathfrak{I}(I) = 0 \ \wedge\ D\mathfrak{J}(I) = I \tag{2.5}$$

We can do this because if $\mathfrak{I}$ and $\mathfrak{J}$ satisfy (2.5) then, by putting

$$F = \left(\frac{x = x}{\mathfrak{I}}\right) \oplus \mathfrak{J}$$

so that

$$dF = D\mathfrak{I} \oplus \left(\frac{x=x}{D\mathfrak{J}}\right)$$

we have that $i[M] \subseteq [dF]$ and

$$\forall N \in \text{pinj-}R,\ dF(N) = 0 \ \Rightarrow\ D\mathfrak{I}(N) = 0 \ \wedge\ D\mathfrak{J}(N) = N$$

That is, $\mathfrak{I}$ and $\mathfrak{J}$ satisfy (2.4) with respect to F.

Conversely, suppose that F is such that $i[M] \subseteq [dF]$ and suppose $\mathfrak{I}$ and $\mathfrak{J}$ satisfy (2.4). Then, since every injective module is pure-injective, we have in particular that $\mathfrak{I}$ and $\mathfrak{J}$ satisfy (2.5).

In order to show that $\mathcal{P}$ is the restriction of Def_R to $\mathrm{Zar}(R^{\mathrm{op}})$, we must produce a natural isomorphism

$$\Psi : R_M \xrightarrow{\cong} \varinjlim_{\mathfrak{I},\mathfrak{J}\ \text{sat. (2.5)}} \mathrm{Nat}(\mathfrak{I}, (R,-)/\mathfrak{J})$$

Let $\mathfrak{a} = (f_1, \ldots, f_n)R$ be τ_M-dense, $\mathfrak{b} = (g_1, \ldots, g_m)R$ be τ_M-torsion, $\varepsilon : \mathfrak{a} \to R/\mathfrak{b}$ be an R-linear map, and $\xi_i \in R$ be elements such that $\overline{\xi_i} = \varepsilon(f_i)$. We will define a map $\Psi(\mathfrak{a}, \mathfrak{b}) : \mathrm{Hom}(\mathfrak{a}, R/\mathfrak{b}) \to \varinjlim \mathrm{Nat}(\mathfrak{I}, (R,-)/\mathfrak{J})$.

We make the following definitions

$$\mathfrak{I} = \exists z_1 \ldots z_n \left(\begin{pmatrix} f_1 & \ldots & f_n \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = x \right)$$

$$\mathfrak{J}_0 = \exists w_1 \ldots w_m \left(\begin{pmatrix} g_1 & \ldots & g_m \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} = y \right)$$

$$\mathfrak{J}_\varepsilon = \exists z_1 \ldots z_n \left(\begin{pmatrix} f_1 & \ldots & f_n \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = 0 \ \wedge\ \begin{pmatrix} \xi_1 & \ldots & \xi_n \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = y \right)$$

Then $\mathfrak{J}_\varepsilon(R) \subseteq \mathfrak{J}_0(R) = \mathfrak{b}$ and $\mathfrak{I}(R) = \mathfrak{a}$.

Let $I \in [M]$. Then since $\mathfrak{a}$ is dense in R, $\mathrm{Hom}(R/\mathfrak{a}, I) = 0$, and so

$$D\mathfrak{I}(I) = \mathrm{ann}_I(\mathfrak{I}(R)) = \mathrm{ann}_I(\mathfrak{a}) = 0$$

(using Proposition 2.3.13). Also, since $\mathfrak{b}$ is a torsion ideal, $\mathrm{Hom}(\mathfrak{b}, I) = 0$, which implies $\mathrm{ann}_I(\mathfrak{b}) = I$ and so

$$D\mathfrak{J}_0(I) = \mathrm{ann}_I(\mathfrak{J}_0(R)) = \mathrm{ann}_I(\mathfrak{b}) = I$$

and

$$I \supseteq (D\mathfrak{J}_\varepsilon)(I) = \mathrm{ann}_I(\mathfrak{J}_\varepsilon(R)) \supseteq \mathrm{ann}_I(\mathfrak{b}) = I$$

So we have that $D\mathfrak{I}(I) = 0$ and $D\mathfrak{J}_0(I) = D\mathfrak{J}_\varepsilon(I) = I$ whenever $I \in [M]$.

Now let

$$\mathfrak{J} = \mathfrak{J}_0 + \sum_{\varepsilon \in \mathrm{Hom}(\mathfrak{a}, R/\mathfrak{b})} \mathfrak{J}_\varepsilon$$

Let $\mathfrak{J}_S$ be the pp-formula $\mathfrak{J}_0 + \sum_{\varepsilon \in S} \mathfrak{J}_\varepsilon$ where S is a finite subset of $\mathrm{Hom}(\mathfrak{a}, R/\mathfrak{b})$. Then $\mathfrak{J}_S(I) = I$ for all $I \in [M]$ and, since $\mathfrak{I}$ is f.p.,

$$\mathrm{Nat}(\mathfrak{I}, (R, -)/\mathfrak{J}) = \varinjlim_{S \text{ fin.} \subseteq \mathrm{Hom}(\mathfrak{a}, R/\mathfrak{b})} \mathrm{Nat}(\mathfrak{I}, (R, -)/\mathfrak{J}_S)$$

We can define a map

$$\Psi(\mathfrak{a}, \mathfrak{b}) : \mathrm{Hom}(\mathfrak{a}, R/\mathfrak{b}) \longrightarrow \mathrm{Nat}(\mathfrak{I}, (R, -)/\mathfrak{J})$$

by

$$\Psi(\mathfrak{a}, \mathfrak{b})(\varepsilon) : \sum f_i x_i \longmapsto \sum \xi_i x_i \mod \mathfrak{J}$$

$\mathfrak{J}_0 \subseteq \mathfrak{J}$ implies that this map is independent of the choice of representatives $\{\xi_i\}_i$ and $\mathfrak{J}_\varepsilon \subseteq \mathfrak{J}$ implies that the map is well-defined.

It is easily checked that when $\mathfrak{a} \leq \mathfrak{a}'$ are τ_M-dense ideals and $\mathfrak{b} \leq \mathfrak{b}'$ are τ_M-torsion, we get a commutative square

$$\begin{tikzcd}
\mathrm{Hom}(\mathfrak{a}, R/\mathfrak{b}) \arrow[r, "\Psi(\mathfrak{a},\mathfrak{b})"] \arrow[d] & \mathrm{Nat}(\mathfrak{I}, (R, -)/\mathfrak{J}) \arrow[d] \\
\mathrm{Hom}(\mathfrak{a}', R/\mathfrak{b}') \arrow[r, "\Psi(\mathfrak{a}',\mathfrak{b}')"] & \mathrm{Nat}(\mathfrak{I}', (R, -)/\mathfrak{J}')
\end{tikzcd}$$

so that $\Psi = \varinjlim \Psi(\mathfrak{a}, \mathfrak{b})$ gives us a map

$$R_M \longrightarrow \varinjlim_{i[M]\subseteq [dF]} \operatorname{End}((R, -)_{\tau_F})$$

<u>CLAIM</u>: Ψ is a ring homomorphism.

To see this, let $\mathfrak{a} = (f_1, \ldots, f_n)R$, $\mathfrak{a}' = (f_1', \ldots, f_{n'}')R$ be τ_M-dense ideals, and $\mathfrak{b} = (g_1, \ldots, g_m)R$, $\mathfrak{b}' = (g_1', \ldots, g_{m'}')R$ be τ_M-torsion. Let $\varepsilon : \mathfrak{a} \to R/\mathfrak{b}$ and $\varepsilon' : \mathfrak{a}' \to R/\mathfrak{b}'$ be two maps representing elements of R_M. Let $\pi_{\mathfrak{b}} : R \to R/\mathfrak{b}$ and $\pi_{\mathfrak{b}'} : R \to R/\mathfrak{b}'$ be the natural projections. We write $[\varepsilon]$ for the class of epsilon in R_M.

We first show that Φ preserves addition. Since R is right coherent, $\mathfrak{a}\cap\mathfrak{a}'$ is finitely generated and $[\varepsilon] + [\varepsilon']$ is represented by the map

$$\mathfrak{a} \cap \mathfrak{a}' \xrightarrow{\varepsilon+\varepsilon'} \frac{R}{\mathfrak{b} + \mathfrak{b}'}$$

Suppose that $\mathfrak{a} \cap \mathfrak{a}' = (a_1, \ldots, a_p)R$. Then there are matrices A and A' such that

$$(a_1 \ldots a_p) = (f_1 \ldots f_n)A = (f_1' \ldots f_{n'}')A'$$

Let $\mathcal{A}$ be the pp-functor

$$\exists z_1 \ldots z_p \left(\begin{pmatrix} a_1 & \ldots & a_p \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix} = x \right)$$

Then $\Psi[\varepsilon + \varepsilon']$ is represented by the map

$$\mathcal{A} \longrightarrow (R, -)/\mathcal{B}$$

(for some functor $\mathcal{B}$ as in the definition of Ψ) given by

$$\sum a_i x_i \longmapsto \begin{pmatrix} \xi_1 & \ldots & \xi_n & \xi_1' & \ldots & \xi_{n'}' \end{pmatrix} \begin{pmatrix} A \\ A' \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \quad \operatorname{mod} \mathcal{B}$$

Now $\Psi[\varepsilon] + \Psi[\varepsilon']$ is represented by the map

$$\mathcal{I} \cap \mathcal{I}' \longrightarrow (R, -)/(\mathcal{J} + \mathcal{J}')$$

which takes $x = \sum f_i x_i = \sum f'_j x'_j$ to $\sum \xi_i x_i + \sum \xi'_j x'_j \mod \mathcal{J} + \mathcal{J}'$. For convenience we call this map $\Psi(\varepsilon) + \Psi(\varepsilon')$. We clearly have that $\mathcal{A} \leq \mathcal{I} \cap \mathcal{I}'$ so we can restrict $\Psi(\varepsilon) + \Psi(\varepsilon')$ to $\mathcal{A}$. We then get that

$$\Psi(\varepsilon)+\Psi(\varepsilon')\left(\begin{pmatrix} a_1 & \dots & a_p \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}\right) = \begin{pmatrix} \xi_1 & \dots & \xi_n \end{pmatrix} A \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} + \begin{pmatrix} \xi'_1 & \dots & \xi'_{n'} \end{pmatrix} A' \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \mod \mathcal{J} + \mathcal{J}'$$

So $\Psi[\varepsilon + \varepsilon']$ and $\Psi[\varepsilon] + \Psi[\varepsilon']$ are represented by the same map

$$\mathcal{A} \longrightarrow (R, -)/(\mathcal{J} + \mathcal{J}' + \mathcal{B})$$

We clearly have that $\Psi(\mathrm{id}_R) = \mathrm{id}_{(R,-)}$ so we need only show that Ψ preserves multiplication to establish the claim. $[\varepsilon'] \cdot [\varepsilon]$ is represented by the map

$$\varepsilon^{-1}(\pi_{\mathfrak{b}}(\mathfrak{a}')) \xrightarrow{\varepsilon} \pi_{\mathfrak{b}}(\mathfrak{a}') \xrightarrow{\cong} \frac{\mathfrak{a}'}{\mathfrak{b} \cap \mathfrak{a}'} \xrightarrow{\overline{\varepsilon'}} \frac{R}{\pi_{\mathfrak{b}'}^{-1}(\varepsilon'(\mathfrak{b} \cap \mathfrak{a}'))}$$

which we call $\varepsilon' \cdot \varepsilon$. Since R is right coherent, $\ker \varepsilon$ and $\pi_{\mathfrak{b}}(\mathfrak{a}') \cap \varepsilon(\mathfrak{a})$ are finitely generated right R-modules. There is an exact sequence

$$0 \longrightarrow \ker \varepsilon \longrightarrow \varepsilon^{-1}(\pi_{\mathfrak{b}}(\mathfrak{a}')) \longrightarrow \pi_{\mathfrak{b}}(\mathfrak{a}') \cap \varepsilon(\mathfrak{a}) \longrightarrow 0$$

so that $\varepsilon^{-1}(\pi_{\mathfrak{b}}(\mathfrak{a}'))$ is a f.g. right ideal. Similarly, right coherence of R implies that $\mathfrak{b} \cap \mathfrak{a}'$ is f.g. which, in turn, implies $\varepsilon'(\mathfrak{b} \cap \mathfrak{a}')$ is f.g. The exact sequence

$$0 \longrightarrow \mathfrak{b} \longrightarrow \pi_{\mathfrak{b}'}^{-1}(\varepsilon'(\mathfrak{b} \cap \mathfrak{a}')) \longrightarrow \varepsilon'(\mathfrak{b} \cap \mathfrak{a}') \longrightarrow 0$$

then implies that the ideal $\pi_{\mathfrak{b}'}^{-1}(\varepsilon'(\mathfrak{b} \cap \mathfrak{a}'))$ is finitely generated.

Suppose that $\varepsilon^{-1}(\pi_{\mathfrak{b}}(\mathfrak{a}')) = (h_1, \dots, h_l)R$. Let $\mathcal{H}$ be the pp-functor

$$\exists z_1 \dots z_l \left(\begin{pmatrix} h_1 & \dots & h_l \end{pmatrix}\begin{pmatrix} z_1 \\ \vdots \\ z_l \end{pmatrix} = x\right)$$

Then $\Psi[\varepsilon' \cdot \varepsilon]$ is represented by a map

$$\mathcal{H} \longrightarrow (R, -)/\mathcal{E}$$

for some $\mathcal{E}$ (as in the definition of Ψ). Our argument does not require specific knowledge of $\mathcal{E}$.

Now $\varepsilon^{-1}(\pi_{\mathfrak{b}}(\mathfrak{a}')) \leq \mathfrak{a}$ so we can write

$$(h_1 \dots h_l) = (f_1 \dots f_n)H$$

for some matrix H. Suppose $\xi_i, \xi'_j, \eta_k \in R$ such that $\overline{\xi_i} = \varepsilon(f_i)$, $\overline{\xi'_j} = \varepsilon'(f'_j)$ and $\overline{\eta_k} = \varepsilon(h_k)$. Then, by applying ε to both sides of the above equation, there is matrix G such that

$$(\xi_1 \dots \xi_n)H = (\eta_1 \dots \eta_l) + (g_1 \dots g_m)G$$

Furthermore,

$$(\eta_1 \dots \eta_l) = (f'_1 \dots f'_{n'}\ g_1 \dots g_m) \begin{pmatrix} H' \\ H'' \end{pmatrix}$$

for some H' and H'' since ε maps each h_k into $\pi_{\mathfrak{b}}(\mathfrak{a}')$.

Now,

$$\begin{aligned}
(\varepsilon' \cdot \varepsilon)(h_1 \dots h_l) &= \overline{\varepsilon'}(\varepsilon(f_1 \dots f_n)H) \\
&= \overline{\varepsilon'}((\xi_1 \dots \xi_n)H \mod \mathfrak{b}) \\
&= \overline{\varepsilon'}((\eta_1 \dots \eta_l) \mod \mathfrak{b}) \\
&= \overline{\varepsilon'}\left((f'_1 \dots f'_{n'}\ g_1 \dots g_m) \begin{pmatrix} H' \\ H'' \end{pmatrix} \mod \mathfrak{b}\right) \\
&= \overline{\varepsilon'}((f'_1 \dots f'_{n'})H' \mod \mathfrak{b}) \\
&= (\xi'_1 \dots \xi'_{n'})H' \mod \pi_{\mathfrak{b}'}^{-1}(\varepsilon'(\mathfrak{b} \cap \mathfrak{a}'))
\end{aligned}$$

We can now write down the action of the canonical representative of $\Psi[\varepsilon' \cdot \varepsilon]$:

$$\sum h_i x_i \longmapsto \begin{pmatrix} \xi'_1 & \dots & \xi'_{n'} \end{pmatrix} H' \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} \mod \mathcal{E}$$

Let $\pi_{\mathcal{I}}$ and $\pi_{\mathcal{J}}$ be the natural projections $(R,-) \to (R,-)/\mathcal{I}$ and $(R,-) \to (R,-)/\mathcal{J}$ respectively. Let $\mathcal{H}'$ be the functor $\Psi(\mathfrak{a},\mathfrak{b})(\varepsilon)^{-1}(\pi_{\mathcal{J}}(\mathcal{I}'))$. Then $\mathcal{H}'$ can be expressed by the pp-formula

$$\exists y_1 \dots y_n, z_1 \dots z_{n'}, w_1 \dots w_m \left(\begin{pmatrix} f_1 & \dots & f_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x \qquad \wedge \right.$$

$$\left. \begin{pmatrix} \xi_1 & \dots & \xi_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} f'_1 & \dots & f'_{n'} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_{n'} \end{pmatrix} + \begin{pmatrix} g_1 & \dots & g_m \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} \right)$$

$\Psi[\varepsilon'] \cdot \Psi[\varepsilon]$ is represented by a map

$$\mathcal{H}' \longrightarrow (R,-)/\mathcal{E}'$$

where $\mathcal{E}'$ is $\pi_{\mathcal{J}'}^{-1}(\Psi(\mathfrak{a}',\mathfrak{b}')(\varepsilon')(\mathcal{J} \cap \mathcal{I}'))$. We claim that $\mathcal{H} \leq \mathcal{H}'$. Suppose that

$$x = \begin{pmatrix} h_1 & \dots & h_l \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} = \begin{pmatrix} f_1 & \dots & f_n \end{pmatrix} H \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix}$$

Put

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = H \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix}$$

Then

$$\begin{pmatrix} \xi_1 & \dots & \xi_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \xi_1 & \dots & \xi_n \end{pmatrix} H \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix}$$

$$= \begin{pmatrix} \eta_1 & \dots & \eta_l \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} + \begin{pmatrix} g_1 & \dots & g_m \end{pmatrix} G \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix}$$

$$= \begin{pmatrix} f'_1 & \dots & f'_{n'} & g_1 & \dots & g_m \end{pmatrix} \begin{pmatrix} H' \\ H'' \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} + \begin{pmatrix} g_1 & \dots & g_m \end{pmatrix} G \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix}$$

$$= \begin{pmatrix} f'_1 & \dots & f'_{n'} \end{pmatrix} H' \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} + \begin{pmatrix} g_1 & \dots & g_m \end{pmatrix} (H'' + G) \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix}$$

So by putting

$$\begin{pmatrix} z_1 \\ \vdots \\ z_{n'} \end{pmatrix} = H' \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} = (H'' + G) \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix}$$

we see that $x \in \mathcal{H}'$.

The action of the above representative of $\Psi[\varepsilon'] \cdot \Psi[\varepsilon]$ is easily calculated to be the following.

$$\sum f_i y_i \longmapsto \begin{pmatrix} \xi'_1 & \dots & \xi'_{n'} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_{n'} \end{pmatrix} \quad \text{mod } \mathcal{E}'$$

where

$$\begin{pmatrix} \xi_1 & \dots & \xi_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} f'_1 & \dots & f'_{n'} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_{n'} \end{pmatrix} + \begin{pmatrix} g_1 & \dots & g_m \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$$

for some $w_1, \dots, w_m$. Since $\mathcal{H} \leq \mathcal{H}'$, we can restrict this map to $\mathcal{H}$ to get

$$\begin{pmatrix} h_1 & \dots & h_l \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} \longmapsto \begin{pmatrix} \xi'_1 & \dots & \xi'_{n'} \end{pmatrix} H' \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} \quad \text{mod } \mathcal{E}'$$

since

$$\begin{pmatrix} h_1 & \dots & h_l \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} = \begin{pmatrix} f_1 & \dots & f_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

where

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = H \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix}$$

and (by the above computation)

$$\begin{aligned} & \begin{pmatrix} \xi_1 & \dots & \xi_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \\ = & \begin{pmatrix} f'_1 & \dots & f'_{n'} \end{pmatrix} H' \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} + \begin{pmatrix} g_1 & \dots & g_m \end{pmatrix} (H'' + G) \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} \end{aligned}$$

So $\Psi[\varepsilon' \cdot \varepsilon]$ and $\Psi[\varepsilon'] \cdot \Psi[\varepsilon]$ are represented by the same map

$$\mathcal{H} \longrightarrow (R, -)/(\mathcal{E} + \mathcal{E}')$$

This establishes the claim.

We will now produce an inverse Φ to Ψ. Let $\mathcal{I}$ and $\mathcal{J}$ be pp-subfunctors of $(R, -)$. Then, since R is right coherent, the sets $\mathcal{I}(R)$ and $\mathcal{J}(R)$ are finitely generated right ideals of R. Suppose now that $\mathcal{I}$ and $\mathcal{J}$ satisfy condition (2.5). So $D\mathcal{I}(I) = 0$ for all $I \in [M]$. Then $D\mathcal{I}(I) = \operatorname{ann}_I(\mathcal{I}(R)) = 0$ which implies that $\operatorname{Hom}(R/\mathcal{I}(R), I) = 0$ for all $I \in [M]$ so that $\mathcal{I}(R)$ is τ_M-dense. Similarly, we have $D\mathcal{J}(I) = I$ for all $I \in [M]$ so that $D\mathcal{J}(I) = \operatorname{ann}_I(\mathcal{J}(R)) = I$ which implies $\operatorname{Hom}(\mathcal{J}(R), I) = 0$. That is, $\mathcal{J}(R)$ is τ_M-torsion.

So the map $\Phi(\mathcal{I}, \mathcal{J}) : \operatorname{Nat}(\mathcal{I}, (R, -)/\mathcal{J}) \to \operatorname{Hom}(\mathcal{I}(R), R/\mathcal{J}(R))$ defined by

$$\theta \longmapsto \theta_R$$

induces, on taking directed colimits, a map

$$\Phi : \varinjlim_{i[M] \subseteq [dF]} \operatorname{End}((R, -)_{\tau_F}) \longrightarrow R_M$$

The map Φ is clearly a ring homomorphism.

It is obvious that $\Phi \circ \Psi = \mathrm{id}$. We need to check that $\Psi \circ \Phi$ is also the identity. Let $\mathfrak{I}$ and $\mathfrak{J}$ be pp-subfunctors of $(R, -)$ satisfying (2.5) and $\theta : \mathfrak{I} \to (R, -)/\mathfrak{J}$ be a natural transformation. So θ is definable. Suppose $\mathfrak{I}(R) = (f_1, \dots, f_n)R$, $\mathfrak{J}(R) = (g_1, \dots, g_m)R$ and $\theta_R(f_i) = \overline{\xi_i}$. Let

$$\mathfrak{I}' = \exists y_1 \dots y_n \left(\begin{pmatrix} f_1 & \dots & f_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x \right)$$

Then $\Psi(\Phi(\theta))$ is represented by the map $\theta' : \mathfrak{I}' \to (R, -)/\mathfrak{J}'$ (for some $\mathfrak{J}'$) defined by

$$\sum f_i x_i \longmapsto \sum \xi_i x_i \mod \mathfrak{J}'$$

Let

$$\mathfrak{J}'' = \exists z_1 \dots z_n \left(\sum f_i z_i \in \mathfrak{I} \ \wedge \ y = \sum \xi_i z_i \ - \ \theta(\sum f_i z_i) \right)$$

Then $\mathfrak{J}''$ is pp-definable since both θ and $\mathfrak{I}$ are and $\mathfrak{J}''(R) = 0$. We have $D\mathfrak{J}''(I) = I$ for all $I \in [M]$ since

$$D\mathfrak{J}''(I) = \mathrm{ann}_I(\mathfrak{J}''(R)) = \mathrm{ann}_I(0) = I$$

So θ and θ' are both equal as maps

$$\mathfrak{I} \cap \mathfrak{I}' \longrightarrow (R, -)/(\mathfrak{J} + \mathfrak{J}' + \mathfrak{J}'')$$

So Ψ and Φ are inverse to each other. They are clearly both natural in $[M]$ and the result is proved. $\square$

This theorem shows, in particular, that the presheaf of definable scalars over a right coherent ring R in some sense generalises the presheaf of localisations of R at finite type hereditary torsion theories on **Mod**-R.

Chapter 3

Imaginaries as coherent sheaves

3.1 Overview

In this chapter we will extend Burke's result (Proposition 2.2.7) and its localised version (Theorem 2.3.14) to general locally finitely presented categories of first order structures. That is, we extend to the non-additive case. More precisely, we replace $\mathbf{Mod}\text{-}R$ by the category of models of a *finite limit theory* T_0. This is a theory axiomatised by sentences of the form $\forall \boldsymbol{x}(\varphi(\boldsymbol{x}) \rightarrow \exists! \boldsymbol{y} \psi(\boldsymbol{x}, \boldsymbol{y}))$ where φ and ψ are conjunctions of atomic formulas. We could equivalently replace $\mathbf{Mod}\text{-}R$ by any LFP category since, by [1, Thm 5.9], a category is LFP if and only if it is the category of models for a finite limit theory in some finitary first-order langauge. Let us take $\mathbf{C}$ to be the category $\operatorname{Mod} T_0$ of models of T_0. The nonadditive analogue of the abelian functor category $(\mathbf{mod}\text{-}R, \mathbf{Ab})$ is the "presheaf topos" $(\operatorname{fp} \mathbf{C}, \mathbf{Set})$, the category of functors $\operatorname{fp} \mathbf{C} \rightarrow \mathbf{Set}$. Any positive existential (pe) formula (that is, a formula built up from atomic formulas using conjunction, disjunction and existential quantification, or equivalently, a finite disjunction of pp-formulas) defines a functor $\mathbf{C} \rightarrow \mathbf{Set}$ and hence an object of $(\operatorname{fp} \mathbf{C}, \mathbf{Set})$. It turns out that in this situation these formulas play the role of pp-formulas in the modules case. The analogue of Burke's result in this context is the following.

Theorem 3.2.12. *The category* $\operatorname{coh}(\operatorname{fp} \mathbf{C}, \mathbf{Set})$ *of coherent functors is equivalent to* *the category with objects φ/θ where φ and θ are pe-formulas and θ defines an equivalence relation on φ, and with maps the pe-definable functions $\varphi/\theta \to \psi/\eta$.*

To obtain a localised version of this result we need to replace the notion of a definable subcategory $\mathbf{D} \subseteq \mathbf{Mod}\text{-}R$ by a category of models of a *coherent theory* T extending T_0. By this we mean that T is of the form $T_0 \cup \{\varphi_\lambda \to \psi_\lambda\}_\lambda$ where this time φ_λ and ψ_λ are positive existential rather than just pp-formulas. Clearly, a definable subcategory of modules is a special case of a category of the form $\operatorname{Mod} T$ for a coherent theory T. Following the terminology for modules, we will refer to such categories as *definable subcategories* of $\mathbf{C}$. In the modules case, given a definable subcategory $\mathbf{D} \subseteq \mathbf{Mod}\text{-}R$, we obtained a finite type Gabriel topology on $\mathbf{mod}\text{-}R$ by defining a left ideal $\mathcal{I}$ of an object $A \in \mathbf{mod}\text{-}R$ to be dense if $\mathcal{I}(D) = \operatorname{Hom}(A, D)$ for all modules $D \in \mathbf{D}$ (see (2.3)). Here we used the fact that we can extend additive functors canonically from $\mathbf{mod}\text{-}R$ to $\mathbf{Mod}\text{-}R$. But we can also canonically extend a set-valued functor on $\operatorname{fp}\mathbf{C}$ to a functor on $\mathbf{C}$ in much the same way (Lemma 3.3.1). Now starting with a definable subcategory $\mathbf{D} \subseteq \mathbf{C} = \operatorname{Mod} T_0$, we get a *finite type Grothendieck topology* on $\operatorname{fp}\mathbf{C}$ in exactly the same way as we obtained a Gabriel topology in the case of modules. We define a subfunctor $S \subseteq \operatorname{Hom}(C, -)$ to *cover* the object $C \in \operatorname{fp}\mathbf{C}$ if $S(D) = \operatorname{Hom}(C, D)$ for all $D \in \mathbf{D}$. We denote the induced finite type Grothendieck topology by $J_{\mathbf{D}}$. The nonadditive analogue of the localisation of $(\mathbf{mod}\text{-}R, \mathbf{Ab})$ at the torsion theory induced by the definable subcategoy $\mathbf{D} \subseteq \mathbf{Mod}\text{-}R$ is the category of *sheaves* for the Grothendieck topology $J_{\mathbf{D}}$, denoted $\operatorname{Sh}(\operatorname{fp}\mathbf{C}, J_{\mathbf{D}})$. The analogue of the localised version of Burke's result is the following.

Theorem 3.3.21. *Let $\mathbf{D} \subseteq \mathbf{C}$ be definable. The category $\operatorname{coh}(\operatorname{Sh}(\operatorname{fp}\mathbf{C}, J_{\mathbf{D}}))$ has as objects pairs of pe-formulas φ/θ where θ defines an equivalence relation on φ on all objects of $\mathbf{D}$. The maps $\varphi/\theta \to \psi/\eta$ are given by pe-formulas ρ which define actual functions $\varphi/\theta \to \psi/\eta$ when restricted to objects of $\mathbf{D}$.*

This result was first proved by Makkai and Reyes in [26, p. 269]. We prove it using a new argument which is close to the argument for modules in its methods and motivations. In contrast, Makkai and Reyes use more topos-theoretic methods which are embedded in a language which may be hard to understand by model-theorists (to whom this thesis is primarily addressed). Our proof establishes characterisations of f.g. and f.p. sheaves along the way. In particular, we see that, unlike the modules case, there are finitely presented sheaves which are not coherent.

We end this chapter by suggesting a categorical definition of the model-theoretic notion of an interpretation as a certain type of functor between categories of coherent sheaves.

Throughout this chapter "definable" means definable without parameters. We will use the term "presheaf" for covariant functors whereas the term is usually reserved for contravariant functors. Likewise, the notions of Grothedieck topology and sheaf are normally defined with arrows going the other way round. So our notion of "sheaf" might be more correctly named a "cosheaf". As long as the reader bears this in mind, no confusion should arise.

3.2 Finitely generated, finitely presented and coherent functors

3.2.1 Functor categories and toposes

The "right" context for module categories and additive functor categories is that of locally finitely presented abelian categories. The context for this chapter is that of locally finitely presented *toposes*. A *Grothendieck topos* is a category of *sheaves* for a *Grothendieck topology* (which will be defined in 3.3.2) on a small category. Roughly, given a small category $\mathbf{A}$, a Grothendieck topology on $\mathbf{A}$ induces a notion of *dense subfunctor* in the category $(\mathbf{A}, \mathbf{Set})$ of set-valued functors. The category of sheaves can be regarded as a *localisation* of $(\mathbf{A}, \mathbf{Set})$ obtained by inverting all maps $\gamma : F \to G$ such that $\operatorname{im}\gamma$ is dense in G and the equaliser of

$$F \times_G F \rightrightarrows F$$

is dense in $F \times_G F$. As in the additive case, this localisation can be instantiated as a full reflexive subcategory of $(\mathbf{A}, \mathbf{Set})$. The abelian analogue of a Grothendieck topos is a Grothendieck abelian category. Recall that this is a cocomplete abelian category with a generator in which directed colimits are exact. Any LFP abelian category is Grothendieck (see 2.3.1). This definition of a Grothendieck abelian category does not refer to localisations of functor categories but uses instead certain exactness and completeness conditions. However, we could have defined a Grothendieck abelian category as a localisation of an (additive) functor category $(\mathbf{A}, \mathbf{Ab})$ (where $\mathbf{A}$ is preadditive). This is the content of the Gabriel-Popescu Theorem (see [35, thm 4.1]). Conversely, we could define a Grothendieck topos as a category with certain exactness and completeness properties. This is the content of Giraud's Theorem (see the appendix of [25]).

A topos generalises the category $\mathbf{Set}$ of sets and functions in the same way that an abelian category generalises the category $\mathbf{Ab}$ of abelian groups. We will be concerned mainly with LFP toposes. One of the simplest examples of an LFP topos is a category of the form $(\mathbf{A}, \mathbf{Set})$ where $\mathbf{A}$ is any (skeletally) small category. This category has a canonical strong generating set of f.p. objects, namely the set of representable functors $\{\mathrm{Hom}(A, -) : A \in \mathbf{A}\}$. In any topos, a morphism which is both epic and monic must be invertible ([25, IV, Prop 2.2]). So, by Proposition 2.2.1, we have the following characterisation of f.g. and f.p. functors.

Corollary 3.2.1. *Let $\mathbf{A}$ be a small category and let F be a set-valued functor on $\mathbf{A}$.*

(a) *F is finitely generated if and only if there are objects $A_1, \ldots, A_n \in \mathbf{A}$ and an epimorphism*

$$\coprod_{i=1}^{n} \mathrm{Hom}(A_i, -) \twoheadrightarrow F$$

(b) *F is finitely presented if and only if there are objects $A_1, \ldots, A_n, B_1, \ldots, B_m \in \mathbf{A}$ and a coequaliser diagram*

$$\coprod_{i=1}^{n} \mathrm{Hom}(A_i, -) \rightrightarrows \coprod_{i=1}^{m} \mathrm{Hom}(B_i, -) \longrightarrow F$$

3.2.2 Functors on finitely presented models

We begin with a locally finitely presented category $\mathbf{C}$. For instance, the category of modules over a ring, the category of groups, or of rings, *et cetera*. In particular, if $\mathcal{L}$ is a (finitary, but possibly many-sorted) first-order language then the category of $\mathcal{L}$-structures is such a category, as is the category of models of any finite limit theory in $\mathcal{L}$ (meaning a theory axiomatised by sentences of the form $\forall \boldsymbol{x}(\varphi(\boldsymbol{x}) \to \exists! \boldsymbol{y}\, \psi(\boldsymbol{x}, \boldsymbol{y}))$ where φ, ψ are $\bigwedge$-atomic). Indeed (see [1, 5.9]), conversely, every locally finitely presented category can be regarded as the category of models for a finite limit theory in a suitable language $\mathcal{L}$. Choose and fix such $\mathcal{L}$.

It may be that the category of definable sets in objects of $\mathbf{C}$ already has disjoint unions in the sense that if U, V are definable sets then there is a definable set W (in some sort) and definable injections $f : U \to W$, $g : V \to W$ such that W is the disjoint union of the images of f and g. If not, then we use the following device. Expand the language, to $\mathcal{L}'$, by adding one new sort and two constants of this new sort $\{c_0, c_1\}$. Add the axiom $c_0 \neq c_1$ and the sentence expressing that there are exactly two elements of the new sort. Variables of the new sort will be denoted by $x'_1, x'_2, \ldots$ etc. This will ensure that we can form definable disjoint unions of (finitely many) definable sets. Let $\mathbf{C}'$ be the category (with the usual homomorphisms) of $\mathcal{L}'$-structures satisfying these sentences and axioms for $\mathbf{C}$ (or just the category $\mathbf{C}$ if we did not add the extra sort). Then $\mathbf{C}'$ is an LFP category with a skeletally small class $\mathrm{fp}\,\mathbf{C}'$ of finitely presented objects (see [1, p. 201]) and these have the form $M \cup \{c_0, c_1\}$ where M is a f.p. object of $\mathbf{C}$. By, e.g. [1, p. 201], an object of $\mathbf{C}'$ is f.p. if it is isomorphic to a quotient of the term algebra on finitely many variables by finitely many term equations and is such that the interpretations of the relation symbols have just finitely many "edges". From now on we write $\mathcal{L}$ for $\mathcal{L}'$, $\mathbf{C}$ for $\mathbf{C}'$ and assume that the language was added to as above if this was necessary. *Throughout this chapter*, $\mathbf{C}$ have this fixed meaning.

A crucial observation in what follows is that a positive existential formula, being preserved by homomorphisms, defines a functor $\mathbf{C} \to \mathbf{Set}$ by associating to each object $C \in \mathbf{C}$ the set it defines and with the obvious action on morphisms. In particular each positive existential formula φ is identified with an element of the functor category $(\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$ (we use the same notation for the formula and the functor).

It is worth singling out the following two properties of the category $\mathbf{C}$. We say that a tuple $\boldsymbol{a}$ of elements of $A \in \mathbf{C}$ *generates* A if for every $b \in A$ there is a term $t(\boldsymbol{x})$ of $\mathfrak{L}$ such that $b = t(\boldsymbol{a})$. Proofs of these properties are easily extracted from [1, Sec. 5A].

FR If $\varphi(\boldsymbol{x})$ is a finite conjunction of atomic formulas, then there is an object $A \in \mathrm{fp}(\mathbf{C})$ and a tuple of generators $\boldsymbol{a} \in \varphi(A)$ for A with the property that if B is any object and $\boldsymbol{b} \in \varphi(B)$ then the map $\boldsymbol{a} \mapsto \boldsymbol{b}$ extends to a morphism $A \to B$. Following the terminology used (for a more general notion) in the additive case ([31, Sec. 8.3]), we say that the pair $(A, \boldsymbol{a})$ is a *free realisation* of φ.

PF If $C \in \mathrm{fp}(\mathbf{C})$ and $\boldsymbol{c}$ is a tuple of generators for C, then there is a finite conjunction of atomic formulas $\varphi(\boldsymbol{x})$ with $\boldsymbol{c} \in \varphi(C)$ such that whenever $D \in \mathbf{C}$ and $\boldsymbol{d} \in \varphi(D)$, the map $\boldsymbol{c} \mapsto \boldsymbol{d}$ extends to a morphism $C \to D$. Such a formula is called a *presentation formula* for C.

Note that FR implies that any finite conjunction of atomic formulas defines a representable functor. Conversely, PF implies that any representable functor is isomorphic to a functor defined by a finite conjunction of atomic formulas.

Let φ be a pe-formula and R a subfunctor of $\varphi \times \varphi$ in $(\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$. Suppose $C \in \mathrm{fp}\,\mathbf{C}$ and $\boldsymbol{a}, \boldsymbol{b}$ are tuples from $\varphi(C)$. Then we write $R(\boldsymbol{a}, \boldsymbol{b})$ when $(\boldsymbol{a}, \boldsymbol{b}) \in R(C)$. If for every $C \in \mathrm{fp}\,\mathbf{C}$ it is the case that $R(C)$ is an equivalence relation on $\varphi(C)$, then we say that R is a *functorial equivalence relation* on φ. We write φ/R for the functor which associates to $C \in \mathrm{fp}\,\mathbf{C}$ the set of equivalence classes of $\varphi(C)$ modulo $R(C)$.

Proposition 3.2.2. *Let F be a functor in* $(\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$.

(a) *Suppose that F is finitely generated. Then there is a positive quantifier-free $\mathfrak{L}$-formula $\varphi(\boldsymbol{x})$ and a functorial equivalence relation $E(\boldsymbol{x},\boldsymbol{y})$ such that*

$$F \cong \frac{\varphi}{E}$$

(b) *Suppose that F is finitely presented. Then there is a positive quantifier-free $\mathfrak{L}$-formula $\varphi(\boldsymbol{x})$ and a positive existential formula $\theta(\boldsymbol{x},\boldsymbol{y})$ such that*

$$F \cong \frac{\varphi}{\tilde{\theta}}$$

where $\tilde{\theta}$ denotes the equivalence relation generated by θ.

Proof. (a) Suppose F is finitely generated. By Corollary 3.2.1, there are $D_1,\ldots,D_m \in \operatorname{fp}\mathbf{C}$ and an epimorphism

$$\coprod_j \operatorname{Hom}(D_j,-) \twoheadrightarrow F$$

Let φ_j be a presentation formula for D_j and assume without loss of generality that the φ_j all have the same number of free variables. For if the largest number of free variables among the φ_j is l and $\varphi_j = \varphi_j(x_1,\ldots,x_r)$ where $r<l$, then simply add to φ_j a conjunction of equations $x_{r+1}=x_1 \wedge \cdots \wedge x_l = x_1$.

Now let $\varphi'_j(x_1,\ldots,x_l,x'_1,\ldots,x'_m)$ be the formula

$$\varphi_j(x_1,\ldots,x_l) \wedge x'_1 = c_0 \wedge \cdots \wedge x'_j = c_1 \wedge \cdots \wedge x'_m = c_0$$

so that $\varphi = \varphi'_1 \vee \cdots \vee \varphi'_m$ defines the disjoint union of the sets defined by $\varphi_1,\ldots,\varphi_m$. In particular, $\varphi \cong \coprod_{j=1}^m \varphi_j$ as a coproduct of functors (note that, in the functor category, coproduct is given pointwise by disjoint union) and we have an epimorphism

$$\varphi \overset{p}{\twoheadrightarrow} F$$

Let $R \subseteq \varphi \times \varphi$ be the functor given by

$$M \longmapsto \{(\boldsymbol{x},\boldsymbol{y}) \in \varphi(M)^2 : p_M(\boldsymbol{x}) = p_M(\boldsymbol{y})\}$$

and let $E(\boldsymbol{x},\boldsymbol{y})$ be the equivalence relation generated by R. Then $F \cong \varphi/E$ as required.

(b) Now suppose that F is f.p., so there is a presentation for F of the form

$$\coprod_{i=1}^{n} \mathrm{Hom}(C_i, -) \underset{q}{\overset{p}{\rightrightarrows}} \coprod_{j=1}^{m} \mathrm{Hom}(D_j, -) \longrightarrow F$$

We can write this as

$$\coprod_{i=1}^{n} \psi_i \underset{q}{\overset{p}{\rightrightarrows}} \varphi \longrightarrow F$$

where $\varphi \cong \coprod_j \varphi_j$ as in part (a) and ψ_i is a presentation formula for C_i.

Since p and q are natural transformations, their components p_i and q_i are determined by their actions at C_i on the identity map id_{C_i}. So if k is such that $(p_i)_{C_i}(\mathrm{id}_{C_i}) \in \mathrm{Hom}(D_k, C_i) \simeq \varphi_k(C_i)$ then p_i is a natural transformation $\psi_i \to \varphi_k$. So since p_i is a natural transformation between two representable functors, it must be induced by a map $f : D_k \to C_i$. In other words $p_i = f^*$ (the latter is the notation we will use for $\mathrm{Hom}(f, -)$).

Let $\boldsymbol{c}$ be a tuple of generators for C_i and $\boldsymbol{d}$ a tuple of generators for D_k. Let $E \in \mathrm{fp}(\mathbf{C})$ and consider the component

$$\psi_i(E) \xrightarrow{p_i} \varphi_k(E)$$

For each d_j from the tuple $\boldsymbol{d}$, there is a term t_j such that $f(d_j) = t_j(\boldsymbol{c})$. Let $\boldsymbol{t}(\boldsymbol{c})$ be the tuple whose jth component is $t_j(\boldsymbol{c})$. Then the action of $(p_i)_E$ on a tuple $\boldsymbol{e} \in \psi_i(E)$ is given simply by

$$\boldsymbol{e} \longmapsto \boldsymbol{t}(\boldsymbol{e})$$

So we see that p_i is a definable map (that is, its graph is, uniformly over $\mathbf{C}$, a definable subset of the product of its domain and codomain). Clearly p_i can be regarded as a definable map into φ'_k by extending the tuple $\boldsymbol{t}(\boldsymbol{x})$ in the obvious way.

Let $\theta(\boldsymbol{x}, \boldsymbol{y})$ be the relation on φ which holds when there is an element $\boldsymbol{a} \in \coprod_{i=1}^{n} \psi_i$ such that $p(\boldsymbol{a}) = \boldsymbol{x}$ and $q(\boldsymbol{a}) = \boldsymbol{y}$. This is expressible by the formula

$$\exists \boldsymbol{z} (\bigvee_{i=1}^{n} (\psi_i(\boldsymbol{z}) \wedge p_i(\boldsymbol{z}) = \boldsymbol{x} \wedge q_i(\boldsymbol{z}) = \boldsymbol{y}))$$

It should be clear that

$$F \cong \frac{\varphi}{\tilde{\theta}}$$

as required. □

In the additive case the corresponding results, see [9, 3.2.5], [31, Chpt. 12], require only positive primitive formulas. (Recall these are the formulas built up from atomic formulas using just conjunction and existential quantification.) To obtain a converse to the above proposition we will need the following lemmas.

Lemma 3.2.3. *Any positive primitive $\mathfrak{L}$-formula defines a finitely presented functor.*

Proof. Let $\varphi(\boldsymbol{x})$ be the positive primitive formula $\exists \boldsymbol{y} \theta(\boldsymbol{x}, \boldsymbol{y})$ where θ is a conjunction of atomic formulas. Let $(C, \boldsymbol{c}, \boldsymbol{d})$ be a free realisation of θ. Clearly C has the property that if E is any other object with $\boldsymbol{e} \in \varphi(E)$, there is a map $f : C \to E$ with $f(\boldsymbol{c}) = \boldsymbol{e}$. So the map

$$\mathrm{Hom}(C, -) \longrightarrow \varphi$$

defined by

$$f \longmapsto f(\boldsymbol{c})$$

is epic. Hence φ is finitely generated.

Now let $\psi(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2)$ be the formula $\theta(\boldsymbol{x}, \boldsymbol{y}_1) \wedge \theta(\boldsymbol{x}, \boldsymbol{y}_2)$. As a finite conjunction of atomic formulas, ψ is representable, hence finitely presented. Define the map $p_i : \psi \to \mathrm{Hom}(C, -)$ to be that whose component at E takes $(\boldsymbol{a}, \boldsymbol{b}_1, \boldsymbol{b}_2) \in \psi(E)$ to the map defined by $(\boldsymbol{c}, \boldsymbol{d}) \mapsto (\boldsymbol{a}, \boldsymbol{b}_i)$. Then the diagram

$$\psi \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathrm{Hom}(C, -) \longrightarrow \varphi$$

is a coequaliser. Since ψ and $\mathrm{Hom}(C, -)$ are f.p., φ is also finitely presented. $\square$

Lemma 3.2.4. *Any positive existential formula defines a finitely presented functor.*

Proof. Any positive existential formula φ is a disjunction of positive primitive formulas. We proceed by induction on the number n of disjuncts. If $n = 1$ then φ is positive primitive and so finitely presented by the above. Now suppose $n > 1$. Write $\varphi = \bigvee_{i=1}^{n} \varphi_i$ where each φ_i is positive primitive. The following diagram, with the obvious maps from the conjunction to the union, is a coequaliser.

$$(\bigvee_{i=1}^{n-1} \varphi_i) \wedge \varphi_n \rightrightarrows (x' = c_0 \wedge \bigvee_{i=1}^{n-1} \varphi_i) \vee (x' = c_1 \wedge \varphi_n) \longrightarrow \varphi$$

Both $\bigvee_{i=1}^{n-1}\varphi_i$ and $(\bigvee_{i=1}^{n-1}\varphi_i)\wedge\varphi_n$ have $n-1$ disjuncts so the induction hypothesis applies. It now easily follows that φ is finitely presented since the functor in the middle is a coproduct of f.p. functors. $\square$

Proposition 3.2.5.

(a) *Any functor of the form φ/E where φ is positive existential and E is a functorial equivalence relation on φ is finitely generated.*

(b) *Any functor of the form $\varphi/\tilde{\theta}$ where φ is a positive existential formula and θ is a positive existential relation on φ is finitely presented.*

Proof. (a) There is a natural epimorphism $\varphi \twoheadrightarrow \varphi/E$. Hence φ/E is f.g.

(b) Consider the diagram

$$\theta \rightrightarrows \varphi \longrightarrow \varphi/\tilde{\theta}$$

where the maps are the obvious ones. This is clearly a coequaliser. So $\varphi/\tilde{\theta}$ is finitely presented as required. $\square$

Combining Propositions 3.2.2 and 3.2.5 we get the following corollary (recall that $\mathbf{C}$ is the LFP category of models of a finite limit theory).

Corollary 3.2.6. *Let F be a functor in* $(\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$.

(a) *F is finitely generated if and only if $F \cong \varphi/E$ where φ is positive existential and E is a functorial equivalence relation.*

(b) *F is finitely presented if and only if it is of the form $\varphi/\tilde{\theta}$ where φ and θ are positive existential.*

We will now characterise the natural transformations between f.g. functors in terms of positive existential formulas. Let φ/E_1 and ψ/E_2 be two f.g. functors. A positive existential formula $\rho(\boldsymbol{x},\boldsymbol{y})$ clearly defines a natural transformation $\varphi/E_1 \to \psi/E_2$ if the following are satisfied for each $A \in \mathrm{fp}\,\mathbf{C}$.

- For every $\boldsymbol{a} \in \varphi(A)$ there is a $\boldsymbol{b} \in \psi(A)$ with $\rho(\boldsymbol{a},\boldsymbol{b})$.
- If $\boldsymbol{a},\boldsymbol{a}' \in \varphi(A)$ are such that $E_1(\boldsymbol{a},\boldsymbol{a}')$ and for some tuples $\boldsymbol{b},\boldsymbol{b}'$ we have $\rho(\boldsymbol{a},\boldsymbol{b})$ and $\rho(\boldsymbol{a}',\boldsymbol{b}')$, then $\boldsymbol{b},\boldsymbol{b}' \in \psi(A)$ and $E_2(\boldsymbol{b},\boldsymbol{b}')$.

As ρ is a natural transformation, ρ_A takes an equivalence class $[\boldsymbol{a}]_{E_1}$ to the equivalence class $[\boldsymbol{b}]_{E_2}$ where $\rho(\boldsymbol{a}, \boldsymbol{b})$ holds.

Proposition 3.2.7. *Every natural transformation in* $\mathrm{fg}(\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$*, the full subcategory of finitely generated functors, is defined by a positive existential formula.*

Proof. Let $\alpha : \varphi/E_1 \to \psi/E_2$ be a natural transformation in $\mathrm{fg}(\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$. The positive existential formula φ will be of the form $\bigvee_i \exists \boldsymbol{z} \sigma_i(\boldsymbol{x}, \boldsymbol{z})$ where σ_i is a conjunction of atomic formulas. Let $(A_i, \boldsymbol{a}_i, \boldsymbol{b}_i)$ be a free realisation of σ_i. For each i choose a representative $\boldsymbol{a}'_i$ for the E_2-equivalence class $\alpha_{A_i}[\boldsymbol{a}_i]_{E_1}$. Since $(\boldsymbol{a}_i, \boldsymbol{b}_i)$ generates A_i, there will be a tuple $\boldsymbol{t}_i$ of terms such that $\boldsymbol{t}_i(\boldsymbol{a}_i, \boldsymbol{b}_i) = \boldsymbol{a}'_i$. Let $\rho(\boldsymbol{x}, \boldsymbol{y})$ be the formula

$$\bigvee_i \exists \boldsymbol{z}(\sigma_i(\boldsymbol{x}, \boldsymbol{z}) \wedge \boldsymbol{t}_i(\boldsymbol{x}, \boldsymbol{z}) = \boldsymbol{y})$$

Suppose that $\boldsymbol{c}, \boldsymbol{c}'$ are tuples from an object C and that $\rho(\boldsymbol{c}, \boldsymbol{c}')$ holds. Then there is an i and a tuple $\boldsymbol{d}$ such that $\sigma_i(\boldsymbol{c}, \boldsymbol{d})$ and $\boldsymbol{t}_i(\boldsymbol{c}, \boldsymbol{d}) = \boldsymbol{c}'$. So there is a map $f : A_i \to C$ defined by $(\boldsymbol{a}_i, \boldsymbol{b}_i) \mapsto (\boldsymbol{c}, \boldsymbol{d})$. Since α is a natural transformation, the following diagram commutes, where f_* denotes the obvious induced map.

$$\begin{tikzcd}
A_i \arrow[d, "f"'] & (\boldsymbol{a}_i, \boldsymbol{b}_i) \arrow[d, mapsto] \\
C & (\boldsymbol{c}, \boldsymbol{d})
\end{tikzcd}
\qquad
\begin{tikzcd}
\frac{\varphi}{E_1}(A_i) \arrow[r, "\alpha_{A_i}"] \arrow[d, "f_*"'] & \frac{\psi}{E_2}(A_i) \arrow[d, "f_*"] \\
\frac{\varphi}{E_1}(C) \arrow[r, "\alpha_C"'] & \frac{\psi}{E_2}(C)
\end{tikzcd}$$

Clearly $\boldsymbol{c} \in \varphi(C)$ so we can investigate the action of α_C on the equivalence class

$[\boldsymbol{c}]_{E_1}$.

$$\begin{aligned}\alpha_C[\boldsymbol{c}]_{E_1} &= \alpha_C[f(\boldsymbol{a}_i)]_{E_1}\\ &= \alpha_C f_*[\boldsymbol{a}_i]_{E_1}\\ &= f_*\alpha_{A_i}[\boldsymbol{a}_i]_{E_1}\\ &= f_*[\boldsymbol{a}_i']_{E_2}\\ &= f_*[\boldsymbol{t}_i(\boldsymbol{a}_i,\boldsymbol{b}_i)]_{E_2}\\ &= [f(\boldsymbol{t}_i(\boldsymbol{a}_i,\boldsymbol{b}_i))]_{E_2}\\ &= [\boldsymbol{t}_i(f(\boldsymbol{a}_i,\boldsymbol{b}_i))]_{E_2}\\ &= [\boldsymbol{t}_i(\boldsymbol{c},\boldsymbol{d})]_{E_2}\\ &= [\boldsymbol{c}']_{E_2}\end{aligned}$$

So we see that if $\rho(\boldsymbol{c},\boldsymbol{c}')$ holds in C, then $\alpha_C[\boldsymbol{c}]_{E_1} = [\boldsymbol{c}']_{E_2}$. We now check that ρ defines the natural transformation α.

Let $\boldsymbol{a} \in \varphi(A)$. Then there is an i and a tuple $\boldsymbol{b}$ such that $\sigma_i(\boldsymbol{a},\boldsymbol{b})$. Clearly $\rho(\boldsymbol{a},\boldsymbol{t}_i(\boldsymbol{a},\boldsymbol{b}))$ holds. By above, this implies that $\alpha_A[\boldsymbol{a}]_{E_1} = [\boldsymbol{t}_i(\boldsymbol{a},\boldsymbol{b})]_{E_2}$ so we must have, in particular, that $\boldsymbol{t}_i(\boldsymbol{a},\boldsymbol{b}) \in \psi(A)$. Now suppose that $E_1(\boldsymbol{a},\boldsymbol{a}')$, $\rho(\boldsymbol{a},\boldsymbol{b})$ and $\rho(\boldsymbol{a}',\boldsymbol{b}')$. Then $[\boldsymbol{b}]_{E_2} = \alpha_A[\boldsymbol{a}]_{E_1} = \alpha_A[\boldsymbol{a}']_{E_1} = [\boldsymbol{b}']_{E_2}$. So we have $\boldsymbol{b},\boldsymbol{b}' \in \psi(A)$ and $E_2(\boldsymbol{b},\boldsymbol{b}')$. Hence ρ defines a natural transformation $\varphi/E_1 \to \psi/E_2$ which is, therefore, α. $\square$

3.2.3 Local coherence of the functor category

We will show that our functor category $(\mathrm{fp}\,\mathbf{C},\mathbf{Set})$ is a locally coherent category in the sense that every finitely generated subobject of a finitely presented object is finitely presented. In the abelian context, but not here, this is equivalent to the class of f.p. functors being closed under finite limits. For, we will see that our f.p. functors are closed under finite products but need not be closed under equalisers.

Proposition 3.2.8. *Any finitely generated subfunctor of a finitely presented functor is itself finitely presented.*

Proof. Let F be f.p. and $G \subseteq F$ be a f.g. subfunctor. By Corollary 3.2.6 above $F \cong \varphi/\tilde{\theta}$ for pe-formulas φ and θ, and $G \cong \psi/E$ for a pe-formula ψ. By Proposition 3.2.7 the embedding of G in F is defined by a pe-formula ρ:

$$\psi/E \overset{\rho}{\rightarrowtail} \varphi/\tilde{\theta}$$

The functor G is isomorphic to the following subfunctor of F:

$$\frac{\exists \boldsymbol{z}(\psi(\boldsymbol{z}) \wedge \rho(\boldsymbol{z}, \boldsymbol{x}))}{\tilde{\theta}}$$

Which is finitely presented. $\square$

Corollary 3.2.9. *Any finitely generated subfunctor of a representable functor is defined by a pe-formula.*

Proposition 3.2.10. *The category* $\mathrm{fp}(\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$ *is closed under finite products but need not be closed under equalisers.*

Proof. Given a positive existential relation $\theta(\boldsymbol{x}, \boldsymbol{y})$, define $\theta_s(\boldsymbol{x}, \boldsymbol{y})$ to be the positive existential relation

$$\theta(\boldsymbol{x}, \boldsymbol{y}) \vee \theta(\boldsymbol{y}, \boldsymbol{x}) \vee \boldsymbol{x} = \boldsymbol{y}$$

Then θ_s is the symmetric and reflexive closure of θ and $\tilde{\theta} = \tilde{\theta}_s$.

Let $\varphi/\tilde{\theta}$ and $\psi/\tilde{\eta}$ be finitely presented functors. We claim that

$$\frac{\varphi(\boldsymbol{x})}{\tilde{\theta}(\boldsymbol{x}, \boldsymbol{x}')} \times \frac{\psi(\boldsymbol{y})}{\tilde{\eta}(\boldsymbol{y}, \boldsymbol{y}')} \cong \frac{\varphi(\boldsymbol{x}) \wedge \psi(\boldsymbol{y})}{\widetilde{\theta_s \wedge \eta_s}}$$

To see this, suppose that $([\boldsymbol{x}], [\boldsymbol{y}]) = ([\boldsymbol{x}'], [\boldsymbol{y}'])$ in the product. Then $\tilde{\theta}(\boldsymbol{x}, \boldsymbol{x}')$ and $\tilde{\eta}(\boldsymbol{y}, \boldsymbol{y}')$. So there is a sequence $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n \in \varphi$ and a sequence $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_n \in \psi$ with $\theta_s(\boldsymbol{x}, \boldsymbol{x}_0) \wedge \cdots \wedge \theta_s(\boldsymbol{x}_n, \boldsymbol{x}')$ and $\eta_s(\boldsymbol{y}, \boldsymbol{y}_0) \wedge \cdots \wedge \eta_s(\boldsymbol{y}_n, \boldsymbol{y}')$. The claim follows.

Consider, with notation as above, an equaliser diagram

$$E \longrightarrow \varphi/\tilde{\theta} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \psi/\tilde{\eta}$$

The natural transformations p_1 and p_2 can be expressed by positive existential formulas ρ_1 and ρ_2 respectively (by Proposition 3.2.7). We then have

$$E = \{[\boldsymbol{x}] \in \varphi/\tilde{\theta} : \exists \boldsymbol{y}, \boldsymbol{y}' \rho_1(\boldsymbol{x}, \boldsymbol{y}) \wedge \rho_2(\boldsymbol{x}, \boldsymbol{y}') \wedge \tilde{\eta}(\boldsymbol{y}, \boldsymbol{y}')\}$$

So E is the functor given by

$$\frac{\varphi(\boldsymbol{x}) \wedge \exists \boldsymbol{y}, \boldsymbol{y}'(\rho_1(\boldsymbol{x}, \boldsymbol{y}) \wedge \rho_2(\boldsymbol{x}, \boldsymbol{y}') \wedge \tilde{\eta}(\boldsymbol{y}, \boldsymbol{y}'))}{\tilde{\theta}}$$

The formula on the top need not be reducible to a first-order formula. Here is an explicit example.

Let $\mathfrak{L}$ be the language of rings and let $\theta(x, y)$ be the relation $x = y + 1$. Let E be the equaliser of the maps

$$x = x \underset{2x}{\overset{x}{\rightrightarrows}} \frac{x = x}{\tilde{\theta}}$$

Then E defines the set of x such that $2x = m + x$ for some $m \in \mathbb{Z}$. So $E(\mathbb{R}) = \mathbb{Z}$. If E where reducible to a finitary first-order formula, then this would imply that $\mathbb{Z}$ was definable in $\mathbb{R}$, which is not the case. So E cannot be finitary first-order. This is in contrast to the additive case where the group structure gives finitary first-order definability of such relations. $\square$

3.2.4 Coherent functors

We are now in a position to characterise the category $\mathrm{coh}(\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$ of coherent functors.

Proposition 3.2.11. $F \in (\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$ *is coherent if and only if* $F \cong \varphi/\theta$ *where* φ *and* θ *are positive existential formulas and* θ *defines an equivalence relation on* φ.

Proof. ($\Rightarrow$) Suppose that F is coherent. So in particular F is f.p. and by Corollary 3.2.6, $F \cong \varphi/\tilde{\theta}$. Since F is coherent, $\varphi \times_F \varphi$ is f.g. But $\varphi \times_F \varphi \cong \tilde{\theta}$ so $\tilde{\theta}$ is f.g. Since $\tilde{\theta} \subseteq \varphi \times \varphi$, the proof of Proposition 3.2.8 implies that $\tilde{\theta} \cong \eta$ for some pe-formula η.

($\Leftarrow$) Now suppose that $F \cong \varphi/\theta$ where φ and θ are pe-formulas. Let G be a f.g. functor and $f : G \to F$ a map. By Corollary 3.2.6 and Proposition 3.2.7 we can write this as

$$\frac{\psi}{E} \xrightarrow{\rho} \frac{\varphi}{\theta}$$

for some pe-formulas ψ and ρ. Let $\sim$ be the equivalence relation on $\psi \times \psi$ defined by

$$(\boldsymbol{x}, \boldsymbol{y}) \sim (\boldsymbol{x}', \boldsymbol{y}') \Leftrightarrow E(\boldsymbol{x}, \boldsymbol{x}') \text{ and } E(\boldsymbol{y}, \boldsymbol{y}')$$

Let $\gamma(\boldsymbol{x}, \boldsymbol{y})$ be the formula

$$\exists \boldsymbol{w}, \boldsymbol{z}(\psi(\boldsymbol{x}) \wedge \psi(\boldsymbol{y}) \wedge \rho(\boldsymbol{x}, \boldsymbol{w}) \wedge \rho(\boldsymbol{y}, \boldsymbol{z}) \wedge \theta(\boldsymbol{w}, \boldsymbol{z}))$$

Then $G \times_F G \cong \gamma/\sim$ which is f.g. □

We collect together the above proposition and Proposition 3.2.7 in the following theorem.

Theorem 3.2.12. *The category* $\mathrm{coh}(\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$ *of coherent functors on* $\mathrm{fp}\,\mathbf{C}$ *is equivalent to the category with objects* φ/θ *where* φ *and* θ *are pe-formulas and* θ *defines an equivalence relation on* φ*, and with maps the pe-definable functions* $\varphi/\theta \to \psi/\eta$*.*

3.3 Finitely generated, finitely presented and coherent sheaves

3.3.1 Extending functors

In defining localisations of our functor category we will need to make use of the notion of an extension of a functor defined on f.p. objects to the category of all objects. The well-definedness of the extension depends on the following lemma (which has an analogue in the additive case; see, e.g. [4, pp. 4-5], [22, 5.6]). We prove this directly (it is, e.g., contained in [1, 2.26]).

Lemma 3.3.1. *Let* $F : \mathrm{fp}(\mathbf{C}) \to \mathbf{Set}$ *be a functor defined on the category of finitely presented objects of* $\mathbf{C}$*. Let* $X, Y \in \mathbf{C}$*.*

(a) *If* $((A_i)_i, (a_{ik} : A_i \to A_k)_{i<k})$ *and* $((B_j)_j, (b_{jl} : B_j \to B_l)_{j<l})$ *are directed systems where* $A_i, B_j \in \mathrm{fp}(\mathbf{C})$ *and with* $X = \varinjlim_i A_i = \varinjlim_j B_j$ *then* $\varinjlim_i F(A_i) \cong \varinjlim_j F(B_j)$*.*

(b) *If* $X = \varinjlim_i X_i$ *and* $Y = \varinjlim_i Y_j$ *and* $f : X \to Y$*, then there is a canonically induced map*

$$f_* : \varinjlim_i F(X_i) \longrightarrow \varinjlim_j F(Y_j)$$

which is functorial in the obvious sense.

Proof. (a) Suppose

$$X = \varinjlim_{i \in I} A_i = \varinjlim_{j \in J} B_j$$

Since the A_i are f.p., for each $i \in I$, there is an $f_i : A_i \to B_{k_i}$ such that the following diagram commutes.

$$\begin{tikzcd}
A_i \arrow[rr, "a_i"] \arrow[dr, "f_i"'] & & X \\
& B_{k_i} \arrow[ur, "b_{k_i}"'] &
\end{tikzcd}$$

Now consider the following diagram.

$$\begin{tikzcd}
A_i \arrow[rr, "a_{ij}"] \arrow[d, "f_i"'] & & A_j \arrow[d, "f_j"] & \\
B_{k_i} \arrow[rr, "b_{k_i k_j}"] \arrow[dr, "b_{k_i}"'] & & B_{k_j} \arrow[dl, "b_{k_j}"'] \arrow[dr, dashed, "b_{k_j l}"] & \\
& X & & B_l \arrow[ll, dashed, "b_l"]
\end{tikzcd}$$

Although the square does not necessarily commute, there will be an $l \in J$ such that $b_{k_j l} b_{k_i k_j} f_i = b_{k_j l} f_j a_{ij}$. This implies that the following diagram is a cocone on the $F(A_i)$.

$$\begin{tikzcd}
F(A_i) \arrow[rr, "F(a_{ij})"] \arrow[d, "F(f_i)"'] & & F(A_j) \arrow[d, "F(f_j)"] \\
F(B_{k_i}) \arrow[dr, "\pi_{k_i}"'] & & F(B_{k_j}) \arrow[dl, "\pi_{k_j}"] \\
& \varinjlim_{j \in J} F(B_j) &
\end{tikzcd}$$

(Here the $\pi_j : F(B_j) \to \varinjlim_{l \in J} F(B_l)$ are the canonical maps to the limit. Similarly we write $\sigma_i : F(A_i) \to \varinjlim_{k \in I} F(A_k)$.) Hence there is a unique ρ such that the following diagram commutes.

$$\begin{tikzcd}
F(A_i) \arrow[rr, "F(a_{ij})"] \arrow[dr, "\sigma_i"] \arrow[dd, "F(f_i)"'] & & F(A_j) \arrow[dl, "\sigma_j"'] \arrow[dd, "F(f_j)"] \\
& \varinjlim_{i\in I} F(A_i) \arrow[dd, dashed, "\rho"] & \\
F(B_{k_i}) \arrow[dr, "\pi_{k_i}"] & & F(B_{k_j}) \arrow[dl, "\pi_{k_j}"'] \\
& \varinjlim_{j\in J} F(B_j) &
\end{tikzcd}$$

Similarly, for each $j \in J$ there will be a map $g_j : B_j \to A_{m_j}$ such that $b_j = a_{m_j} g_j$ and one gets a map $\rho' : \varinjlim_j F(B_j) \to \varinjlim_{i\in I} F(A_i)$ such that $\rho'\pi_j = \sigma_{m_j} F(g_j)$.

Now consider the following diagram.

$$\begin{tikzcd}
A_i \arrow[r, "f_i"] \arrow[dr, "a_i"'] & B_{k_i} \arrow[r, "g_{k_i}"] \arrow[d, "b_{k_i}"] & A_{m_{k_i}} \arrow[dl, "a_{m_{k_i}}"] \\
& X &
\end{tikzcd}$$

By essential uniqueness, there is an $l_i \in I$ such that $a_{k_i l_i} g_{k_i} f_i = a_{i l_i}$. Then

$$\begin{aligned}
\rho'\rho\sigma_i &= \rho'\pi_{k_i} F(f_i) \\
&= \sigma_{m_{k_i}} F(g_{k_i}) F(f_i) \\
&= \sigma_{m_{k_i}} F(g_{k_i} f_i) \\
&= \sigma_{l_i} F(a_{k_i l_i}) F(g_{k_i} f_i) \\
&= \sigma_{l_i} F(a_{i l_i}) \\
&= \sigma_i
\end{aligned}$$

So $\rho'\rho$ is the identity on $\varinjlim F(A_i)$. By symmetry, $\rho\rho'$ is the identity on $\varinjlim F(B_j)$ so ρ is an isomorphism.

(b) The argument for this is similar to that in part (a). $\square$

We are now in a position to define the extension of a functor $F \in (\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$ to the whole of $\mathbf{C}$. We will denote this extension of F also by F, and put for $C \in \mathbf{C}$,

$$F(C) = \varinjlim F(C_i)$$

where $C = \varinjlim C_i$ is a directed colimit of f.p. objects. The action of F on maps of $\mathbf{C}$ is given canonically as in the above lemma.

One can also extend a natural transformation $\sigma : F \to G$ in the category $(\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$ to a natural transformation between the extensions by defining, for $C = \varinjlim C_i \in \mathbf{C}$, a directed colimit of f.p. objects,

$$\sigma_C = \varinjlim \sigma_{C_i}$$

(One may check that this is independent of representation of C as a directed colimit of finitely presented objects.) Any f.p. functor $\varphi/\tilde{\theta}$ in $(\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$ extends to the functor $\varphi/\tilde{\theta} : \mathbf{C} \to \mathbf{Set}$ defined by the same formulas and this, it is easily checked, commutes with directed colimits. Similarly, a natural transformation given by a pe-formula ρ extends to a functor given by the same formula and which commutes with directed colimits.

3.3.2 Definable subcategories and finite type topologies

Let $\mathbf{A}$ be a small category. Any subfunctor S of $\mathrm{Hom}(A, -)$ may be thought of as a collection of maps with domain A which is closed under composition on the left: i.e. if $f \in S$ and g is a composable map (in the sense that $\mathrm{dom}\, g = \mathrm{cod}\, f$), then $gf \in S$. We say that S is a *(left) ideal* of A. Given an ideal S of A and a map $f : A \to B$ in $\mathbf{A}$, one gets an ideal $f_*(S)$ of B defined to be $\{g : \mathrm{dom}\, g = B \text{ and } gf \in S\}$. A *(left) Grothendieck topology* J on $\mathbf{A}$ is a function which associates to any object $A \in \mathbf{A}$ a set $J(A)$ of left ideals of A. The elements of $J(A)$ are called *covers* of A or *dense subfunctors* of $\mathrm{Hom}(A, -)$ and are required to satisfy the following conditions.

G1 $\mathrm{Hom}(A, -) \in J(A)$ for every $A \in \mathbf{A}$;

G2 If $S \in J(A)$ and $f : A \to B$ is an arrow of $\mathbf{A}$, then $f_*(S) \in J(B)$;

G3 If $S \in J(A)$ and $R \subseteq \mathrm{Hom}(A, -)$ is such that, for any arrow $f \in S$, $f_*(R) \in J(\mathrm{cod}\, f)$, then $R \in J(A)$.

A Grothendieck topology has an additive analogue (see 2.3.1).

Fix a subcategory $\mathbf{D}$ of $\mathbf{C}$. We associate to $\mathbf{D}$ a Grothendieck topology $J_{\mathbf{D}}$ on $\mathrm{fp}(\mathbf{C})$ defined by

$$S \in J_{\mathbf{D}}(A) \quad \Leftrightarrow \quad S(D) = \mathrm{Hom}(A, D) \text{ for all } D \in \mathbf{D}$$

In this definition we are using the extension of S from $\mathrm{fp}\,\mathbf{C}$ to $\mathbf{C}$ as explained in the previous section. One easily verifies that this extension does satisfy the above axioms for a Grothendieck topology.

Conversely, to a given Grothendieck topology J on $\mathrm{fp}(\mathbf{C})$ one can associate a full subcategory $V(J)$ of $\mathbf{C}$ by

$$D \in V(J) \quad \Leftrightarrow \quad S(D) = \mathrm{Hom}(A, D) \text{ for all } A \in \mathrm{fp}(\mathbf{C}) \text{ and } S \in J(A)$$

The following notion (central in the model theory of modules) is needed to describe the "closed" points of this correspondence.

If $C \subseteq D$ in $\mathbf{C}$ then C is a *pure* subobject of D if for every commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r, tail] & D \end{tikzcd}$$

with A and B finitely presented, the map g factors through f. The following proposition is proved in [1, 5.15] but we include a proof here for completeness.

Proposition 3.3.2. *The subobject $C \subseteq D$ is pure if and only if, for every positive existential formula $\varphi(\boldsymbol{x})$ and any tuple $\boldsymbol{c}$ from C, $C \models \varphi(\boldsymbol{c})$ if $D \models \varphi(\boldsymbol{c})$.*

Proof. It is clearly sufficient to prove the statement for positive primitive formulas.

($\Rightarrow$) Suppose that $C \subseteq D$ is pure. Let $\varphi(\boldsymbol{x})$ be the positive primitive formula $\exists \boldsymbol{y} \sigma(\boldsymbol{x}, \boldsymbol{y})$ where σ is a conjunction of atomic formulas. Suppose $D \models \varphi(\boldsymbol{c})$ where $\boldsymbol{c}$ is a tuple from C. Then there is a tuple $\boldsymbol{d}$ from D such that $D \models \sigma(\boldsymbol{c}, \boldsymbol{d})$. Let $(A, \boldsymbol{a}, \boldsymbol{b})$ be a free realisation of σ. We get the following commutative diagram, where $\mathrm{Fr}(\boldsymbol{x})$ is the free $\mathfrak{L}$-structure generated by a tuple of variables $\boldsymbol{x}$ (e.g. see [1, p. 202]).

$$\begin{tikzcd} & \boldsymbol{x} \arrow[r, mapsto] & \boldsymbol{a} & \\ \boldsymbol{x} \arrow[d, mapsto] & \mathrm{Fr}(\boldsymbol{x}) \arrow[r] \arrow[d] & A \arrow[d] & (\boldsymbol{a}, \boldsymbol{b}) \arrow[d, mapsto] \\ \boldsymbol{c} & C \arrow[r] & D & (\boldsymbol{c}, \boldsymbol{d}) \end{tikzcd}$$

By purity, we get a map $f : A \to C$ such that $f(\boldsymbol{a}) = \boldsymbol{c}$. Since $\boldsymbol{a} \in \varphi(A)$ we must have $\boldsymbol{c} \in \varphi(C)$ as required.

($\Leftarrow$) Suppose we have a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, tail] & D \end{tikzcd}$$

where A and B are finitely presented. Let $\boldsymbol{a}$ be a tuple of generators for A and $\boldsymbol{b}$ a tuple of generators for B. Let φ be a presentation formula for B.

There is a tuple of terms $\boldsymbol{t}$ such that $f(\boldsymbol{a}) = \boldsymbol{t}(\boldsymbol{b})$. Let $\psi(\boldsymbol{x})$ be the positive primitive formula

$$\exists \boldsymbol{y}(\varphi(\boldsymbol{y}) \wedge \boldsymbol{t}(\boldsymbol{y}) = \boldsymbol{x})$$

Then $D \models \psi(gf(\boldsymbol{a}))$ since $D \models \varphi(g(\boldsymbol{b}))$ and

$$\boldsymbol{t}(g(\boldsymbol{b})) = g(\boldsymbol{t}(\boldsymbol{b})) = gf(\boldsymbol{a})$$

Since $gf(\boldsymbol{a}) = h(\boldsymbol{a})$ lies in C, the hypothesis implies that $C \models \psi(h(\boldsymbol{a}))$. So there is a tuple $\boldsymbol{c} \in \varphi(C)$ such that $\boldsymbol{t}(\boldsymbol{c}) = h(\boldsymbol{a})$. By the definition of presentation formula, there is a map $f' : B \to C$ with $f'(\boldsymbol{b}) = \boldsymbol{c}$. So

$$h(\boldsymbol{a}) = \boldsymbol{t}(\boldsymbol{c}) = \boldsymbol{t}(f'(\boldsymbol{b})) = f'(\boldsymbol{t}(\boldsymbol{b})) = f'f(\boldsymbol{a})$$

and we see that h factors through f as required. $\square$

Following terminology used in the additive case, we will say that a category $\mathbf{D} \subseteq \mathbf{C}$ closed in $\mathbf{C}$ under ultraproducts and pure subobjects is a *definable subcategory* of $\mathbf{C}$. As the following result shows, this means definable by sentences of a particular sort.

Theorem 3.3.3. *The following are equivalent.*

(i) $\mathbf{D}$ *is a definable subcategory;*

(ii) $\mathbf{D}$ *can be axiomatised by a coherent theory.*

This result is a direct consequence of a result of Keisler's [20]. Recall that a coherent theory is one whose axioms are of the form $\forall \boldsymbol{x}(\varphi(\boldsymbol{x}) \to \psi(\boldsymbol{x}))$ where φ and

ψ are positive existential. Keisler's result is that an elementary class of structures is closed under directed colimits if and only if it can be axiomatised by sentences of the form

$$\forall \boldsymbol{x} \exists \boldsymbol{y} \bigwedge_{i \in I} \bigvee_{j \in J} (g_{ij}(\boldsymbol{x}) \rightarrow h_{ij}(\boldsymbol{x}, \boldsymbol{y}))$$

where the g_{ij} and h_{ij} are atomic formulas and I and J are finite. But the above sentence is equivalent to the following collection of coherent sentences

$$\left\{ \forall \boldsymbol{x} \left(\bigwedge_{i \in I'} \bigwedge_{j \in J} g_{ij}(\boldsymbol{x}) \rightarrow \exists \boldsymbol{y} \bigwedge_{i \in I'} \bigvee_{j \in J} h_{ij}(\boldsymbol{x}, \boldsymbol{y}) \right) \right\}_{I' \subseteq I}$$

Now a definable subcategory $\mathbf{D}$ is an elementary class closed under directed colimits. The fact that $\mathbf{D}$ is elementary follows (by [21, cor 6.1.16]) from the fact that $\mathbf{D}$ is closed under ultraproducts and its complement is closed under ultrapowers (since every structure is purely, even elementarily, embeddable in any of its ultrapowers via the diagonal map). The fact that $\mathbf{D}$ is closed under directed colimits follows from the fact that for any directed system of structures $(D_i : i \in I)$ there is an ultrafilter $\mathcal{U}$ on I and a pure embedding of $\varinjlim D_i$ in $\prod_i D_i/\mathcal{U}$ (see, e.g., [34]). Hence, Keisler's result gives us that a definable subcategory $\mathbf{D}$ can be axiomatised by a coherent theory. The converse is easy.

Using Theorem 3.3.3, we see that the class of integral domains forms a definable subcategory of the category of commutative rings: this class is defined by the coherent axiom

$$\forall x, y (xy = 0 \rightarrow x = 0 \vee y = 0)$$

We shall say that a Grothendieck topology J is of *finite type* if every cover has a finitely generated cover below it (that is, every dense ideal contains a f.g. dense ideal). In our context this is equivalent to there being a positive existential formula dense below every cover (by Corollary 3.2.9). If J has the stronger property that any cover has a positive primitive formula dense below it, then we shall say it is of *finite pp type*.

Proposition 3.3.4.

(a) *If* $\mathbf{D}$ *is definable, then* $J_{\mathbf{D}}$ *is of finite type. If in addition* $\mathbf{D}$ *is closed under finite products, then* $J_{\mathbf{D}}$ *is of finite pp type.*

(b) *If* J *is of finite type, then* $V(J)$ *is definable. If* J *is of finite pp type, then* $V(J)$ *is also closed under products.*

Proof. (a) Suppose $\mathbf{D}$ is definable. Let $S \subseteq \operatorname{Hom}(C, -)$ be a cover with respect to $J_{\mathbf{D}}$. Let φ be a presentation formula for C, so $\operatorname{Hom}(C, -) \cong \varphi$. If $\{\varphi_\lambda : \lambda \in \Lambda\}$ is the set of positive existential formulas below S, then clearly

$$S = \bigcup_{\lambda \in \Lambda} \varphi_\lambda$$

Suppose that none of them is dense. So for each λ there is a $D_\lambda \in \mathbf{D}$ such that $\varphi_\lambda(D_\lambda) \subsetneqq \varphi(D_\lambda)$. Let p be the partial type

$$\{\varphi \wedge \neg\varphi_\lambda : \lambda \in \Lambda\}.$$

Then p is closed under finite conjunctions. Define

$$\langle\lambda\rangle = \{\mu \in \Lambda : \varphi_\lambda \subseteq \varphi_\mu\}.$$

Then

$$\langle\lambda\rangle \cap \langle\mu\rangle = \langle\nu\rangle$$

where $\varphi_\nu = \varphi_\lambda \vee \varphi_\mu$. So the set $\{\langle\lambda\rangle : \lambda \in \Lambda\}$ has the finite intersection property and there is an ultrafilter $\mathcal{U}$ containing it. The ultraproduct $D^* = \prod_\lambda D_\lambda/\mathcal{U}$ realises p. More concretely, the tuple $[(\boldsymbol{a}_\lambda)_\lambda]$ where $\boldsymbol{a}_\lambda \in \varphi(D_\lambda) \setminus \varphi_\lambda(D_\lambda)$ is a realisation. To see this, take any $\varphi \wedge \neg\varphi_\lambda \in p$. Then

$$\langle\lambda\rangle \subseteq \{\mu \in \lambda : D_\mu \models \varphi \wedge \neg\varphi_\lambda(\boldsymbol{a}_\mu)\}$$

and so the right hand side is in $\mathcal{U}$. So $D^* \models p([(\boldsymbol{a}_\lambda)_\lambda])$. But this means that $S(D^*) \neq \operatorname{Hom}(C, D^*)$ which is a contradiction. So $J_{\mathbf{D}}$ must be of finite type.

Now suppose that in addition $\mathbf{D}$ is closed under finite products. We know from above that any cover $S \subseteq \varphi \cong \operatorname{Hom}(C, -)$ has a dense positive existential formula ψ below it. Write ψ as $\psi_1 \vee \cdots \vee \psi_n$ where each ψ_i is positive primitive. Suppose that none of the ψ_i is dense. Then for each i there is $D_i \in \mathbf{D}$ and $\boldsymbol{a}_i \in \varphi(D_i) \setminus \psi_i(D_i)$. Let $D = \prod_{i=1}^n D_i$. Then

$$\psi(D) = \bigcup_{k=1}^{n} \left(\prod_{i=1}^{n} \psi_k(D_i) \right) \subseteq \prod_{i=1}^{n} \varphi(D_i) = \varphi(D)$$

Since $D \in \mathbf{D}$ and ψ is dense, we must have $\psi(D) = \varphi(D)$. But $\boldsymbol{a}_k \in \varphi(D_k) \setminus \psi_k(D_k)$ implies that

$$(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) \in \prod_{i=1}^{n} \varphi(D_i) \setminus \prod_{i=1}^{n} \psi_k(D_i)$$

for each k which is a contradiction.

(b) Suppose J is a finite type topology. For each $A \in \operatorname{fp} \mathbf{C}$, let φ_A be a presentation formula for A. Now let T be the coherent theory with axioms

$$\{\varphi_A \to \psi : A \in \operatorname{fp} \mathbf{C},\ \psi \in J(A)\}$$

Then $V(J) = \operatorname{Mod} T$ (the class of models of T). By Theorem 3.3.3, $V(J)$ is definable.

Suppose now that J is of finite pp type. Let $\psi \subseteq S \subseteq \varphi$ be dense where ψ is positive primitive. Let $D = \prod_i D_i$ where each D_i is in $V(J)$.

$$\psi(D) = \prod_i \psi(D_i) = \prod_i \varphi(D_i) = \varphi(D)$$

so $S(D) = \varphi(D)$ and we must have $D \in V(J)$ as required.

$\square$

Proposition 3.3.5. *Suppose* $\mathbf{D}$ *is a definable subcategory. Then* $\mathbf{D} = V(J_{\mathbf{D}})$.

Proof. Obviously we have $\mathbf{D} \subseteq V(J_{\mathbf{D}})$. For the other direction we use Theorem 3.3.3.

There is a coherent theory T' such that $\mathbf{D} = \operatorname{Mod} T'$. Suppose $C \in V(J_{\mathbf{D}})$. We want $C \models T'$. So consider an arbitrary formula $\forall \boldsymbol{x}(\varphi(\boldsymbol{x}) \to \psi(\boldsymbol{x}))$ from T'. We may clearly assume that φ is positive primitive and ψ is positive existential. Since φ is positive primitive, there is a finitely presented object A and an epimorphism

$$\operatorname{Hom}(A, -) \xrightarrow{f} \varphi$$

Consider the pullback diagram

$$\begin{tikzcd}
f^{-1}(\psi \wedge \varphi) \arrow[r, two heads] \arrow[d, tail] & \psi \wedge \varphi \arrow[d, tail] \\
\operatorname{Hom}(A, -) \arrow[r, two heads] & \varphi
\end{tikzcd}$$

Let $D \in \mathbf{D}$. Then $(\psi \wedge \varphi)(D) = \varphi(D)$, hence $f^{-1}(\psi \wedge \varphi)(D) = \mathrm{Hom}(A, D)$. So $f^{-1}(\psi \wedge \varphi) \in J_{\mathbf{D}}(A)$. But this means $f^{-1}(\psi \wedge \varphi)(C) = \mathrm{Hom}(A, C)$ which implies that $(\psi \wedge \varphi)(C) = \varphi(C)$. So $C \models \forall \boldsymbol{x}(\varphi(\boldsymbol{x}) \rightarrow \psi(\boldsymbol{x}))$ as required. □

Note that it follows from the above that if $\mathbf{D}$ is a definable subcategory closed under finite products, then it is closed under arbitrary products. This fact can also be found in [10, 6.7].

To obtain a converse to Proposition 3.3.5 we will first need some sheaf theory.

3.3.3 Some general sheaf theory

Fix a Grothendieck topology J on a skeletally small category $\mathbf{A}$. A presheaf $P \in (\mathbf{A}, \mathbf{Set})$ is called a *separated presheaf* if whenever $A \in \mathbf{A}$ and $f, g : \mathrm{Hom}(A, -) \rightarrow P$ are maps such that $f \restriction_S = g \restriction_S$ for some cover $S \in J(A)$, then $f = g$. A presheaf $F \in (\mathbf{A}, \mathbf{Set})$ is called a *sheaf* if for any object $A \in \mathbf{A}$ and any cover $S \in J(A)$, every map $\alpha : S \rightarrow F$ has a unique extension to a map $\mathrm{Hom}(A, -) \rightarrow F$.

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d] & F \\ \mathrm{Hom}(A, -) \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

Given a Grothendieck topology J on $\mathbf{A}$, one can form the category of sheaves $\mathrm{Sh}(\mathbf{A}, J)$ which will be a reflexive subcategory of $(\mathbf{A}, \mathbf{Set})$ (see [25, III, §4]). So we have an adjoint pair

$$\mathrm{Sh}(\mathbf{A}, J) \underset{a}{\overset{i}{\rightleftarrows}} (\mathbf{A}, \mathbf{Set})$$

where a is the associated sheaf functor and i is the inclusion functor. The functor a is left adjoint to i and commutes with finite limits.

The construction of the associated sheaf is usually done by two applications of Grothendieck's plus functor. This functor takes a presheaf P in $(\mathbf{A}, \mathbf{Set})$ to the presheaf P^+ which is defined on objects $C \in \mathbf{A}$ as follows.

$$P^+C = \varinjlim_{R \in J(C)} \mathrm{Nat}(R, P)$$

where the directed colimit is taken over the decreasing family of covers. Note that since a finite intersection of covers is a cover, we do indeed have a directed system over which we can take a limit. Here $\mathrm{Nat}(R, P) = \mathrm{Hom}(R, P)$ is the set of natural transformations $R \to P$. The elements of P^+C can be represented by pairs (R, α) where $\alpha : R \to P$ is a natural transformation and $R \in J(C)$. We factor by the equivalence relation

$$(R, \alpha) \sim (S, \beta) \quad \Leftrightarrow \quad \text{there is a cover } T \subseteq R \cap S \text{ s.t. } \alpha \restriction_T = \beta \restriction_T$$

Given a map $f : C \to C'$, the induced map $P^+f : P^+C \to P^+C'$ is defined as follows.

$$[R, \alpha] \longmapsto [f_*R, \alpha f^*]$$

Here $f^* : \mathrm{Hom}(C', -) \to \mathrm{Hom}(C, -)$ is the induced map and f_*R is the cover $(f^*)^{-1}R$ (since an inverse image of a cover is a cover).

The plus functor takes a natural transformation $\varphi : P \to Q$ to the natural transformation $\varphi^+ : P^+ \to Q^+$ which is defined as follows.

$$\varphi^+_C[R, \alpha] = [R, \varphi \circ \alpha]$$

The canonical map $\eta : P \to P^+$ is defined as follows.

$$\eta_C(x) = [\mathrm{Hom}(C, -), x]$$

On the right hand side we identify the element $x \in P(C)$ with the corresponding element of $\mathrm{Nat}(\mathrm{Hom}(C, -), P)$. Clearly η is a monomorphism if and only if P is a separated presheaf. For any presheaf P, P^+ is separated. If P is separated, then P^+ is a sheaf. So P^{++} is always a sheaf. The associated sheaf functor is equivalent to two applications of the plus functor, so that $aP = P^{++}$.

The following lemmas will be important for us. The first is direct from the definition of P^+.

Lemma 3.3.6. *For every map* $\mathrm{Hom}(A, -) \to P^+$*, there is a cover* $S \in J(A)$ *and a map* $S \to P$ *such that the following square commutes.*

$$\begin{tikzcd} S \arrow[r] \arrow[d] & P \arrow[d] \\ \mathrm{Hom}(A, -) \arrow[r] & P^+ \end{tikzcd}$$

Lemma 3.3.7. *Let $f : a(\mathrm{Hom}(A,-)) \to aP$ be a map of sheaves. Then there is a cover $S \in J(A)$ such that for every $h \in S$ there is an $x_h \in P(C = \operatorname{cod} h)$ such that $a(x_h) = f \circ a(h^*)$ as in the following diagram (which uses the Yoneda identification of PC with $\mathrm{Nat}(\mathrm{Hom}(C,-),P)$).*

$$\begin{tikzcd} a(\mathrm{Hom}(C,-)) \arrow[rr, "a(x_h)"] \arrow[dr, "a(h^*)"'] & & aP \\ & a(\mathrm{Hom}(A,-)) \arrow[ur, "f"'] & \end{tikzcd}$$

Proof. By applying Lemma 3.3.6 to the composition

$$\mathrm{Hom}(A,-) \longrightarrow a(\mathrm{Hom}(A,-)) \xrightarrow{f} aP$$

there is a cover $R \in J(A)$ and a map $R \to P^+$ s.t. the following diagram commutes.

$$\begin{tikzcd} R \arrow[rr] \arrow[d] & & P^+ \arrow[d] \\ \mathrm{Hom}(A,-) \arrow[r] & a(\mathrm{Hom}(A,-)) \arrow[r, "f"] & aP \end{tikzcd}$$

Let $g \in R$ and put $D = \operatorname{cod} g$. Then, again by Lemma 3.3.6 there is a cover $R_g \in J(D)$ and a map $R_g \to P$ such that

$$\begin{tikzcd} R_g \arrow[rr] \arrow[d] & & P \arrow[d] \\ \mathrm{Hom}(D,-) \arrow[r, "g^*"'] & R \arrow[r] & P^+ \end{tikzcd}$$

commutes.

Let $S \subseteq \mathrm{Hom}(A,-)$ be defined by $S = \{g'g : g \in R, g' \in R_g\}$. By G3 of the axioms for a Grothendieck topology $S \in J(A)$. Suppose $h \in S$. Then $h = g'g$ for some $g \in R$ and $g' \in R_g$. The map

$$\mathrm{Hom}(C = \operatorname{cod} h,-) \xrightarrow{(g')^*} R_g \longrightarrow P$$

corresponds to an element $x_h \in P(C)$ and

$$\begin{tikzcd} \mathrm{Hom}(C,-) \arrow[rr, "x_h"] \arrow[d, "h^*"] & & P \arrow[dd] \\ S \arrow[d] & & \\ \mathrm{Hom}(A,-) \arrow[r] & a(\mathrm{Hom}(A,-)) \arrow[r, "f"] & aP \end{tikzcd}$$

commutes. Hence $a(x_h) = f \circ a(h^*)$ as required. ∎

As described above, the associated sheaf functor a is equivalent to two applications of the plus functor, but it is also equivalent to one application of the "sep-functor" followed by one application of the plus functor. The sep-functor will be relevant to us so we describe it now.

We define an equivalence relation on $P(C)$ by

$$x \sim y \quad \Leftrightarrow \exists S \in J(C) \text{ s.t. } x \restriction_S = y \restriction_S$$

Define $P_{\text{sep}}C = P(C)/\sim$. A map $f : C \to C'$ induces a map $f_* : P_{\text{sep}}(C) \to P_{\text{sep}}(C')$ in the obvious way. This association is functorial and so P_{sep} is a well defined element of $(\mathbf{A}, \mathbf{Set})$. Moreover, the map

$$P \longmapsto P_{\text{sep}}$$

is also functorial as one can check (the induced maps are the obvious ones). There is a canonical epimorphism $\sigma : P \to P_{\text{sep}}$.

Lemma 3.3.8. *P_{sep} is separated.*

Proof. Let $S \in J(C)$ and let $x, y \in P(C)$ with $\bar{x}, \bar{y}$ the corresponding equivalence classes in $P_{\text{sep}}C$. Suppose that $\bar{x} \restriction_S = \bar{y} \restriction_S$. So for each $f \in S$, there is an $R_f \in J(\operatorname{cod} f)$ such that

$$f_*(x) \restriction_{R_f} = f_*(y) \restriction_{R_f}$$

Let S' be the set

$$\{gf : f \in S, g \in R_f\}$$

Then $S' \in J(C)$ (by Grothendieck topology axiom G3) and $x \restriction_{S'} = y \restriction_{S'}$. So $\bar{x} = \bar{y}$ and P_{sep} is separated as required. $\square$

Since P_{sep} is separated, the canonical map $\eta : P_{\text{sep}} \to (P_{\text{sep}})^+$ is a monomorphism and $(P_{\text{sep}})^+$ is a sheaf. Since σ is universal from P to separated presheaves and η is universal from P to sheaves (these statements are easily verified), it follows that $(-)^+ \circ (-)_{\text{sep}} = a$. We have the following picture.

$$P \twoheadrightarrow P_{\text{sep}} \rightarrowtail aP$$

Let P be any presheaf. A subobject $P' \subseteq P$ is *dense* (w.r.t. the ambient Grothendieck topology J) if for any representable functor $\mathrm{Hom}(A, -)$ and any map $f : \mathrm{Hom}(A, -) \to P$, the inverse image $f^{-1}(P')$ is in $J(A)$. For any presheaf P, we use the notation $\mathcal{D}(P)$ for its set of dense subobjects. This set is closed under finite intersections so forms a downward directed family.

Lemma 3.3.9.

$$\mathrm{Nat}(aP, aQ) \cong \varinjlim_{P' \in \mathcal{D}(P)} \mathrm{Nat}(P', Q_{\mathrm{sep}})$$

Proof. First note that $\mathrm{Nat}(aP, aQ) \cong \mathrm{Nat}(P, aQ)$. Define the map

$$\Phi : \mathrm{Nat}(P, aQ) \longrightarrow \varinjlim_{P' \in \mathcal{D}(P)} \mathrm{Nat}(P', Q_{\mathrm{sep}})$$

by

$$\Phi(\alpha) = \left[\alpha^{-1} Q_{\mathrm{sep}}, \alpha \restriction_{\alpha^{-1} Q_{\mathrm{sep}}}\right]$$

In the other direction, we define the map

$$\Psi : \varinjlim_{P' \in \mathcal{D}(P)} \mathrm{Nat}(P', Q_{\mathrm{sep}}) \longrightarrow \mathrm{Nat}(P, aQ)$$

by

$$\Psi \left[P', \alpha\right]_C (x) = \left[x^{-1} P', \alpha \circ x\right]$$

Note that the morphism $\alpha \circ x : x^{-1} P' \to Q_{\mathrm{sep}}$ has a unique extension to a morphism $\mathrm{Hom}(C, -) \to aQ$, that is, to an element of $aQ(C)$. It is easily checked that, since Q_{sep} is separated, Ψ is well defined. We claim that $\Psi = \Phi^{-1}$.

We first show that $\Psi \circ \Phi = \mathrm{id}$.

$$\begin{aligned} \Psi(\Phi(\alpha))_C(x) &= \Psi\left[\alpha^{-1} Q_{\mathrm{sep}}, \alpha\right]_C (x) \\ &= \left[x^{-1}(\alpha^{-1} Q_{\mathrm{sep}}), \alpha \circ x\right] \end{aligned}$$

Write $R = x^{-1}(\alpha^{-1} Q_{\mathrm{sep}})$. Suppose that $\alpha_C(x) = [S, \beta]$. We want to show that there is a cover $T \subseteq R \cap S$ such that $\alpha \circ x \restriction_T = \beta \restriction_T$.

For any $f : C \to D$ in R,

$$\begin{aligned}(\alpha \circ x)(f) &= \alpha_D(f_*(x)) \\ &= f_*(\alpha_C(x)) \\ &= f_*[S, \beta] \\ &= [f_*S, \beta f^*]\end{aligned}$$

But we know that $(\alpha \circ x)(f)$ is the element $[\mathrm{Hom}(D, -), (\alpha \circ x)(f)] \in aQ$. So there is a cover $S_f \subseteq f_*S$ such that

$$\beta f^* \restriction_{S_f} = (\alpha \circ x)(f) \restriction_{S_f}$$

So setting

$$T = \{gf : f \in R, g \in S_f\}$$

we see that T is a cover such that $\alpha \circ x \restriction_T = \beta \restriction_T$ as required.

We now show that $\Phi \circ \Psi = \mathrm{id}$. Let $\beta = \Psi[P', \alpha]$, so

$$\beta_C(x) = \left[x^{-1}P', \alpha \circ x\right]$$

Then we have

$$\Phi(\beta) = \left[\beta^{-1}Q_{\mathrm{sep}}, \beta\right].$$

Now, $P' \subseteq \beta^{-1}Q_{\mathrm{sep}}$ since if $x \in P'C$ then $\beta_C(x) = [\mathrm{Hom}(C, -), \alpha \circ x] \in Q_{\mathrm{sep}}$. Clearly $\beta \restriction_{P'} = \alpha$ and the result follows. □

3.3.4 Geometric morphisms, points, and coherent toposes

The class of all toposes can be made into a category (actually it is a *2-category*, see [8, Vol. 1, 7.1]). A map of toposes $\varphi : \mathcal{E} \to \mathcal{F}$ is defined to be a *geometric morphism*, meaning an adjoint pair

$$\mathcal{E} \underset{\varphi^*}{\overset{\varphi_*}{\rightleftarrows}} \mathcal{F}$$

where φ^* is left adjoint to φ_* and φ^* is left exact (i.e. commutes with finite limits). Composition can be defined by composing the adjoints (see [24, p. 101]). A *point* p

of a topos $\mathcal{E}$ is a geometric morphism $p : \mathbf{Set} \to \mathcal{E}$. A topos $\mathcal{E}$ is said to have *enough points* if for any two subobjects A, B of a given object $E \in \mathcal{E}$, we have that

$$A \subseteq B \quad \Leftrightarrow \quad p^*(A) \subseteq p^*(B) \text{ for all points } p : \mathbf{Set} \to \mathcal{E}$$

A *site* is a pair $(\mathbf{A}, J)$ consisting of a small category $\mathbf{A}$ with a Grothendieck topology J. The site is called *coherent* if J is of finite type. So if $\mathbf{C}$ is an LFP category and $\mathbf{D} \subseteq \mathbf{C}$ a definable subcategory, then the site $(\operatorname{fp}\mathbf{C}, J_{\mathbf{D}})$ is coherent (by Proposition 3.3.4(a)). A *coherent topos* is a topos which can be realised as the category of sheaves for a coherent site. The following theorem is proved in [25, IX, §11].

Theorem 3.3.10 (Deligne). *A coherent topos has enough points.*

Suppose that p is a point of the presheaf topos $(\mathbf{A}, \mathbf{Set})$. Since every presheaf is a colimit of representables, and p^*, as a left adjoint, commutes with all colimits, p^* is determined by its action on those representables. So p^* is determined by its composition with the Yoneda embedding

$$\mathbf{A}^{\mathrm{op}} \longrightarrow (\mathbf{A}, \mathbf{Set})$$

Conversely, any functor $F : \mathbf{A}^{\mathrm{op}} \to \mathbf{Set}$ determines a functor $(\mathbf{A}, \mathbf{Set}) \to \mathbf{Set}$ by associating to a presheaf $P \in (\mathbf{A}, \mathbf{Set})$ the set

$$\frac{\coprod_{A \in \mathbf{A}} F(A) \times P(A)}{\sim}$$

where $(xu, y) \sim (x, uy)$ for all $u : A \to B$ in A, $x \in F(B)$ and $y \in P(A)$ (here xu is defined to be $F(u)(x)$ and uy is $P(u)(y)$). This functor has an obvious action on maps and is denoted by $F \otimes_{\mathbf{A}} -$ (see [25, VII, §2]). Like the tensor product for modules, it has a right adjoint $\mathbf{Set} \to (\mathbf{A}, \mathbf{Set})$ given by

$$X \longmapsto \operatorname{Hom}(F(-), X)$$

So F determines a geometric morphism $\mathbf{Set} \to (\mathbf{A}, \mathbf{Set})$ if and only if $F \otimes_{\mathbf{A}} -$ is left exact. A functor $F : \mathbf{A}^{\mathrm{op}} \to \mathbf{Set}$ such that $F \otimes_{\mathbf{A}} -$ is left exact is called a *flat* functor. Now every functor $F \in (\mathbf{A}^{\mathrm{op}}, \mathbf{Set})$ is a colimit of representable functors in a canonical way: the indexing category is the category of *elements* of F whose objects are pairs (A, x) where $A \in \mathbf{A}$ and $x \in F(A)$ and whose maps $(A, x) \to (B, y)$ are those maps $f : A \to B$ such that $F(f)(y) = x$. Let U be the forgetful functor from the category of elements of F to $\mathbf{A}$. Then the maps $\mathrm{Hom}(-, U(A, x)) \to F$ corresponding to $x \in F(A)$ give a cocone on F. This cocone is in fact a colimit and will be said to be *canonical*. If the category of elements of F is a filtered category (see [1, p. 13]), then we can express F canonically as a filtered colimit of representable functors. It turns out that this property is equivalent to flatness: F is flat if and only if its category of elements is filtered ([25, VII, Thm 9.1]).

We have seen that a flat functor $F \in (\mathbf{A}^{\mathrm{op}}, \mathbf{Set})$ determines a point of $(\mathbf{A}, \mathbf{Set})$. In fact, all points of $(\mathbf{A}, \mathbf{Set})$ arise in this way (see [25, VII, Thm 5.2]). Now, suppose that J is a Grothendieck topology on $\mathbf{A}$ and p is a point of $\mathrm{Sh}(\mathbf{A}, J)$. The inclusion of sheaves in presheaves has a left exact left adjoint and so gives a geometric morphism

$$\mathrm{Sh}(\mathbf{A}, J) \to (\mathbf{A}, \mathbf{Set})$$

By composition with this arrow, the point p gives a point of $(\mathbf{A}, \mathbf{Set})$ and so must be induced by a flat functor $F : \mathbf{A}^{\mathrm{op}} \to \mathbf{Set}$. The functor F then has the property that, whenever $S \subseteq \mathrm{Hom}(A, -)$ is a cover, the induced monomorphism $F \otimes_{\mathbf{A}} S \to F \otimes_{\mathbf{A}} \mathrm{Hom}(A, -)$ is an isomorphism. Such a functor is said to be *J-continuous* or simply *continous* if the context is clear. Conversely, given a flat J-continuous functor F, the induced point of $(\mathbf{A}, \mathbf{Set})$ factors through the sheaf category and gives a point of $\mathrm{Sh}(\mathbf{A}, J)$. Every point of $\mathrm{Sh}(\mathbf{A}, J)$ arises in this way (see [25, VII, Lemma 5.3]).

Lemma 3.3.11. *Let J be a Grothendieck topology on $\mathrm{fp}\,\mathbf{C}$. A functor $F : (\mathrm{fp}\,\mathbf{C})^{\mathrm{op}} \to \mathbf{Set}$ is flat and J-continous if and only if it is of the form $\mathrm{Hom}(-, D)$ for some object $D \in V(J)$.*

Proof. Firstly, suppose that $D \in V(J)$. The functor $\mathrm{Hom}(-, D) \otimes_{\mathrm{fp}\,\mathbf{C}} -$ is isomorphic to ev_D (the "evaluation at D" functor). That is, if $P \in (\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$, then $(\mathrm{Hom}(-, D) \otimes P) \cong P(D)$ in a natural way. In an LFP category, directed colimits commute with finite limits (see [1, Prop 1.59]). Since $\mathbf{C}$ is LFP and the extension of P to $\mathbf{C}$ is computed via directed colimits (see Lemma 3.3.1), we have that ev_D commutes with finite limits. So $\mathrm{Hom}(-, D) \otimes -$ is left exact which implies that $\mathrm{Hom}(-, D)$ is flat. Now, if $S \subseteq \mathrm{Hom}(C, -)$ is a cover, then $\mathrm{Hom}(-, D) \otimes S \to \mathrm{Hom}(-, D) \otimes \mathrm{Hom}(C, -)$ is the inclusion $S(D) \subseteq \mathrm{Hom}(C, D)$. Since $D \in V(J)$, this inclusion is an equality. Hence $\mathrm{Hom}(-, D)$ is continuous.

Conversely, suppose that F is a continous flat functor $(\mathrm{fp}\,\mathbf{C})^{\mathrm{op}} \to \mathbf{Set}$. Since F is flat, we can write $F = \varinjlim_i \mathrm{Hom}(-, A_i)$ as a filtered colimit of representables. Since, for any $C \in \mathrm{fp}\,\mathbf{C}$ we have that $\mathrm{Hom}(C, -)$ commutes with filtered colimits in $\mathbf{C}$ (see [1, Cor 1.5]), we have that $F \cong \mathrm{Hom}(-, \varinjlim A_i)$. Let $D = \varinjlim_i A_i$. Then $(F \otimes -) \cong \mathrm{ev}_D$ and for any object $C \in \mathrm{fp}\,\mathbf{C}$ and any cover $S \subseteq \mathrm{Hom}(C, -)$, we have that $SD = \mathrm{Hom}(C, D)$ since F is continous. So $D \in V(J)$ as required. □

We can use Deligne's theorem and the above lemma to prove the following.

Proposition 3.3.12. *Let J be a finite type topology on* $\mathrm{fp}\,\mathbf{C}$. *Then $J = J_{V(J)}$.*

Proof. We clearly have that $J \subseteq J_{V(J)}$ so we need only establish that $J_{V(J)} \subseteq J$. Let $C \in \mathrm{fp}\,\mathbf{C}$ and $S \in J_{V(J)}(C)$. Let $a : (\mathrm{fp}\,\mathbf{C}, \mathbf{Set}) \to \mathrm{Sh}(\mathrm{fp}\,\mathbf{C}, J)$ be the associated sheaf functor. Consider the inclusion $aS \subseteq a(\mathrm{Hom}(C, -))$. If p is a point of $\mathrm{Sh}(\mathrm{fp}\,\mathbf{C}, J)$, by Lemma 3.3.11 and the description of points of $\mathrm{Sh}(\mathrm{fp}\,\mathbf{C}, J)$, there is an object $D \in V(J)$ such that for any presheaf P, we have $p^*(aP) \cong (\mathrm{Hom}(-, D) \otimes P) \cong PD$. So we have that $p^*(aS) = S(D) = \mathrm{Hom}(C, D) = p^*(a\mathrm{Hom}(C, -))$ since $S \in J_{V(J)}(C)$ and $D \in V(J)$. Since J is of finite type, the topos $\mathrm{Sh}(\mathrm{fp}\,\mathbf{C}, J)$ has enough points (by Deligne's Theorem). Hence we have that $aS = a(\mathrm{Hom}(C, -))$. But this implies that $S \in J(C)$. So we have that $J_{V(J)} = J$ as required. □

Combining Propositions 3.3.5 and 3.3.12 we get the following corollary which is completely analogous to the additive case (see [30, 13.3.6]).

Corollary 3.3.13. *There is a bijection between definable subcategories of* $\mathbf{C}$ *and finite type topologies on* $\mathrm{fp}\,\mathbf{C}$.

That Deligne's Theorem could be used to prove Proposition 3.3.12 was suggested to us by Tibor Beke.

3.3.5 Finitely presented sheaves

Let J be a Grothendieck topology on $\mathrm{fp}\,\mathbf{C}$. We have an adjunction

$$\mathrm{Sh}(\mathrm{fp}\,\mathbf{C}, J) \underset{a}{\overset{i}{\rightleftarrows}} (\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$$

as before, where the associated sheaf functor a commutes with finite limits. Note that a monomorphism f is a dense embedding of presheaves if and only if $a(f)$ is invertible in the category of sheaves. The next result is general: if a functor with a left adjoint commutes with directed colimits then its left adjoint preserves finitely presented objects.

Lemma 3.3.14. *Suppose i commutes with directed colimits. Then for any finitely presented (resp. finitely generated) functor P in the presheaf category $(\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$, aP is finitely presented (resp. finitely generated) in the category of sheaves.*

Proof. Let P be a f.p. (resp. f.g.) presheaf and let $\varinjlim G_i$ be a directed colimit (resp. directed union) of sheaves. Note that since i preserves monomorphisms, directed unions are taken to directed unions. Then:

$$\mathrm{Nat}(aP, \varinjlim G_i) \cong \mathrm{Nat}(P, i(\varinjlim G_i)) \cong \mathrm{Nat}(P, \varinjlim iG_i) \cong \varinjlim \mathrm{Nat}(P, iG_i) \cong \varinjlim \mathrm{Nat}(aP, G_i)$$

$\square$

Proposition 3.3.15. *The topology J is of finite type if and only if the inclusion functor i commutes with directed colimits.*

Proof. $(\Rightarrow)$ Suppose J is of finite type. Let $\{F_i\}_i$ be directed system of sheaves and let $\varinjlim F_i$, with limit maps $f_k : F_k \to \varinjlim F_i$ be their directed colimit as presheaves. Let ψ be a finitely presented cover of an object $A \in \mathrm{fp}(\mathbf{C})$. Consider a natural transformation $\alpha : \psi \to \varinjlim F_i$. Since ψ is f.p. α factors essentially uniquely through some F_k as $f_k\beta$, as shown. But since F_k is a sheaf, β extends uniquely to a natural transformation ρ on $\mathrm{Hom}(A,-)$ as in the following diagram.

$$\begin{tikzcd}
\psi \arrow[r, "\alpha"] \arrow[d] \arrow[dr, "\beta"] & \varinjlim F_i \\
\mathrm{Hom}(A,-) \arrow[r, dashed, "\rho"'] & F_k \arrow[u, "f_k"']
\end{tikzcd}$$

Clearly $f_k\rho$ is the unique extension of α to $\mathrm{Hom}(A,-)$, so $\varinjlim F_i$ is a sheaf.

($\Leftarrow$) Now suppose that i commutes with directed colimits. Let $S \subseteq \mathrm{Hom}(A,-)$ be a cover. Write

$$S = \bigcup_\lambda \varphi_\lambda$$

as a directed union of finitely presented subobjects. Since a is a left adjoint it commutes with directed colimits. It also commutes with finite limits and so preserves monomorphisms. Hence

$$a(\mathrm{Hom}(A,-)) = aS = \bigcup_\lambda a(\varphi_\lambda)$$

is a directed union in the sheaf category. Since i commutes with directed colimits, we have, by Lemma 3.3.14, that $a(\mathrm{Hom}(A,-))$ is finitely presented in the sheaf category. So $a(\mathrm{Hom}(A,-)) = a(\varphi_\lambda)$ for some λ. But this means that $\varphi_\lambda \subseteq \mathrm{Hom}(A,-)$ is dense. This proves that J is of finite type. $\square$

The above proposition has an additive analogue (see [30, Prop 12.2.3])

Proposition 3.3.16. *Let J be a finite type topology.*

(a) *If F is a finitely generated sheaf, there is a finitely generated presheaf P such that $F \cong aP$;*

(b) *IF F is a finitely presented sheaf, then there is a finitely presented presheaf P such that $F \cong aP$.*

Proof. (a) By Lemma 3.3.14 if J is a finite type topology, then $\mathrm{Sh}(\mathrm{fp}\,\mathbf{C}, J)$ is an LFP topos with a generating set

$$\{a(\mathrm{Hom}(A,-)) : A \in \mathrm{fp}\,\mathbf{C}\}$$

of finitely presented objects. By Proposition 2.2.1 there is an epimorphism

$$\coprod_i a(\mathrm{Hom}(A_i, -)) \twoheadrightarrow F \cong aF$$

where the coproduct is finite. By Lemma 3.3.7, for each component $a(\mathrm{Hom}(A_i, -)) \to aF$ there is a cover S_i of A_i such that for each $C \in \mathrm{fp}\,\mathbf{C}$ and $h \in S_iC$, the composition

$$a(\mathrm{Hom}(C, -)) \longrightarrow a(\mathrm{Hom}(A_i, -)) \longrightarrow aF$$

is of the form $a(f_{i,h})$ for some $f_{i,h} : \mathrm{Hom}(C, -) \to F$. Since the topology J is of finite type, we can assume that S_i is finitely presented, so there are finitely many C_k and $h_k \in S_iC_k$ and an epimorphism

$$\coprod_k \mathrm{Hom}(C_k, -) \twoheadrightarrow S_i$$

This gives an epimorphism

$$\coprod_k a(\mathrm{Hom}(C_k, -)) \twoheadrightarrow a(\mathrm{Hom}(A_i, -))$$

Given all this, and since a commutes with coproducts, it is clear that we can choose the A_i such that our epimorphism

$$\coprod_i a(\mathrm{Hom}(A_i, -)) \twoheadrightarrow aF$$

is of the form $a(f)$ for some $f : \coprod_i \mathrm{Hom}(A_i, -) \to F$. Put $P = \mathrm{im}\, f$. Then P is f.g. and $aP \cong aF \cong F$.

(b) Now suppose that F is f.p., so there is a presentation for F of the form

$$\coprod_i a(\mathrm{Hom}(A_i, -)) \underset{q}{\overset{p}{\rightrightarrows}} \coprod_j a(\mathrm{Hom}(B_j, -)) \longrightarrow F$$

where all coproducts are finite. The argument of part (a) applies here to give us that, when the A_i are chosen suitably, $p = a(f)$ and $q = a(g)$ for maps

$$\coprod_i \mathrm{Hom}(A_i, -) \underset{g}{\overset{f}{\rightrightarrows}} \coprod_j \mathrm{Hom}(B_j, -)$$

Since a commutes with colimits, $F \cong aP$ where P is the, clearly finitely presented, coequaliser of the above diagram. $\square$

For the remainder of this section, let $J = J_{\mathbf{D}}$ for a definable subcategory $\mathbf{D} \subseteq \mathbf{C}$. In this case we have the following neat characterisation of dense subobjects of presheaves (involving the extension of functors from $\mathrm{fp}\,\mathbf{C}$ to $\mathbf{C}$ defined earlier).

Lemma 3.3.17. *$P' \subseteq P$ is a dense inclusion of presheaves if and only if $P'(D) = P(D)$ for all $D \in \mathbf{D}$.*

Proof. Let $D \in \mathbf{D}$ and $x \in P(D)$. Then there is a f.p. object $C \in \mathbf{C}$, an element $y \in P(C)$ and a map $f : C \to D$ s.t. $f_*(y) = x$. Since P' is dense in P, we have that $(y^{-1}P')(D) = \mathrm{Hom}(C, D)$. So $f_*(y) \in P'(D)$, i.e. $x \in P'(D)$ as required.

For the converse, assume $P'(D) = P(D)$ for all $D \in \mathbf{D}$. Let $A \in \mathrm{fp}\,\mathbf{C}$, and $f : \mathrm{Hom}(A, -) \to P$ a natural transformation. Then our assumption implies that $(f^{-1}P')(D) = f^{-1}(P'D) = f^{-1}(PD) = \mathrm{Hom}(A, D)$. So by definition of $J_{\mathbf{D}}$, we have that $f^{-1}P'$ is dense in $\mathrm{Hom}(A, -)$. This implies that P' is dense in P. □

We are now in a position to prove a characterisation result for the category of finitely presented sheaves $\mathrm{fp}(\mathrm{Sh}(\mathrm{fp}\,\mathbf{C}, J_{\mathbf{D}}))$ where $\mathbf{D} \subseteq \mathbf{C}$ is a definable subcategory. Let P and Q be finitely presented presheaves. So $P \cong \varphi/\tilde{\theta}$ and $Q \cong \psi/\tilde{\eta}$ say. Let $E \subseteq \psi \times \psi$ be the functorial equivalence relation defined by

$$E(C) = \{(\boldsymbol{x}, \boldsymbol{y}) \in \psi(C) \times \psi(C) : \exists R \in J_{\mathbf{D}}(C) \text{ s.t. } \forall g \in R,\ \tilde{\eta}(g(\boldsymbol{x}), g(\boldsymbol{y}))\}$$

It is clear that $Q_{\mathrm{sep}} = \psi/E$.

Note that $\tilde{\eta} \subseteq E$ is dense. To see this, let $C \in \mathrm{fp}\,\mathbf{C}$ and suppose that $(\boldsymbol{x}, \boldsymbol{y}) \in E(C)$. We need to show that $(\boldsymbol{x}, \boldsymbol{y})^{-1}\tilde{\eta}$ is a cover of C. That is, $S = \{f \in \mathrm{Hom}(C, -) : \tilde{\eta}(f(\boldsymbol{x}), f(\boldsymbol{y}))\}$ is in $J_{\mathbf{D}}(C)$. Let $f : C \to D$ be a map with $D \in \mathbf{D}$. We know there is an $R \in J_{\mathbf{D}}(C)$ such that for all $g \in R$, $\tilde{\eta}(g(\boldsymbol{x}), g(\boldsymbol{y}))$ holds. But since $R(D) = \mathrm{Hom}(C, D)$ we have $f \in R(D)$ and so $\tilde{\eta}(f(\boldsymbol{x}), f(\boldsymbol{y}))$ which implies $f \in S$. This shows that S is a cover of C and that $\tilde{\eta} \subseteq E$ is dense.

We claim that E is in fact the closure of $\tilde{\eta}$ in $\psi \times \psi$ (that is, E is the largest subobject of $\psi \times \psi$ in which $\tilde{\eta}$ is dense). Let $E \subseteq F \subseteq \psi \times \psi$ be such that $E(D) = F(D)$ for all $D \in \mathbf{D}$. We want to show that $E = F$. So take $(\boldsymbol{x}, \boldsymbol{y}) \in FC$. This element defines a natural transformation $\alpha : \mathrm{Hom}(C,-) \to F$ and we can take the pullback as in the following diagram.

$$\begin{tikzcd} \alpha^{-1}E \arrow[r] \arrow[d, tail] & E \arrow[d, tail] \\ \mathrm{Hom}(C,-) \arrow[r] & F \end{tikzcd}$$

Since $E \subseteq F$ is dense, $\alpha^{-1}E$ is in $J_{\mathbf{D}}(C)$. For all $f \in \alpha^{-1}E$, $\alpha(f) = f_*(\boldsymbol{x},\boldsymbol{y}) = (f(\boldsymbol{x}), f(\boldsymbol{y})) \in E(\operatorname{cod} f)$. So there is a cover R_f such that for all $g \in R_f$, $\tilde{\eta}(gf(\boldsymbol{x}), gf(\boldsymbol{y}))$. So putting

$$T = \{gf : f \in \alpha^{-1}E, g \in R_f\}$$

we have that $T \in J_{\mathbf{D}}(C)$ and for all $h \in T$, $\tilde{\eta}(h(\boldsymbol{x}), h(\boldsymbol{y}))$. So $(\boldsymbol{x},\boldsymbol{y}) \in E(C)$ which means that $E = F$. This shows that E is indeed the closure of $\tilde{\eta}$ in $\psi \times \psi$.

Now let us write E as the directed union of positive existential formulas η_λ such that $\eta \subseteq \eta_\lambda \subseteq E$. Then $\psi/E = \varinjlim_\lambda \psi/\tilde{\eta}_\lambda$ as a directed colimit of finitely presented functors. Since $J_{\mathbf{D}}$ is a finite type topology, any dense subobject P' of P will have a dense f.p. object (of the form given by Corollary 3.2.9) below it. This can be seen from the following diagram.

$$\begin{tikzcd} \varphi' \arrow[r, two heads] \arrow[d, tail] & \alpha(\varphi') \arrow[d, tail] \\ \alpha^{-1}P' \arrow[r, two heads] \arrow[d, tail] & P' \arrow[d, tail] \\ \mathrm{Hom}(C,-) \arrow[r, "\alpha"'] & P \end{tikzcd}$$

Here $\alpha(\varphi')$ is the image of the restriction of α to the positive existential formula $\varphi' \in J_{\mathbf{D}}(C)$. Since $\alpha(\varphi')$ is the quotient of a f.p. functor, it must be f.g. But since P is f.p. and our functor category is locally coherent, $\alpha(\varphi')$ is in fact f.p. Moreover, it is easy to see that it is dense in P.

So we can assume that $\mathcal{D}(P)$ is the filter generated by $\{\varphi_\mu/\tilde{\theta}\}_\mu$ where $\varphi_\mu/\tilde{\theta}$ is dense in $\varphi/\tilde{\theta}$. Then

$$\mathrm{Nat}(aP, aQ) \cong \varinjlim_\mu \varinjlim_\lambda \mathrm{Nat}\left(\frac{\varphi_\mu}{\tilde{\theta}}, \frac{\psi}{\tilde{\eta}_\lambda}\right)$$

We know that any map $\varphi_\mu/\tilde{\theta} \to \psi/\tilde{\eta}_\lambda$ is given by a positive existential formula ρ (which defines a map on the whole of $\mathbf{C}$). Restricted to $\mathbf{D}$, ρ will define a map $\varphi/\tilde{\theta} \to \psi/\tilde{\eta}$. Moreover, any other representative for the class of ρ in the directed colimit will define the same map $\varphi/\tilde{\theta} \to \psi/\tilde{\eta}$ on $\mathbf{D}$.

Conversely, suppose that $\rho : \varphi/\tilde{\theta} \to \psi/\tilde{\eta}$ defines a map on $\mathbf{D}$. Without loss, we can asssume that $\forall \boldsymbol{x}, \boldsymbol{y}(\rho(\boldsymbol{x}, \boldsymbol{y}) \to \psi(\boldsymbol{y}))$ holds on $\mathbf{C}$. Let $\varphi_0(\boldsymbol{x})$ be the positive existential formula

$$\varphi(\boldsymbol{x}) \wedge \exists \boldsymbol{y} \rho(\boldsymbol{x}, \boldsymbol{y})$$

Then clearly $\varphi_0/\tilde{\theta}$ is dense in $\varphi/\tilde{\theta}$. Now let $\eta_0(\boldsymbol{y}, \boldsymbol{z})$ be the formula

$$\exists \boldsymbol{x}, \boldsymbol{x}'(\theta(\boldsymbol{x}, \boldsymbol{x}') \wedge \rho(\boldsymbol{x}, \boldsymbol{y}) \wedge \rho(\boldsymbol{x}', \boldsymbol{z})) \vee \eta(\boldsymbol{y}, \boldsymbol{z})$$

It is clear that $\tilde{\eta}_0(D) = \tilde{\eta}(D)$ for all $D \in \mathbf{D}$ so that $\eta \subseteq \eta_0 \subseteq E$. Note that ρ defines a map

$$\varphi_0/\tilde{\theta} \to \psi/\tilde{\eta}_0$$

Thus we obtain the following theorem.

Theorem 3.3.18. *Let* $\mathbf{D} \subseteq \mathbf{C}$ *be a definable subcategory. The category* $\mathrm{fp}(\mathrm{Sh}(\mathrm{fp}\,\mathbf{C}, J_{\mathbf{D}}))$ *has the following logical characterisation. Its objects are pairs of positive existential formulas of the form* $\varphi/\tilde{\theta}$ *and its morphisms are positive existential formulas which define maps* $\varphi/\tilde{\theta} \to \psi/\tilde{\eta}$ *when restricted to* $\mathbf{D}$.

3.3.6 Finitely generated and coherent sheaves

The above argument which yielded the characterisation result for f.p. sheaves may also be applied to f.g. sheaves. In fact the argument is slightly simpler. If $P \cong \varphi/E_1$ and $Q \cong \psi/E_2$ are two f.g. presheaves, then

$$\mathrm{Nat}(aP, aQ) \cong \varinjlim_{\lambda} \mathrm{Nat}(\varphi_\lambda/E_1, \psi/E)$$

where φ_λ is dense in φ, and $E \supseteq E_2$ is the closure of E_2 in $\psi \times \psi$ (so $\psi/E \cong Q_{\mathrm{sep}}$). Any map $\varphi_\lambda/E_1 \to \psi/E$ is given by a pe-formula (Proposition 3.2.7) which, when restricted to $\mathbf{D}$, defines a map $\varphi/E_1 \to \psi/E_2$. Two formulas representing the same maps in the directed colimit will define the same map on $\mathbf{D}$.

Conversely if ρ is a pe-formula which defines a map $\varphi/E_1 \to \psi/E_2$ when restricted to $\mathbf{D}$, then the formula φ_0 expressing (as above) that this map is total is dense in φ, and the equivalence relation generated by

$$E_2(\boldsymbol{y}, \boldsymbol{z}) \vee (\exists \boldsymbol{x}, \boldsymbol{x}'(E_1(\boldsymbol{x}, \boldsymbol{x}') \wedge \rho(\boldsymbol{x}, \boldsymbol{y}) \wedge \rho(\boldsymbol{x}', \boldsymbol{z})))$$

is contained in E (since E is the closure of E_2), so ρ defines a natural transformation $\varphi_0/E_1 \to \psi/E$. This gives us the following result.

Proposition 3.3.19. *Let $\mathbf{D} \subseteq \mathbf{C}$ be a definable subcategory. The category of finitely generated sheaves* $\mathrm{fg}(\mathrm{Sh}(\mathrm{fp}\,\mathbf{C}, J_{\mathbf{D}}))$ *is the category with objects φ/E where φ is a pe-formula and E is a functorial equivalence relation on φ, and with maps pe-formulas $\rho : \varphi/E_1 \to \psi/E_2$ which define actual maps on the subcategory* $\mathbf{D}$.

With our characterisation of the category of finitely generated sheaves $\mathrm{fg}(\mathrm{Sh}(\mathrm{fp}\,\mathbf{C}, J_{\mathbf{D}}))$ we can deduce a logical characterisation of the category of coherent sheaves $\mathrm{coh}(\mathrm{Sh}(\mathrm{fp}\,\mathbf{C}, J_{\mathbf{D}}))$.

Proposition 3.3.20. *The coherent objects of* $\mathrm{Sh}(\mathrm{fp}\,\mathbf{C}, J_{\mathbf{D}})$ *are those isomorphic to one of the form φ/θ where φ, θ are pe and θ defines an equivalence relation on φ when restricted to objects of the definable subcategory* $\mathbf{D}$.

Proof. Let F be a coherent sheaf. Then F is f.g. so we can write F in the form φ/E. Since F is coherent, there is a pullback diagram of the form

$$\begin{tikzcd} \theta'/E' \arrow[r, "\rho_1"] \arrow[d, "\rho_2"'] & \varphi \arrow[d] \\ \varphi \arrow[r] & \varphi/E \end{tikzcd}$$

in the sheaf category. By the previous proposition, we can assume that the maps ρ_i are pe-formulas which define functions on $\mathbf{D}$. So

$$E(\boldsymbol{x}, \boldsymbol{y}) \Leftrightarrow \exists \boldsymbol{z}(\theta'(\boldsymbol{z}) \wedge \rho_1(\boldsymbol{z}, \boldsymbol{x}) \wedge \rho_2(\boldsymbol{z}, \boldsymbol{y}))$$

holds on $\mathbf{D}$. Let $\theta(\boldsymbol{x}, \boldsymbol{y})$ be the pe-formula on the right hand side. Then θ defines an equivalence relation on $\mathbf{D}$ and $\varphi/\tilde{\theta} \cong \varphi/E$ in the sheaf category. We can write $\varphi/\tilde{\theta}$ simply as φ/θ where θ defines an equivalence relation on $\mathbf{D}$.

Now suppose $F \cong \varphi/\theta$ where θ defines an equivalence relation on $\mathbf{D}$. We want to show that F is coherent. Let $G \cong \psi/E$ be a f.g. sheaf and let $\rho : \psi/E \to \varphi/\theta$ be a pe-formula which defines a map on $\mathbf{D}$. If $\gamma(\boldsymbol{x}, \boldsymbol{y})$ is the formula

$$\exists \boldsymbol{w}, \boldsymbol{z}(\psi(\boldsymbol{x}) \wedge \psi(\boldsymbol{y}) \wedge \rho(\boldsymbol{x}, \boldsymbol{w}) \wedge \rho(\boldsymbol{y}, \boldsymbol{z}) \wedge \theta(\boldsymbol{w}, \boldsymbol{z}))$$

then $G \times_F G$ is a quotient of γ in the sheaf category, so $G \times_F G$ is finitely generated. Therefore F is coherent as required. $\square$

We describe the category $\operatorname{coh}(\operatorname{Sh}(\operatorname{fp}\mathbf{C}, J_{\mathbf{D}}))$ in the following theorem.

Theorem 3.3.21. *Let $\mathbf{D} \subseteq \mathbf{C}$ be definable. The category $\operatorname{coh}(\operatorname{Sh}(\operatorname{fp}\mathbf{C}, J_{\mathbf{D}}))$ has as objects pairs of pe-formulas φ/θ where θ defines an equivalence relation on φ on all objects of $\mathbf{D}$. The maps $\varphi/\theta \to \psi/\eta$ are given by pe-formulas ρ which define actual functions $\varphi/\theta \to \psi/\eta$ when restricted to objects of $\mathbf{D}$.*

This result can be found in [26] (see p. 269), although it is not so easy to recognise it as the same result. Makkai and Reyes' proof is very different from ours, as is their terminology.

Definable subcategories $\mathbf{D} \subseteq \mathbf{C}$ are categories of models of coherent theories. Given a coherent theory T, the category of sheaves $\operatorname{Sh}(\operatorname{fp}\mathbf{C}, J_{\operatorname{Mod} T})$ is a certain "universal" topos called the *classifying topos* of T (see [25, VIII] and [26, p. 293]). We have shown that the full subcategory of coherent objects in this topos is equivalent to the category with objects

$$\{\varphi/\theta : \varphi, \theta \text{ pe-formulas s.t. } T \vdash (\theta \text{ is an equivalence relation on } \varphi)\}$$

and with hom-sets $\operatorname{Hom}(\varphi/\theta, \psi/\eta)$ of the form

$$\{\rho : \rho \text{ is a pe-formula s.t. } T \vdash (\rho \text{ defines a function } \varphi/\theta \to \psi/\eta)\}$$

We will denote this category by $T^{\mathrm{eq}+}$ or $\mathbf{D}^{\mathrm{eq}+}$ where $\mathbf{D} = \operatorname{Mod} T$.

3.4 Interpretations

3.4.1 Interpretations as functors

Fix two first-order signatures $\mathcal{L}$ and $\mathcal{K}$. For simplicity of notation we write as if $\mathcal{L}$ and $\mathcal{K}$ are one-sorted but all our definitions and results are easily generalised to many-sorted languages. Let $\mathbf{C}$ be the category of $\mathcal{L}$-structures and $\mathbf{A}$ the category of $\mathcal{K}$-structures. Let $\mathbf{D} \subseteq \mathbf{C}$ and $\mathbf{B} \subseteq \mathbf{A}$ be definable subcategories. We define an *interpretation* of $\mathbf{B}$ in $\mathbf{D}$ to be an exact (i.e. commuting with finite limits and finite colimits) functor

$$\Gamma : \mathbf{B}^{\mathrm{eq+}} \longrightarrow \mathbf{D}^{\mathrm{eq+}}$$

Note that although the eq+ categories are closed under finite limits, they need not be closed under finite colimits. However it still makes sense to talk of an exact functor between such categories, meaning one which preserves finite limits and such finite colimits as do exist. In practice specific interpretations are usually defined by specifying data as in the next result.

Proposition 3.4.1. *Fix an object φ/θ of $\mathbf{D}^{\mathrm{eq+}}$. Associate to each n-ary function symbol $f \in \mathcal{K}$ a map*

$$\Gamma_f : \left(\frac{\varphi}{\theta}\right)^n \longrightarrow \frac{\varphi}{\theta}$$

of $\mathbf{D}^{\mathrm{eq+}}$, and to each m-ary relation symbol $R \in \mathcal{K}$ an object

$$\Gamma_R \subseteq \left(\frac{\varphi}{\theta}\right)^m$$

such that for each $D \in \mathbf{D}$ the $\mathcal{K}$-structure $(\frac{\varphi}{\theta}(D), \Gamma_f(D), \Gamma_R(D))_{f,R\in\mathcal{K}}$ is in $\mathbf{B}$. Then the function

$$\begin{aligned} x = x &\longmapsto \varphi/\theta \\ f &\longmapsto \Gamma_f \\ R &\longmapsto \Gamma_R \end{aligned}$$

has a unique extension to an interpretation $\Gamma : \mathbf{B}^{\mathrm{eq+}} \to \mathbf{D}^{\mathrm{eq+}}$.

(Constants can be treated as special cases of the above.)

Proof. We extend the function as follows. First put $\Gamma_{(x=x)^n} = (\varphi/\theta)^n$ for each n. Now, any term $t(x_1, \ldots, x_n)$ of $\mathcal{K}$ defines a map

$$t : (x = x)^n \longrightarrow x = x$$

of $\mathbf{B}^{\mathrm{eq}+}$. Suppose that t is the term $f(f_1(\boldsymbol{x}), \ldots, f_m(\boldsymbol{x}))$ where $f, f_1, \ldots, f_m$ are function symbols of $\mathcal{K}$. Then we put

$$\Gamma_t = \Gamma_f \circ (\Gamma_{f_1}, \ldots, \Gamma_{f_m})$$

We now define the action of Γ on a term equation $t(\boldsymbol{x}) = s(\boldsymbol{x})$ where $\boldsymbol{x} = (x_1, \ldots, x_n)$. The functor $t = s$ is an equaliser as shown.

$$t = s \longrightarrow (x = x)^n \underset{s}{\overset{t}{\rightrightarrows}} x = x$$

So we define $\Gamma_{t=s}$ to be the equaliser of Γ_t and Γ_s. Note that $\Gamma_{t=s}$ will indeed be an object of $\mathbf{D}^{\mathrm{eq}+}$ since this category has all finite limits.

We now consider an atomic formula $R(t_1, \ldots, t_k)$ in m free variables. Define $\Gamma_{R(t_1,\ldots,t_k)}$ to be $\Gamma^{-1}_{(t_1,\ldots,t_k)}(\Gamma_R)$ as in the following diagram.

$$\begin{tikzcd}
\Gamma^{-1}_{(t_1,\ldots,t_k)}(\Gamma_R) \arrow[r] \arrow[d] & \Gamma_R \arrow[d] \\
\left(\dfrac{\varphi}{\theta}\right)^m \arrow[r, "\Gamma_{(t_1,\ldots,t_k)}"'] & \left(\dfrac{\varphi}{\theta}\right)^k
\end{tikzcd}$$

Again, $\Gamma_{R(t_1,\ldots,t_k)}$ will be an object of $\mathbf{D}^{\mathrm{eq}+}$.

For a functor ψ of the form $\bigvee_{i=1}^n \bigwedge_{j=1}^m \psi_{ij}$ where ψ_{ij} is atomic, we define

$$\Gamma_\psi = \bigcup_i \bigcap_j \Gamma_{\psi_{ij}}$$

Suppose now that $\psi = \psi(\boldsymbol{x}, \boldsymbol{y})$ where $\boldsymbol{x}$ is of length n and $\boldsymbol{y}$ of length m. Then Γ_ψ embeds in $(\varphi/\theta)^{n+m}$. We define $\Gamma_{\exists \boldsymbol{y}\psi}$ to be the image of Γ_ψ under the projection map

$$(\varphi/\theta)^{n+m} \longrightarrow (\varphi/\theta)^n$$

So far, we have extended Γ to all positive existential formulas (as well as maps defined by terms). This map corresponds exactly to the syntactic interpretation map of the Reduction Theorem as described in [16, p. 214].

To complete the definition of Γ, we need to define its action on pairs of positive existential formulas ψ/η and maps between them. Note that ψ/η is the coequaliser of the two projection maps

$$\eta \rightrightarrows \psi$$

so we can define $\Gamma_{\psi/\eta}$ to be the coequaliser of the corresponding projection maps

$$\Gamma_\eta \rightrightarrows \Gamma_\psi$$

The fact that η is an equivalence relation on ψ is expressible by a $\mathcal{K}$-formula. Since the above process takes structures in $\mathbf{D}$ to structures in $\mathbf{B}$, the Reduction Theorem gives us that the corresponding $\mathcal{L}$-formula holds on $\mathbf{D}$ so that Γ_η is an equivalence relation on Γ_ψ. Hence $\Gamma_{\psi/\eta} \cong \Gamma_\psi/\Gamma_\eta$.

So how should Γ act on a positive existential map

$$\rho : \psi/\eta \to \chi/\zeta$$

of $\mathbf{B}^{\mathrm{eq}+}$? Note that ρ defines a subfunctor of $\psi \times \chi$ which satisfies the $\mathcal{K}$-formulas

$$\forall \boldsymbol{x}(\psi(\boldsymbol{x}) \to \exists \boldsymbol{y} \rho(\boldsymbol{x}, \boldsymbol{y}))$$
$$\forall \boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}, \boldsymbol{y}'(\eta(\boldsymbol{x}, \boldsymbol{x}') \wedge \rho(\boldsymbol{x}, \boldsymbol{y}) \wedge \rho(\boldsymbol{x}', \boldsymbol{y}') \to \zeta(\boldsymbol{y}, \boldsymbol{y}'))$$

on the definable subcategory $\mathbf{B}$. Regarding ρ as an object of $\mathbf{B}^{\mathrm{eq}+}$, Γ_ρ defines a subfunctor of $\Gamma_\psi \times \Gamma_\chi$. The two $\mathcal{K}$-formulas above have obvious translations to $\mathcal{L}$-formulas which say exactly that Γ_ρ defines a function

$$\Gamma_\rho : \Gamma_{\psi/\eta} \longrightarrow \Gamma_{\chi/\zeta}$$

These $\mathcal{L}$-formulas will hold on $\mathbf{D}$ by the Reduction Theorem. It is easy to check that this action of Γ on maps ρ is functorial.

This extension to a functor $\Gamma : \mathbf{B}^{\mathrm{eq}+} \to \mathbf{D}^{\mathrm{eq}+}$ is clearly the unique such extension to a functor commuting with finite limits and finite colimits. □

The above proof shows, amongst other things, that the interpretation functor Γ takes positive existential formulas to their syntactic translations in the sense of the

Reduction Theorem (see [16]). We will henceforth refer to the following restricted form of this theorem as the Reduction Theorem. Its proof is a direct consequence of the above proof and the proof in [16]. The notation Γ_φ is as in the proof of Proposition 3.4.1.

Theorem 3.4.2 (Reduction Theorem). *Let $\Gamma : \mathbf{B}^{\mathrm{eq}+} \to \mathbf{D}^{\mathrm{eq}+}$ be an exact functor. Fix $D \in \mathbf{D}$ and let $B = (\Gamma_{x=x}(D), \Gamma_f(D), \Gamma_R(D))_{f,R\in\mathcal{K}}$. Then, for any $([\boldsymbol{a}_1], \ldots, [\boldsymbol{a}_n]) \in \Gamma_{x=x}(D)$, and any pe $\mathcal{K}$-formula $\varphi(x_1, \ldots, x_n)$ we have*

$$B \models \varphi([\boldsymbol{a}_1], \ldots, [\boldsymbol{a}_n]) \Leftrightarrow D \models \Gamma_\varphi(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$$

From this it follows that an exact functor $\Gamma : \mathbf{B}^{\mathrm{eq}+} \to \mathbf{D}^{\mathrm{eq}+}$ induces a functor $\mathbf{D} \to \mathbf{B}$ which does indeed correspond to an interpretation in the sense of model theory: we are interpreting $\mathbf{B}$ in $\mathbf{D}$ in a definable way.

Proposition 3.4.3. *Let $\mathbf{B}$ and $\mathbf{D}$ be definable subcategories as above. An exact functor $\Gamma : \mathbf{B}^{\mathrm{eq}+} \to \mathbf{D}^{\mathrm{eq}+}$ induces a functor $\Gamma_{x=x} : \mathbf{D} \to \mathbf{B}$ via*

$$D \longmapsto (\Gamma_{x=x}(D), \Gamma_f(D), \Gamma_R(D))_{f,R\in\mathcal{K}}$$

Moreover, this functor commutes with directed colimits and pure embeddings.

Proof. We first need to show that for every $D \in \mathbf{D}$ the $\mathcal{K}$-structure $(\Gamma_{x=x}(D), \Gamma_f(D), \Gamma_R(D))_{f,R\in\mathcal{K}}$ is in $\mathbf{B}$. Suppose $\mathbf{B} = \operatorname{Mod} T$ for a coherent theory T. Choose a coherent axiom $\varphi \to \psi$ of T. Let $\rho(\boldsymbol{x}, \boldsymbol{y})$ be the relation defined by the pe-formula

$$\varphi(\boldsymbol{x}) \wedge \psi(\boldsymbol{y}) \wedge \boldsymbol{x} = \boldsymbol{y}$$

Then ρ defines a monomorphism $\varphi \to \psi$ in $\mathbf{B}^{\mathrm{eq}+}$. By left exactness of Γ, ρ is taken to a monomorphism $\Gamma_\rho : \Gamma_\varphi \to \Gamma_\psi$ in $\mathbf{D}^{\mathrm{eq}+}$, so

$$\mathbf{D} \models \Gamma_\varphi \to \Gamma_\psi$$

By the reduction theorem,

$$(\Gamma_{x=x}(D), \Gamma_f(D), \Gamma_R(D))_{f,R\in\mathcal{K}} \models \varphi \to \psi$$

for every $D \in \mathbf{D}$.

That the association

$$D \longmapsto (\Gamma_{x=x}(D), \Gamma_f(D), \Gamma_R(D))_{f \in \mathcal{K}}$$

is functorial is obvious. So we indeed have a functor $\Gamma_{x=x} : \mathbf{D} \to \mathbf{B}$. It remains to show that this functor commutes with directed colimits and pure embeddings.

Since $\Gamma_{x=x}$, Γ_f and Γ_R have expressions as quotients of pe-formulas, which we know commute with directed colimits, we have the first statement. As for the second, suppose that $D \subseteq D'$ is a pure embedding in $\mathbf{D}$. Then $\Gamma_{x=x}(D) \to \Gamma_{x=x}(D')$ is a monomorphism. For suppose $\Gamma_{x=x} \cong \varphi/\theta$. Let $\boldsymbol{a}, \boldsymbol{b} \in \varphi(D)$ such that $D \models \theta(\boldsymbol{a}, \boldsymbol{b})$. By purity, $D \models \theta(\boldsymbol{a}, \boldsymbol{b})$, so $\varphi/\theta(D) \to \varphi/\theta(D')$ is monic.

To show that the embedding $\varphi/\theta(D) \to \varphi/\theta(D')$ is pure, let ψ be a positive existential $\mathcal{K}$-formula and let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \varphi(X)$ such that

$$\left(\frac{\varphi}{\theta}(D'), \Gamma_f(D'), \Gamma_R(D')\right)_{f,R \in \mathcal{K}} \models \psi([\boldsymbol{a}_1], \ldots, [\boldsymbol{a}_n])$$

We can now apply the reduction theorem to get

$$D' \models \Gamma_\psi(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$$

Since $D \subseteq D'$ is pure, we have

$$D \models \Gamma_\psi(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$$

and so, by applying the reduction theorem once more, we have that

$$\left(\frac{\varphi}{\theta}(D), \Gamma_f(D), \Gamma_R(D)\right)_{f,R \in \mathcal{K}} \models \psi([\boldsymbol{a}_1], \ldots, [\boldsymbol{a}_n])$$

as required. □

It is easy to see that interpretations behave well with respect to composition. For suppose that we have interpretations

$$\mathbf{B}^{\mathrm{eq+}} \xrightarrow{\Gamma} \mathbf{D}^{\mathrm{eq+}} \xrightarrow{\Delta} \mathbf{F}^{\mathrm{eq+}}$$

where $\Gamma_{x=x} = \varphi/\theta$ and $\Delta_{x=x} = \psi/\eta$. Then one can check that $\Gamma_{x=x} \circ \Delta_{x=x} = \Delta_{\varphi/\theta}$ so

$$(\Delta \circ \Gamma)_{x=x} = \Gamma_{x=x} \circ \Delta_{x=x}$$

So now that we have a nice definition of interpretation, we can give a relatively simple and natural definition of a bi-interpretation. We say that a pair (Γ, Δ) is a *bi-interpretation* between definable subcategories $\mathbf{B}$ and $\mathbf{D}$ if

$$\mathbf{B}^{\mathrm{eq}+} \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \mathbf{D}^{\mathrm{eq}+}$$

are interpretations such that

$$(\Gamma \circ \Delta)_{x=x} \cong x = x$$
$$(\Delta \circ \Gamma)_{x=x} \cong x = x$$

as objects of $\mathbf{D}^{\mathrm{eq}+}$ and $\mathbf{B}^{\mathrm{eq}+}$ respectively. So these natural isomorphisms will be given by positive existential formulas, corresponding to the notion of definable isomorphism in the classical definition of a bi-interpretation.

We shall give some examples of interpretations and bi-interpretations in the next section. But first we show that one can recover a definable subcategory $\mathbf{D}$ as the category of exact functors on $\mathbf{D}^{\mathrm{eq}+}$. Our notation for this category of exact functors is $\mathrm{Ex}(\mathbf{D}^{\mathrm{eq}+}, \mathbf{Set})$. As mentioned earlier, in the additive case this was noticed by Herzog but in the category-theoretic literature such results go back at least as far as [26].

Proposition 3.4.4. *Given a definable subcategory $\mathbf{D}$ in a first-order language $\mathcal{L}$,*

$$\mathbf{D} \simeq \mathrm{Ex}(\mathbf{D}^{\mathrm{eq}+}, \mathbf{Set})$$

Proof. We define a functor

$$\mathrm{ev} : \mathbf{D} \longrightarrow (\mathbf{D}^{\mathrm{eq}+}, \mathbf{Set})$$

by $\mathrm{ev}_D(F) = F(D)$ and $\mathrm{ev}_f(F) = F(f)$. It is easily checked that ev_D is an exact functor $\mathbf{D}^{\mathrm{eq}+} \to \mathbf{Set}$, so ev maps to $\mathrm{Ex}(\mathbf{D}^{\mathrm{eq}+}, \mathbf{Set})$. It is also easy to see that ev is a faithful functor. For suppose $f, g : X \to Y$ are parallel arrows in $\mathbf{D}$ such that $\mathrm{ev}_f = \mathrm{ev}_g$. Then $\mathrm{ev}_f(x = x) = \mathrm{ev}_g(x = x)$, so $f = g$.

Now suppose that we have a map $\Phi : \mathrm{ev}_D \to \mathrm{ev}_E$ in $(\mathbf{D}^{\mathrm{eq}+}, \mathbf{Set})$. Evaluate Φ at $(x = x) \in \mathbf{D}^{\mathrm{eq}+}$ to obtain a map $f : D \to E$ of sets. For each n consider the map

$\bigwedge_{i=1}^{n}(x_i = x_i) \to (x_k = x_k)$ of $\mathbf{D}^{\mathrm{eq}+}$ corresponding to projection onto the kth factor. The naturality of Φ gives us the following commutative square.

$$\begin{tikzcd} D^n \arrow[r, "\Phi_{(x=x)^n}"] \arrow[d, "\pi_k"'] & E^n \arrow[d, "\pi_k"] \\ D \arrow[r, "f"'] & E \end{tikzcd}$$

By letting k vary we see that $\Phi_{(x=x)^n} = f^n$.

Now, for every n-ary function symbol $\sigma : (x = x)^n \to (x = x)$ naturality of Φ gives a commutative square

$$\begin{tikzcd} D^n \arrow[r, "f^n"] \arrow[d, "\sigma_D"'] & E^n \arrow[d, "\sigma_E"] \\ D \arrow[r, "f"'] & E \end{tikzcd}$$

and, for every relation symbol $R \subseteq (x = x)^m$, a commutative square

$$\begin{tikzcd} R(D) \arrow[r, "\Phi_R"] \arrow[d, tail] & R(E) \arrow[d, tail] \\ D^m \arrow[r, "f^m"'] & E^m \end{tikzcd}$$

Thus f is a homomorphism $D \to E$. That is, $f \in \mathbf{D}$.

Since every pe-formula φ in n variables defines a subobject of $(x = x)^n$ in $\mathbf{D}^{\mathrm{eq}+}$, it is easy to see that Φ_φ is the restriction of f^n to $\varphi(X)$. Also since φ/θ is a quotient object of φ in $\mathbf{D}^{\mathrm{eq}+}$, one can deduce that $\Phi_{\varphi/\theta}$ takes an equivalence class $[\boldsymbol{x}]$ in $\varphi/\theta(X)$ to the equivalence class $[f^n(\boldsymbol{x})]$ in $\varphi/\theta(Y)$. Hence $\Phi = \mathrm{ev}_f$. That is, the functor ev is full.

We now show that ev is an equivalence. Let $\Gamma : \mathbf{D}^{\mathrm{eq}+} \to \mathbf{Set}$ be any exact functor. Let D be the $\mathfrak{L}$-structure $(\Gamma(x = x), \Gamma(f), \Gamma(R))_{f,R \in \mathfrak{L}}$. Using the exactness of Γ, it is easily checked that $\Gamma(\varphi) = \varphi(D)$ for every pe-formula φ. We claim that $D \in \mathbf{D}$. Suppose that $\mathbf{D} = \mathrm{Mod}\, T$ for a coherent theory T. Let $\varphi \to \psi$ be a coherent axiom of T. Let $\rho(\boldsymbol{x}, \boldsymbol{y})$ be the relation defined by

$$\varphi(\boldsymbol{x}) \wedge \psi(\boldsymbol{y}) \wedge \boldsymbol{x} = \boldsymbol{y}$$

Then ρ defines a monomorphism $\varphi \to \psi$ in $\mathbf{D}^{\mathrm{eq}+}$. By exactness of Γ, $\Gamma(\rho) : \Gamma(\varphi) \to \Gamma(\psi)$ is a monomorphism of sets, so $\Gamma(\varphi) \subseteq \Gamma(\psi)$, i.e. $\varphi(D) \subseteq \psi(D)$, that is

$$D \models \varphi \to \psi$$

Thus $D \in \mathbf{D} = \mathrm{Mod}\, T$. Clearly $\Gamma \cong \mathrm{ev}_D$, so ev is indeed an equivalence as required.

$\square$

3.4.2 Examples of interpretations

Let $\mathcal{L} = \mathcal{K}$ be the language of rings. Let $\mathbf{D}$ be the category of real closed fields and $\mathbf{B}$ the category of algebraically closed fields. It is easy to see that $\mathbf{D}$ and $\mathbf{B}$ are definable subcategories. The standard two dimensional interpretation of an algebraically closed field in a real closed field is an instance of an interpretation in our sense.

In fact any classical interpretation by positive existential formulas of a $\mathcal{K}$-structure B in an $\mathcal{L}$-structure D can be thought of as an interpretation in our sense between the category of $\mathcal{K}$-structures and a suitable definable subcategory $\mathbf{D}$ of the category of $\mathcal{L}$-structures. For suppose that

$$B \cong \left(\frac{\varphi}{\theta}(D), \Gamma_f(D), \Gamma_R(D)\right)_{f,R\in\mathcal{K}}$$

where $\theta(D)$ is an equivalence relation on $\varphi(D)$. Let $\mathbf{D}$ be the category of $\mathcal{L}$-structures on which θ is an equivalence relation and the Γ_f are definable maps $(\varphi/\theta)^n \to \varphi/\theta$, where n is the arity of f. The sentences expressing these facts can be checked to be coherent sentences. This implies that $\mathbf{D}$ is a definable subcategory and it can easily be seen that this is an instance of an interpretation $(\mathcal{K}\text{-}\mathbf{Str})^{\mathrm{eq}+} \to \mathbf{D}^{\mathrm{eq}+}$ via the map

$$\begin{array}{ccc} x = x & \longmapsto & \varphi/\theta \\ (f \in \mathcal{K}) & \longmapsto & \Gamma_f \end{array}$$

We now look at an example of a bi-interpretation. This example shows that our theory is also well-suited to what is traditionally regarded as the domain of abelian model/category theory. Let Q be the quiver

$$1 \bullet \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \bullet\, 2$$

and let k be any field. Then the path algebra of Q over k, written kQ, has a basis $\{e_1, e_2, \alpha, \beta\}$ over k. For any kQ-module M, left multiplication by α and β restricted to e_1M define maps of e_1M into e_2M. Let $\mathbf{D}$ be the definable subcategory of kQ-**Mod** consisting of those modules in which this restriction of α is invertible. Define the interpretation

$$\Gamma : (k[X]\text{-}\mathbf{Mod})^{\mathrm{eq}+} \longrightarrow \mathbf{D}^{\mathrm{eq}+}$$

by putting $\Gamma_{x=x} = \exists z(x = e_2 z)$ and $\Gamma_X(x, y) = \exists z(\alpha e_1 z = x \wedge \beta e_1 z = y)$. In other words, $\Gamma_{x=x}(M) = e_2M$ and $\Gamma_X = \beta\alpha^{-1} : e_2M \to e_2M$.

Now let Δ be the interpretation

$$\Delta : \mathbf{D}^{\mathrm{eq}+} \longrightarrow (k[X]\text{-}\mathbf{Mod})^{\mathrm{eq}+}$$

defined by $\Delta_{x=x}(M) = (x = x \wedge y = y)$ and

$$\begin{aligned}
\Delta_\alpha : (x, y) &\longmapsto (0, x)\\
\Delta_\beta : (x, y) &\longmapsto (0, X(x))\\
\Delta_{e_1} : (x, y) &\longmapsto (x, 0)\\
\Delta_{e_2} : (x, y) &\longmapsto (0, y)
\end{aligned}$$

One can check that with this definition one does indeed get a kQ-module structure on the direct sum $M \oplus M$ of the $k[X]$-module M. We claim that the pair (Γ, Δ) gives a bi-interpretation between the categories $\mathbf{D}$ and $k[X]$-**Mod**. To see this, note that $\Delta_{x=x} \circ \Gamma_{x=x}(M) = e_2M \oplus e_2M$ and the natural isomorphism

$$M \xrightarrow{\cong} e_2M \oplus e_2M$$

is given by $x \mapsto (\alpha e_1 x, e_2 x)$.

Conversely, $\Gamma_{x=x} \circ \Delta_{x=x}(M) = M$. So we do indeed have a bi-interpretation.

3.4.3 Interpretations and coherent morphisms

Recall that a coherent topos is a category of sheaves for a coherent site. It is a result of topos theory (see [19, Vol. 2, p. 910]) that any coherent topos is the classifying topos of a coherent theory. Hence, any coherent topos will be of the form $\mathrm{Sh}(\mathrm{fp}\,\mathbf{C}, J_{\mathbf{D}})$ for an LFP category $\mathbf{C}$ and a definable subcategory $\mathbf{D} \subseteq \mathbf{C}$, since this category of sheaves is precisely the classifying topos for the coherent theory defining $\mathbf{D}$ (see [26, p. 293]). It follows that the results of this chapter apply to general coherent toposes.

Let $\mathcal{E}$ be a coherent topos and let J be the finite type *right* Grothendieck topology on $\mathrm{coh}\,\mathcal{E}$ generated by finite epimorphic families. More precisely, we say that $S \subseteq \mathrm{Hom}(-, C)$ is in $J(C)$ if and only if there are finitely many maps $s_i \in S$ such that $\coprod_i s_i$ is an epimorphism onto C. Then, it turns out that $\mathcal{E} \simeq \mathrm{Sh}(\mathrm{coh}\,\mathcal{E}, J)$ (see [18, 7.3.1], [27, p. 1]). Suppose now that $\mathcal{E}$ and $\mathcal{F}$ are two coherent toposes and $\Gamma : \mathrm{coh}\,\mathcal{E} \to \mathrm{coh}\,\mathcal{F}$ is an interpretation. Then since Γ commutes with finite colimits, it will take finite epimorphic families to finite epimorphic families. Let J and K be the finite type right Grothendieck topologies generated by finite epimorphic families on $\mathrm{coh}\,\mathcal{E}$ and $\mathrm{coh}\,\mathcal{F}$ respectively. Then Γ has the property that if $S \in J(C)$, then the set

$$\{\Gamma(s) : s \in S\}$$

generates a cover of $\Gamma(C)$ (i.e. the smallest subfunctor of $\mathrm{Hom}(-, \Gamma(C))$ containing the set is a cover of $\Gamma(C)$).

Whenever there are two small categories with finite limits $\mathbf{A}$ and $\mathbf{B}$, both endowed with a Grothendieck topology, and a left exact functor $F : \mathbf{A} \to \mathbf{B}$ such that for any $A \in \mathbf{A}$ and cover S of A the set $\{F(s) : s \in S\}$ generates a cover of $F(A)$, F is said to be a *morphism of sites*. So Γ above is a morphism of sites $(\mathrm{coh}\,\mathcal{E}, J) \to (\mathrm{coh}\,\mathcal{F}, K)$. As such, it will induce a map of sheaves

$$\Phi : \mathrm{Sh}(\mathrm{coh}\,\mathcal{F}, K) \longrightarrow \mathrm{Sh}(\mathrm{coh}\,\mathcal{E}, J)$$

The direct image functor Φ_* is simply $\Phi_*(F) = F \circ \Gamma$. The inverse image functor Φ^* is slightly more complicated and its definition requires left Kan extensions (see [19, Vol. 1, A4.1.4]). However, in this case $\mathcal{E} \simeq \mathrm{Sh}(\mathrm{coh}\,\mathcal{E}, J)$, and $\Phi^* \restriction_{\mathrm{coh}\,\mathcal{E}} = \Gamma$. In particular Φ^* takes coherent objects to coherent objects. Any geometric morphism between coherent toposes which has the property that the inverse image functor restricts to a functor on coherent objects is called a *coherent morphism*. So any interpretation $\Gamma : \mathrm{coh}\,\mathcal{E} \to \mathrm{coh}\,\mathcal{F}$ induces a coherent morphism $\mathcal{F} \to \mathcal{E}$, and conversely, any coherent morphism $\Phi : \mathcal{F} \to \mathcal{E}$ induces an interpretation $\Phi^* \restriction_{\mathrm{coh}\,\mathcal{E}} \mathrm{coh}\,\mathcal{E} \to \mathrm{coh}\,\mathcal{F}$ (since the inverse image functor preserves finite limits by definition and preserves colimits by the fact that it is a left adjoint).

In summary, our interpretation functors defined at the start of this section correspond exactly to coherent geometric morphisms between coherent toposes (objects which have been studied in the category-theoretic literature). As we have seen, these abstractly defined maps can be regarded as genuine model-theoretic interpretations by positive existential formulas. The relationship between our interpretations and coherent morphisms was suggested to us by Tibor Beke.

Chapter 4

An example: G-sets

4.1 Overview

A good candidate for a nonadditive category where ideas related to the Ziegler spectrum for modules may be developed is the category of left G-sets for a group G. (This is more or less the category of vector spaces over a field without the abelian group structure.) We denote this category by G-**Set**. It can be thought of as the category of covariant set-valued functors on the category $\{G\}$ with one object such that $\mathrm{Hom}(G, G) = G$ and composition of maps is simply group multiplication. It is a locally finitely presented topos (by [1, 1.12]) where the regular left G-set (which is the functor $\mathrm{Hom}(G, -) : \{G\} \to \mathbf{Set}$ represented by the unique object G) is a strong f.p. generator. A G-set is said to be *indecomposable* if it is not the coproduct of two proper sub-G-sets. Coproducts in G-**Set** are simply disjoint unions. Clearly a G-set is indecomposable if and only if it is transitive, and every G-set decomposes into a coproduct of indecomposable G-sets.

In this chapter we will investigate notions of purity and pure-injectivity of G-sets and the possiblity of defining a Ziegler spectrum in this context. In the case of modules, one shows that definable subcategories are determined by the indecomposable pure-injective modules they contain. In particular, a non-trivial definable subcategory of modules must contain a (non-trivial) indecomposable pure-injective module. One can show this by taking any non-trivial module M in the definable subcategory **D**. The *pure-injective hull* of M, denoted $H(M)$ will also be in **D**. If $H(M)$ decomposes into a direct sum of indecomposables, then $H(M)$ will contain an indecomposable pure-injective direct summand which will also be in **D**. If $H(M)$ does not decompose into indecomposables, it will be elementarily equivalent to a pure-injective module which does so decompose. That module will also be in **D** and it will contain an indecomposable pure-injective direct summand which is in **D**.

Unfortunately this argument doesn't work in the case of G-sets. Despite the fact that every G-set decomposes into indecomposables, there is a pure-injective G-set which does not have an indecomposable pure-injective component (Corollary 4.3.17). We will show that there are non-trivial definable subcategories which contain no non-trivial indecomposables whatsoever (Proposition 4.4.2). Although this implies that the set of indecomposable pure-injectives is not the right set of G-sets on which to define a Ziegler topology (whose closed sets correspond to coherent theories), there is a larger set which does work. However, an explicit description of the resulting space in specific examples appears to be hard.

4.2 Basic facts about the category G-Set

4.2.1 Finitely presented G-sets

By Proposition 2.2.1 a G-set X is finitely presented if and only if there is a coequaliser of the form

$$G^{(m)} \rightrightarrows G^{(n)} \longrightarrow X$$

where for a G-set Y, $Y^{(k)}$ is the coproduct or equivalently, disjoint union, of k copies of Y.

Here is another characterisation (which we have not found in the literature).

Proposition 4.2.1. *A G-set X is f.p. if and only if X has finitely many orbits and the stabiliser of every element is finitely generated.*

Proof. Suppose that X is finitely presented. So there is a coequaliser of the form

$$\coprod_{i=1}^{m} G_i \underset{p_2}{\overset{p_1}{\rightrightarrows}} \coprod_{j=1}^{n} G_j \xrightarrow{q} X$$

where $G_i = G_j = G$ for all i, j. Since q is epic it is clear that X has at most n orbits. Now we must show that the stabiliser of every element is finitely generated.

Let q_j be the jth component of q and let $p_{1,i}$ and $p_{2,i}$ be the ith components of p_1 and p_2 respectively. $p_{1,i}$ and $p_{2,i}$ are maps

$$G_i(= G) \longrightarrow G_{j_1(i)}(= G) \quad \text{and} \quad G_i(= G) \longrightarrow G_{j_2(i)}(= G)$$

respectively. It is easy to see that $\mathrm{Hom}(G, G) \cong G$ where each $g \in G$ corresponds to the right multiplication map $r(g) : G \to G$. We fix the notation $p_{1,i} = r(g_{1,i})$ and $p_{2,i} = r(g_{2,i})$.

We claim that we can assume that $j_1(i) = j_2(i)$ for each i. For suppose that $j_1(i) \neq j_2(i)$ for some i. Let

$$f = r(g_{1,i}^{-1} g_{2,i}) : G_{j_1(i)} \longrightarrow G_{j_2(i)}$$

Then

$$p_{2,i} = f p_{1,i} \quad \text{and} \quad q_{j_1(i)} = q_{j_2(i)} f$$

Since f is an isomorphism, we can regard it as an *identification* of $G_{j_1(i)}$ with $G_{j_2(i)}$ so that, under this identification, $p_{1,i} = p_{2,i}$ and $q_{j_1(i)} = q_{j_2(i)}$. So we can eliminate $G_{j_1(i)}$ say from the coequaliser and replace all maps $p_{1,i'}$ (and $p_{2,i'}$) such that $j_1(i') = j_1(i)$ (resp. $j_2(i') = j_1(i)$) with $fp_{1,i'}$ (resp. $fp_{2,i'}$). If we go through this process of elimination step by step (modifying the coequaliser every time we make an elimination as above) we will reach the stage where $j_1(i) = j_2(i)$ for each i.

So assume from now on that $j_1(i) = j_2(i)$ for each i. We write simply $j(i)$ for $j_1(i) = j_2(i)$. For each $j \in \{1, \dots, n\}$ let H_j be the subgroup of G generated by the elements

$$\{g_{1,i}^{-1} g_{2,i} : j(i) = j\}$$

Note that H_j is trivial when $j(i) \neq j$ for all i. The group H_j stabilises $q_j(1_j)$, since if $j(i) = j$, then

$$g_{1,i}^{-1} g_{2,i} q_j(1_j) = g_{1,i}^{-1} q_j(g_{2,i}) = g_{1,i}^{-1} q_j p_{2,i}(1_i) = g_{1,i}^{-1} q_j p_{1,i}(1_i) = g_{1,i}^{-1} q_j(g_{1,i}) = q_j(1_j)$$

But the converse is also true: if $gq_j(1_j) = q_j(1_j)$, then $g \in H_j$. To see this, let $\varphi_j : G_j \to G/H_j$ be the map which sends 1_j to H_j and let $\varphi = \coprod_j \varphi_j$. Then $\varphi p_1 = \varphi p_2$ since for each i,

$$\varphi p_{1,i}(1_i) = \varphi_{j(i)}(g_{1,i}) = g_{1,i}H_{j(i)} = g_{2,i}H_{j(i)} = \varphi_{j(i)}(g_{2,i}) = \varphi p_{2,i}(1_i)$$

By the universal property of the coequaliser, φ factors through q as $\varphi = \varphi' q$ say. We have

$$\varphi'(q_j(1_j)) = \varphi_j(1_j)) = H_j$$

So $\mathrm{Stab}(q_j(1_j)) \leq \mathrm{Stab}(H_j) = H_j$. So $\mathrm{Stab}(q_j(1_j)) = H_j$ which is finitely generated. Now any other element $x \in X$ will be in the orbit of some $q_j(1_j)$ and so its stabiliser will be a conjugate of H_j and hence also be finitely generated. This proves one direction of the proposition.

For the other direction, assume X has finitely many orbits and that the stabiliser of each element is finitely generated. So the decomposition of X into transitive G-sets looks like

$$\coprod_{j=1}^{n} G/H_j$$

where each H_j is a f.g. subgroup of G. Suppose H_j is generated by the elements $\{x_{j,1}, \ldots, x_{j,n_j}\}$. For each j and $k \in \{1, \ldots, n_j\}$ let

$$p_{jk} : G_{jk}(= G) \longrightarrow G_j(= G)$$

be the map $r(x_{j,k})$ and let id_{jk} be the identity $G_{jk} \to G_j$. Let q_j be the quotient map $G_j \to G/H_j$. Put $p_1 = \coprod_{jk} p_{jk}$, $p_2 = \coprod_{jk} \mathrm{id}_{jk}$ and $q = \coprod_j q_j$. Then the diagram below is a coequaliser.

$$\coprod_{j=1}^{n} \left(\coprod_{k=1}^{n_j} G_{jk} \right) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \coprod_{j=1}^{n} G_j \xrightarrow{q} X$$

It is easy to see that $qp_1 = qp_2$ so we need only show that q is universal with this property. To this end, let Y be an arbitrary G-set and φ a map $\coprod_j G_j \to Y$ such that $\varphi p_1 = \varphi p_2$. Let φ_j be the jth component of φ. Then for every k, $\varphi_j(x_{j,k}) = \varphi_j(1_j)$ so that $H_j \leq \mathrm{Stab}(\varphi_j(1_j))$. But this means there is a unique map $\varphi'_j : G/H_j \to Y$ such that $\varphi_j = \varphi'_j q_j$. By taking coproducts over j, we get a unique map $\varphi' : X \to Y$ such that $\varphi = \varphi' q$ as required. $\square$

4.2.2 Projective and injective G-sets

In order to define a Ziegler spectrum for the category G-**Set** we need to study the pure-injective objects. We will begin by studying the injective and, dually, the projective ones.

Proposition 4.2.2.

(a) *A G-set X is injective if and only if there is some element $x \in X$ with* $\mathrm{Stab}(x) = G$.

(b) *X is projective if and only if for every element $x \in X$,* $\mathrm{Stab}(x) = 1$.

Proof. (a) Suppose that X is injective. Let $\{*\}$ be the one-point G-set with $\mathrm{Stab}(*) = G$. X embeds into $X \coprod \{*\}$. By injectivity of X there is a map $f : X \coprod \{*\} \to X$ of G-sets such that $f \restriction_X = \mathrm{id}_X$. Let $x = f(*)$. Then for any $g \in G$, $gx = f(g*) = f(*) = x$. So $\mathrm{Stab}(x) = G$.

Conversely, suppose that $x_0 \in X$ has G as its stabiliser. Let $X \subseteq Y$ be an inclusion of G-sets. Then we can define a map $f : Y \to X$ by $f(x) = x$ for all $x \in X$ and $f(y) = x_0$ for $y \in Y \setminus X$. Clearly f is a map of G-sets which splits the embedding $X \subseteq Y$.

(b) Suppose that X is projective. Let $Y = \coprod_{x \in X} G_x$ be the disjoint union of copies of the regular left G-set G indexed by X. Then there is an epimorphism $f : Y \to X$ which takes the identity in G_x to $x \in X$. Since X is projective, there is a map $f' : X \to Y$ such that $ff' = \mathrm{id}_X$. Suppose $gx = x$ in X. Then $gf'(x) = f'(gx) = f'(x)$ in G_x which implies that $g = 1$.

Conversely, suppose that the stabiliser of every element in X is trivial. Then X must decompose as $\coprod_i G_i$ where $G_i = G$ for every i. Suppose we have an epimorphism $f : Y \to X$. Let 1_i be the identity of G_i and let $f'(1_i)$ be an element of $f^{-1}\{1_i\}$. Then $f' : X \to Y$ which takes $g \in G_i$ to $gf'(1_i) \in Y$ is a section of f. So X must be projective. □

Corollary 4.2.3.

(a) *$\{*\}$ is the unique indecomposable injective G-set and every other injective G-set has the form $X \coprod \{*\}$.*

(b) *The regular left G-set G is the unique indecomposable projective object (which is also a projective generator for G-**Set**) and any other projective G-set is isomorphic to a disjoint union of copies of G.*

Given a G-set X which is not injective, one can always embed X into the injective object $X \coprod \{*\}$. This is in fact the *injective hull* of X. The *injective hull* of an object X is an injective object which contains X as an *essential subobject.* An inclusion $X \subseteq Y$ is *essential* if any map $f : Y \to Z$ with $f \restriction_X$ an embedding is itself an embedding. It is easy to deduce that the injective hull of X is unique up to isomorphism over X (thereby justifying our use of the definite article).

Proposition 4.2.4. *Let X be a noninjective G-set. Then, $X \coprod \{*\}$ is the injective hull of X.*

Proof. Let $f : X \coprod \{*\} \to Y$ be a map such that $f \restriction_X$ is an injection. Suppose f is not an injection. Then there is an $x_0 \in X$ with $f(*) = f(x_0)$. Since X is not injective, there is a $g \in G$ such that $gx_0 \neq x_0$. Now $f(x_0) = f(*) = f(g*) = gf(*) = gf(x_0) = f(gx_0)$. But since $f \restriction_X$ is an injection, we must have $x_0 = gx_0$ which is a contradiction. So the embedding $X \subseteq X \coprod \{*\}$ is essential. □

4.2.3 Formulas in the language of G-sets

As in the case of modules over a ring, there is a natural language for G-sets in which we can prove a quantifier elimination result. This language has function symbols corresponding to the elements of G as its only nonlogical symbols. We denote this language by $\mathcal{L}_G$. We use the same notation for these function symbols as for elements of G and no confusion should arise. The theory of G-sets, T_G, is axiomatised by the sentences

$$\forall x(1x = x)$$

$$\forall x((gh)x = g(h(x)))$$

In any G-set we can define the following sets.

$$\operatorname{supp}(g) = \{x : gx \neq x\}$$
$$\operatorname{fix}(g) = \{x : gx = x\}$$

For a tuple $\boldsymbol{g}$ from G, we define

$$\operatorname{supp}(\boldsymbol{g}) = \bigcap_i \operatorname{supp}(g_i)$$
$$\operatorname{fix}(\boldsymbol{g}) = \bigcap_i \operatorname{fix}(g_i)$$

Theorem 4.2.5. *Every formula of $\mathfrak{L}_G$ is equivalent modulo T_G to a Boolean combination of formulas of the form "$|\operatorname{supp}(\boldsymbol{g}) \cap \operatorname{fix}(\boldsymbol{h})| \geq k$", "$x \in \operatorname{supp}(\boldsymbol{g}) \cap \operatorname{fix}(\boldsymbol{h})$", and $gx = y$.*

Proof. We can eliminate one existential quantifier at a time. First consider the formula

$$\exists x \, (x \in \operatorname{supp}(\boldsymbol{g}) \cap \operatorname{fix}(\boldsymbol{h}) \wedge s_1 y = x \wedge s_2 y = x \wedge s'_1 y' = x \wedge s'_2 y' = x)$$

with free variables y, y'. With a little thought it can be seen that this formula is equivalent to

$$y \in \operatorname{supp}(s_1^{-1} \boldsymbol{g} s_1) \cap \operatorname{fix}(s_1^{-1} \boldsymbol{h} s_1, s_2^{-1} s_1, s_1^{-1} s'_2 (s'_1)^{-1} s_2) \wedge y = s_1^{-1} s'_1 y'$$

With this example in mind, it should be clear that it is sufficient to show that a formula $\psi(y_1, \ldots, y_n)$ of the form

$$\exists x \left(x \in \operatorname{supp}(\boldsymbol{g}) \cap \operatorname{fix}(\boldsymbol{h}) \wedge \bigwedge_{i=1}^{n} (y_i \neq s_{i1} x \wedge \cdots \wedge y_i \neq s_{i\nu_i} x) \right)$$

is of the required form. Suppose that this formula is true in a G-set X. Suppose that for each $l \in \{1, \ldots, n\}$, J_l is a set of indices from $\{1, \ldots, \nu_l\}$ such that the elements $s_{lj}^{-1} y_l$ for $j \in J_l$, $l = 1, \ldots, n$ are all distinct and such that each $s_{lj}^{-1} y_l$ is in $\operatorname{supp}(\boldsymbol{g}) \cap \operatorname{fix}(\boldsymbol{h})$. Moreover, take the collection of sets J_l to be a maximal such collection in the sense that for any l, if $j \notin J_l$, then either $s_{lj}^{-1} y_l \notin \operatorname{supp}(\boldsymbol{g}) \cap \operatorname{fix}(\boldsymbol{h})$ or $s_{lj}^{-1} y_l = s_{ik}^{-1} y_i$ for some i and $k \in J_i$. Then the formula ψ is true in X when

$|\mathrm{supp}(\boldsymbol{g}) \cap \mathrm{fix}(\boldsymbol{h})| \geq \sum_{l=1}^{n} |J_l| + 1$. But, since there are only finitely many different variations on this particular situation, we can enumerate all the possibilities so that ψ can be rewritten as the disjunction of all formulas

$$\bigwedge_{l=1}^{n} \bigwedge_{j \in J_l} \left(s_{lj}^{-1} y_l \in \mathrm{supp}(\boldsymbol{g}) \cap \mathrm{fix}(\boldsymbol{h}) \right)$$
$$\wedge \bigwedge_{l \neq m} \bigwedge_{j \neq j' \in J_l} \bigwedge_{k \in J_m} \left(s_{lj}^{-1} y_l \neq s_{mk}^{-1} y_m \wedge s_{lj}^{-1} y_l \neq s_{lj'}^{-1} y_l \right)$$
$$\bigwedge_{l=1}^{n} \bigwedge_{j \notin J_l} \left(s_{lj}^{-1} y_l \notin \mathrm{supp}(\boldsymbol{g}) \cap \mathrm{fix}(\boldsymbol{h}) \vee \bigvee_{m=1}^{n} \bigvee_{k \in J_m} s_{lj}^{-1} y_l = s_{mk}^{-1} y_m \right)$$
$$\wedge\ |\mathrm{supp}(\boldsymbol{g}) \cap \mathrm{fix}(\boldsymbol{h})| \geq \left(\sum_{l=1}^{n} |J_l| \right) + 1$$

where each J_l ranges over subsets of $\{1, \ldots, \nu_l\}$. Since $ty \in \mathrm{supp}(\boldsymbol{g}) \cap \mathrm{fix}(\boldsymbol{h})$ is equivalent to $y \in \mathrm{supp}(t^{-1}\boldsymbol{g}t) \cap \mathrm{fix}(t^{-1}\boldsymbol{h}t)$ and $t_1 y_1 = t_2 y_2$ is equivalent to $y_1 = t_1^{-1} t_2 y_2$, the above formula is of the required form.

□

This result looks rather like the "pp elimination of quantifiers" result for modules (see [31, Cor 2.13]). It is probably known but we were unable to find a reference.

Corollary 4.2.6. *In L_G, every sentence is equivalent modulo T_G to a Boolean combination of sentences of the form*

$$|\mathrm{supp}(\boldsymbol{g}) \cap \mathrm{fix}(\boldsymbol{h})| \geq k$$

Corollary 4.2.7. *For any G-set X,* $\mathrm{Th}(X)$ *has quantifier elimination.*

4.3 Purity and pure-injectivity in G-Set

4.3.1 Pure embeddings

Recall that an inclusion of G-sets $X \subseteq Y$ is *pure* if for every positive primitive formula $\varphi(\boldsymbol{x})$ and tuple $\boldsymbol{a}$ from X, $Y \models \varphi(\boldsymbol{a}) \Rightarrow X \models \varphi(\boldsymbol{a})$. Now a positive primitive formula in $\mathcal{L}_G$ is equivalent to a formula of the form

$$x_1 \in \mathrm{fix}(\boldsymbol{g}_1) \wedge \cdots \wedge x_n \in \mathrm{fix}(\boldsymbol{g}_n) \wedge \mathrm{fix}(\boldsymbol{h}_1) \neq \emptyset \wedge \cdots \wedge \mathrm{fix}(\boldsymbol{h}_m) \neq \emptyset$$

So we have the following proposition (which appears as Lemma 1 in [7]).

Proposition 4.3.1. *The extension $X \subseteq Y$ is pure if and only if for any finite tuple $\boldsymbol{g}$ from G, $\mathrm{fix}_Y(\boldsymbol{g}) \neq \emptyset$ implies $\mathrm{fix}_X(\boldsymbol{g}) \neq \emptyset$.*

Let R-**Mod** be the category of left R-modules and **mod**-R the category of finitely presented right R-modules. For any left R-module M, one has an object (functor) $T_M := - \otimes M \in (\mathbf{mod}\text{-}R, \mathbf{Ab})$. T is a fully faithful functor

$$R\text{-}\mathbf{Mod} \rightarrowtail (\mathbf{mod}\text{-}R, \mathbf{Ab})$$

which takes pure monomorphisms in R-**Mod** to monomorphisms in $(\mathbf{mod}\text{-}R, \mathbf{Ab})$. Conversely if $- \otimes f : - \otimes M \to - \otimes N$ is a monomorphism in $(\mathbf{mod}\text{-}R, \mathbf{Ab})$ then $f : M \to N$ is pure in R-**Mod** (see Propositions 2.3.7 and 2.3.12).

It is natural to ask whether there is a similar picture for G-sets. Let X be a right G-set and Y a left G-set. Then $X \otimes_G Y$ is the set of elements $x \otimes y$ factored by the equivalence relation

$$x \otimes y \sim x' \otimes y' \Leftrightarrow \exists g \in G \text{ s.t. } x' = xg \text{ and } y' = g^{-1}y$$

(see [25, p. 380]). The operation $\otimes$ behaves like a tensor product in the sense that there is a Hom-tensor adjunction. Given a G-set Y and a set E, we can get a G^{op}-set (right G-set) $\mathrm{Hom}(Y, E)$ where the action of G is defined by $(fg)(y) = f(gy)$. If $\varphi : X \otimes Y \to E$ is a map of sets, we can get a map $\Gamma_\varphi : X \to \mathrm{Hom}(Y, E)$ of G^{op}-sets defined by

$$\Gamma_\varphi(x)(y) = \varphi(x \otimes y)$$

Conversely, if $\psi : X \to \mathrm{Hom}(Y, E)$ is a map of G^{op}-sets then there is a map of sets $\Delta_\psi : X \otimes Y \to E$ defined by

$$\Delta_\psi(x \otimes y) = \psi(x)(y)$$

These maps give an adjunction

$$\mathrm{Hom}(X \otimes Y, E) \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \mathrm{Hom}(X, \mathrm{Hom}(Y, E))$$

As for modules, for any (left) G-set Y, we get an object $-\otimes Y \in (G^{\mathrm{op}}\text{-}\mathbf{set}, \mathbf{Set})$ where G^{op}-**set** is the category of finitely presented G^{op}-sets. For any $F \in (G^{\mathrm{op}}\text{-}\mathbf{set}, \mathbf{Set})$, FG is a G-set with action defined by $gx = F(l_g)(x)$ where l_g is the G^{op}-endomorphism of G given by left multiplication by g. The map $g \otimes y \mapsto gy$ gives an isomorphism $G \otimes Y \cong Y$ of G-sets and the map $f \mapsto f(1)$ gives an isomorphism $\mathrm{Hom}(G, X) \cong X$ of G^{op}-sets. Moreover, the map which evaluates a natural transformation at G, gives a natural isomorphism

$$\mathrm{Hom}(-\otimes Y, F) \cong \mathrm{Hom}(Y, FG)$$

where the right-hand Hom is of (left) G-sets.

With the isomorphism $Y \cong G \otimes Y$ it is easy to see that the functor $Y \mapsto -\otimes Y$ is fully faithful so that G-**Set** embeds into $(G^{\mathrm{op}}\text{-}\mathbf{set}, \mathbf{Set})$. The situation looks remarkably similar to the modules case. So can we characterise pure embeddings of G-sets in terms of the tensor product? Unfortunately the answer to this question is negative.

Proposition 4.3.2. *For any $X \in G^{op}$-**set**, the functor $X \otimes -$ preserves monomorphisms.*

Proof. Let $f : Y \to Y'$ be a monomorphism of G-sets. $(X \otimes f)(x \otimes y) = x \otimes f(y)$. Suppose that $x \otimes f(y) = x' \otimes f(y')$. Then there is a $g \in G$ such that $x' = xg$ and $f(y') = g^{-1}f(y)$. But $f(g^{-1}y) = g^{-1}f(y) = f(y')$ implies that $g^{-1}y = y'$ since f is assumed to be a monomorphism. Hence $x \otimes y = x' \otimes y'$.

□

So the functor $Y \mapsto -\otimes Y$ takes all embeddings of G-sets, not just the pure ones, to embeddings of functors.

We say that a G-set X is *pure-injective* if every pure embedding $f : X \to Y$ is split, i.e. there is a map $g : Y \to X$ such that $gf = \mathrm{id}_X$.

Corollary 4.3.3. *Let X be a pure-injective, noninjective G-set. Then $-\otimes X$ is not an injective functor.*

Proof. Since X is noninjective, there is a G-set Y and a monomorphism f such that $f : X \to Y$ is not split. We claim that $- \otimes f$ is a non-split embedding. By above it is certainly an embedding. Suppose it were split. So there is a $\theta : - \otimes Y \to - \otimes X$ such that $\theta \circ (- \otimes f) = \text{id}$. Now $\theta = - \otimes \theta_G$ where $\theta_G : Y \to X$. We have that $\theta_G f = \text{id}_X$ which is a contradiction. □

So, unlike the modules case, the functor category is not a natural setting for the study of purity and pure-injectivity of G-sets.

4.3.2 Pure-injective G-sets and pure-injective hulls

A standard result of model theory gives us that a G-set X is pure-injective if and only if every atomic type $\Phi(\boldsymbol{x})$ (where $\boldsymbol{x}$ may be infinite) with parameters from X is realised in X (note this is equivalent to every pp-type in finitely many variables being realised, see e.g. Hodges [16, Thm 10.7.1]). It is clear that we need only consider parameter-free types since if $a \in X$ and $gx = a$ is an equation, then $x = g^{-1}a$ is uniquely determined in X. So let $\Phi(x_i)_i$ be a parameter-free atomic type of X. Assume that Φ is closed under deductive consequences of equations. Define the equivalence relation $\sim$ on the set of variables $\{x_i\}_i$ by

$$x_j \sim x_k \Leftrightarrow \exists g \in G \text{ s.t. } gx_j = x_k \in \Phi$$

Suppose that $\sim$ is transitive on $\{x_i\}_i$. Then Φ is equivalent to the type $\Phi'(x_0) = \{gx_0 = x_0 : gx_0 = x_0 \in \Phi\}$ for any chosen x_0. For if we have the formula $gx_j = x_k \in \Phi$, then since $\sim$ is transitive, there are $h, h' \in G$ such that $x_j = h'x_0$ and $x_k = hx_0$ and so since Φ is assumed to contain all its equational consequences, we have that $h^{-1}gh'x_0 = x_0 \in \Phi$. Hence this equation is in Φ' and implies $gx_j = x_k$. So Φ is equivalent to a type in one variable which corresponds to a subset $S \subseteq G$ with the property that for any finite tuple $\boldsymbol{s}$ from S, $\text{fix}_X(\boldsymbol{s}) \neq \emptyset$. If $\sim$ is not transitive on $\{x_i\}_i$, then Φ is equivalent to a family of types in one variable (one for each equivalence class). So we have the following characterisation of pure-injectivity for G-sets (which appears as Proposition 1 in [7]).

Proposition 4.3.4. *The G-set X is pure-injective (or* equationally compact*) if whenever $S \subseteq G$ is such that for any finite tuple $\boldsymbol{s}$ from S, $\mathrm{fix}_X(\boldsymbol{s}) \neq \emptyset$, then $\mathrm{fix}_X(S) := \bigcap_{s\in S} \mathrm{fix}_X(s) \neq \emptyset$.*

Equivalently, a G-set X is pure-injective if and only if it satisfies the following condition.

(3.2.1) If $H \leq G$ is a subgroup such that $\mathrm{fix}_X(H') \neq \emptyset$ for every finitely generated subgroup $H' \leq H$, then $\mathrm{fix}_X(H) \neq \emptyset$.

We say that a subgroup S can be *conjugated into* a subgroup H if one of the conjugates of S is contained in H. Since a subgroup of G fixes a point of the transitive G-set G/H if and only if it can be conjugated into H (because $\mathrm{Stab}(gH) = gHg^{-1}$) we have the following

Corollary 4.3.5. *The transitive G-set G/H is pure-injective if and only if any subgroup $S \leq G$ whose finitely generated subgroups can be conjugated into H, can itself be conjugated into H.*

We shall say that a subgroup $H \leq G$ is PIP (the *pure-injectivity property*) if G/H is a pure-injective G-set. Clearly 1 and G are both PIP. Also, any finite or normal subgroup of G is PIP.

An extension $X \subseteq Y$ is *pure essential* if it is pure and whenever $f : Y \to Z$ is a map such that $f \restriction_X$ is a pure embedding, then f is an embedding. By our characterisation of pure embeddings for G-sets, the f of this definition will in fact be a pure embedding. A G-set Y is the *pure-injective hull* of X if Y is pure-injective and contains X as a pure-essential subobject. The pure-injective hull of X is unique up to isomorphism over X (see [37]). Banaschewski [7] shows that every G-set X has a pure-injective hull which we denote by $H(X)$. We will now show that $H(X)$ is elementary equivalent to X (as in the case of modules).

Lemma 4.3.6. *The pure-injective hull $H(X)$ of X is elementarily equivalent to X and the embedding of X in $H(X)$ is elementary.*

Proof. To see this embed X elementarily in a $\emptyset$-saturated extension Y (so in particular Y is pure-injective and the embedding is pure). The hull $H(X)$ will sit in between X and Y (see Wenzel [37]) and the embedding of $H(X)$ in Y will be pure (since $X \subseteq H(X)$ is pure-essential).

$$X \underset{\text{pure}}{\rightarrowtail} H(X) \underset{\text{pure}}{\rightarrowtail} Y$$

Let φ be a sentence of the form $|\mathrm{supp}(\boldsymbol{g}) \cap \mathrm{fix}(\boldsymbol{h})| \geq k$. Clearly $X \models \varphi$ implies $H(X) \models \varphi$. For the converse, if $H(X) \models \varphi$ then so does $Y \models \varphi$. But now since $X \subseteq Y$ is elementary, $X \models \varphi$. By Corollary 4.2.6 $X \equiv H(X)$.

Since complete theories of G-sets have quantifier elimination, the embedding of X in its pure-injective hull $H(X)$ is elementary (as in the case of modules). $\square$

The paper [7] contains a reasonably explicit construction of the pure-injective hull which we now describe. Let $\mathrm{FF}(X)$ be the set of subgroups $H \leq G$ such that for any finitely generated $H' \leq H$, $\mathrm{fix}_X(H') \neq \emptyset$. By Zorn's Lemma, any such subgroup H is contained in a maximal one. Let $\mathbb{M}_X$ be the set of maximal subgroups H in $\mathrm{FF}(X)$ such that $\mathrm{fix}_X(H) = \emptyset$. Note that if X is pure-injective, then $\mathbb{M}_X = \emptyset$. Clearly each of the sets $\mathrm{FF}(X)$ and $\mathbb{M}_X$ is closed under conjugacy in G. Let $\mathrm{I}\mathbb{M}_X$ be a set of representatives of conjugacy classes of subgroups in $\mathbb{M}_X$. Then Banaschewski [7] constructs the pure-injective hull of X as

$$X \coprod \left(\coprod_{H \in \mathrm{I}\mathbb{M}_X} G/H \right)$$

In the modules case, if a pure-injective module has an indecomposable direct summand, then it has an indecomposable pure-injective direct summand. So, since every G-set decomposes into indecomposables, one might expect that every pure-injective G-set contains an indecomposable pure-injective component. This leads one to ask first of all whether some $H \in \mathbb{M}_X$ is PIP. We will now show that this is equivalent to a problem about ultrafilters.

For convenience we will write, for any set A, $A^{<\omega}$ for the set of finite tuples from A. Fix an infinite G-set X. Define

$$\mathbb{L}_X = \{\mathrm{fix}_X(\boldsymbol{g}) : \boldsymbol{g} \in G^{<\omega}\} \cup \{\emptyset\}$$

Then $\mathbb{L}_X$ is a bounded meet-semilattice (a poset with meets, a top, and a bottom element). If $G = \mathrm{Sym}(X)$ then $\mathbb{L}_X$ is the full powerset of X. On the other hand if G acts trivially on X, so that $G_x = G$ for every $x \in X$, then $\mathbb{L}_X = \{X, \emptyset\}$.

If $\mathcal{F}$ is a filter of $\mathbb{L}_X$, then we define the subgroup

$$H_{\mathcal{F}} = \{g \in G : \mathrm{fix}(g) \in \mathcal{F}\}$$

which is an object of $\mathrm{FF}(X)$. The fact that $H_{\mathcal{F}}$ is a subgroup follows from the identity $\mathrm{fix}(g) = \mathrm{fix}(g^{-1})$ and the inclusion $\mathrm{fix}(g,h) \subseteq \mathrm{fix}(gh)$ for any $g, h \in G$. That $H_{\mathcal{F}}$ is in $\mathrm{FF}(X)$ follows from the fact that $\emptyset \notin \mathcal{F}$.

Conversely, if we have a subgroup $H \in \mathrm{FF}(X)$, then we can define the filter

$$\mathcal{F}(H) = \{A \in \mathbb{L}_X : A \supseteq \mathrm{fix}(\boldsymbol{h}) \text{ for some } \boldsymbol{h} \in H^{<\omega}\}$$

This is the smallest filter containing the set $\{\mathrm{fix}(h) : h \in H\}$.

Now,

$$\begin{aligned}
\mathcal{F}(H_{\mathcal{F}}) &= \{A \in \mathbb{L}_X : A \supseteq \mathrm{fix}(\boldsymbol{h}) \text{ some } \boldsymbol{h} \in H_{\mathcal{F}}^{<\omega}\} \\
&= \{A \in \mathbb{L}_X : A \supseteq \mathrm{fix}(\boldsymbol{h}) \text{ where } \mathrm{fix}(\boldsymbol{h}) \in \mathcal{F}\}
\end{aligned}$$

so that, clearly,

$$\mathcal{F}(H_{\mathcal{F}}) = \mathcal{F}$$

Also,

$$\begin{aligned}
H_{\mathcal{F}(H)} &= \{g \in G : \mathrm{fix}(g) \in \mathcal{F}(H)\} \\
&= \{g \in G : \mathrm{fix}(g) \supseteq \mathrm{fix}(\boldsymbol{h}) \text{ where } \boldsymbol{h} \in H^{<\omega}\}
\end{aligned}$$

so that

$$H \subseteq H_{\mathcal{F}(H)}$$

Lemma 4.3.7. *If H is maximal in $\mathrm{FF}(X)$, then $H = H_{\mathcal{F}(H)}$*

We shall say that a filter $\mathcal{F}$ is *principal* if there is a nonempty subset $Y \subseteq X$ such that

$$\mathcal{F} = \{A \in \mathbb{L}_X : A \supseteq Y\}$$

The notation for such a filter is $\langle Y \rangle$. Note that Y need not be in $\mathbb{L}_X$ so our terminology deviates slightly from the standard one.

Lemma 4.3.8. *If $\mathcal{F}$ is a principal filter then* $\mathrm{fix}(H_{\mathcal{F}}) \neq \emptyset$.

Proof. Suppose $\mathcal{F} = \langle Y \rangle$. Since $\mathrm{fix}(H_{\mathcal{F}}) = \bigcap \mathcal{F}$, we have $\mathrm{fix}(H_{\mathcal{F}}) \supseteq Y \neq \emptyset$.

$\square$

An *ultrafilter* is a maximal filter. By Zorn's Lemma, any filter is contained in an ultrafilter.

Lemma 4.3.9. *An ultrafilter $\mathcal{U}$ of $\mathbb{L}_X$ is principal if and only if* $\mathrm{fix}(H_{\mathcal{U}}) \neq \emptyset$.

Proof. Suppose that $\mathrm{fix}(H_{\mathcal{U}}) \neq \emptyset$. Then $\mathcal{U}$ is contained in the proper principal filter $\langle \mathrm{fix}(H_{\mathcal{U}}) \rangle$ and hence must be equal to it. $\square$

We shall write $G_{(Y)}$ for the point stabiliser of Y.

Lemma 4.3.10. *Let $H = G_{(Y)}$ then*

(a) $H = H_{\mathcal{F}(H)}$.

(b) $\mathcal{F}(H) = \langle Y \rangle$

Proof. (a) Suppose $h \in H_{\mathcal{F}(H)}$. Then $\mathrm{fix}(h) \in \mathcal{F}(H)$ so that $\mathrm{fix}(h) \supseteq \mathrm{fix}(\boldsymbol{g})$ for some $\boldsymbol{g} \in G_{(Y)}^{<\omega}$. But this means $\mathrm{fix}(h) \supseteq Y$ which in turn means that $h \in G_{(Y)} = H$.

(b) Suppose $A \in \mathcal{F}(H)$. Then $A \supseteq \mathrm{fix}(\boldsymbol{h})$ where $\boldsymbol{h} \in G_{(Y)}^{<\omega}$. So $A \supseteq \mathrm{fix}(\boldsymbol{h}) \supseteq Y$. That is $A \in \langle Y \rangle$. Conversely if $A = \mathrm{fix}(\boldsymbol{g}) \supseteq Y$ then $\boldsymbol{g} \in G_{(Y)}^{<\omega}$ so that $A \in \mathcal{F}(H)$. $\square$

Lemma 4.3.11. *If H is a maximal subgroup in* $\mathrm{FF}(X)$ *then $\mathcal{F}(H) = \langle Y \rangle$ if and only if $H = G_{(Y)}$.*

Proof. Suppose that $\mathcal{F}(H) = \langle Y \rangle$. Let $h \in H$. Then $\mathrm{fix}(h) \in \mathcal{F}(H) = \langle Y \rangle$ so that $h \in G_{(Y)}$. So $H \subseteq G_{(Y)}$ and by maximality $H = G_{(Y)}$. The converse is Lemma 4.3.10(b). $\square$

Lemma 4.3.12. *Let $\mathcal{U}$ be an ultrafilter. Then $H_{\mathcal{U}}$ is maximal in* $\mathrm{FF}(X)$.

Proof. Suppose $H_{\mathcal{U}} \leq H$ where $H \in \mathrm{FF}(X)$. We then have $\mathcal{U} = \mathcal{F}(H_{\mathcal{U}}) \subseteq \mathcal{F}(H)$ so that $\mathcal{U} = \mathcal{F}(H)$. Now let $h \in H$. Then $\mathrm{fix}(h) \in \mathcal{F}(H) = \mathcal{U}$ so that $h \in H_{\mathcal{U}}$. So we have $H = H_{\mathcal{U}}$. □

Lemma 4.3.13. *If H is maximal in* $\mathrm{FF}(X)$ *then $\mathcal{F}(H)$ is an ultrafilter.*

Proof. Let $\mathcal{U}$ be an ultrafilter containing $\mathcal{F}(H)$. Then $H \subseteq H_{\mathcal{F}(H)} \subseteq H_{\mathcal{U}}$. By maximality, $H = H_{\mathcal{U}}$. So $\mathcal{F}(H) = \mathcal{F}(H_{\mathcal{U}}) = \mathcal{U}$. □

The following proposition now follows easily from these lemmas.

Proposition 4.3.14. *There is a bijection between ultrafilters of $\mathbb{L}_X$ and maximal subgroups in* $\mathrm{FF}(X)$. *The nonprincipal ultrafilters correspond to those subgroups in $\mathbb{M}_X$.*

The group G acts on the space of ultrafilters of $\mathbb{L}_X$ by $g\mathcal{U} = \{gA : A \in \mathcal{U}\}$. For any $h_1, \ldots, h_n \in G$, $g\mathrm{fix}(h_1, \ldots, h_n) = \mathrm{fix}(gh_1g^{-1}, \ldots, gh_ng^{-1})$. This implies that when H is a maximal subgroup in $\mathrm{FF}(X)$,

$$g\mathcal{F}(H) = \mathcal{F}(gHg^{-1})$$

Proposition 4.3.15. *A maximal subgroup $H \in \mathrm{FF}(X)$ is PIP if and only if, for any filter $\mathcal{F} \subseteq \mathbb{L}_X$ such that $\mathcal{F} \subseteq \bigcup_{g\in G} g\mathcal{F}(H)$, there is a $g_0 \in G$ such that $\mathcal{F} \subseteq g_0\mathcal{F}(H)$.*

Proof. ($\Rightarrow$) Suppose H is a PIP maximal subgroup in $\mathrm{FF}(X)$. Let $\mathcal{F} \subseteq \mathbb{L}_X$ be a filter such that $\mathcal{F} \subseteq \bigcup_{g\in G} g\mathcal{F}(H)$. Then $H_{\mathcal{F}}$ is locally conjugated into H. For suppose $s_1, \ldots, s_n \in H_{\mathcal{F}}$. Then $\mathrm{fix}(s_1, \ldots, s_n) \in \mathcal{F}$ and so there is a $g \in G$ such that $\mathrm{fix}(s_1, \ldots, s_n) \in g\mathcal{F}(H) = \mathcal{F}(gHg^{-1})$. So $s_1, \ldots, s_n \in H_{\mathcal{F}(gHg^{-1})} = gHg^{-1}$. Since H is PIP, there is a $g_0 \in G$ such that $H_{\mathcal{F}} \subseteq g_0Hg_0^{-1}$. So $\mathcal{F} = \mathcal{F}(H_{\mathcal{F}}) \subseteq \mathcal{F}(g_0Hg_0^{-1}) = g_0\mathcal{F}(H)$.

($\Leftarrow$) Suppose $S \leq G$ is locally conjugated into H. Then in particular $S \in \mathrm{FF}(X)$. We claim that $\mathcal{F}(S) \subseteq \bigcup_{g\in G} g\mathcal{F}(H)$. For suppose $A \supseteq \mathrm{fix}(s_1, \ldots, s_n)$ where $s_1, \ldots, s_n \in S$. By assumption, there is a $g \in G$ such that $s_1, \ldots, s_n \in gHg^{-1}$. So $\mathrm{fix}(s_1, \ldots, s_n) \in \mathcal{F}(gHg^{-1}) = g\mathcal{F}(H)$ and A must be in $g\mathcal{F}(H)$ which establishes the claim. But this means there is a $g_0 \in G$ with $\mathcal{F}(S) \subseteq g_0\mathcal{F}(H) = \mathcal{F}(g_0Hg_0^{-1})$. Hence $S \subseteq H_{\mathcal{F}(S)} \subseteq H_{\mathcal{F}(g_0Hg_0^{-1})} = g_0Hg_0^{-1}$. □

So the PIP subgroups in $\mathbb{M}_X$ correspond to the nonprincipal ultrafilters $\mathcal{U}$ of $\mathbb{L}_X$ with the property that

(3.2.2) Whenever $\mathcal{F} \subseteq \mathbb{L}_X$ is a filter such that $\mathcal{F} \subseteq \bigcup_{g\in G} g\mathcal{U}$, there is a $g_0 \in G$ such that $\mathcal{F} \subseteq g_0\mathcal{U}$.

Proposition 4.3.16. *Let X be a countable set and $G = \mathrm{Sym}(X)$. Then none of the subgroups in $\mathbb{M}_X$ are PIP.*

Proof. In this case $\mathbb{L}_X$ is the full powerset of X. Let $\mathcal{U}$ be a nonprincipal ultrafilter on X. Let $\mathcal{F}$ be a nonprincipal ultrafilter which is not in the set

$$\mathfrak{X} = \{g\mathcal{U} : g \in G\}$$

There clearly must be a nonprincipal ultrafilter $\mathcal{F} \notin \mathfrak{X}$ since there are $2^{2^{\aleph_0}}$ nonprincipal ultrafilters on X (see [21]) but $|\mathfrak{X}| = 2^{\aleph_0}$.

Let $A \in \mathcal{F}$. If A is cofinite, then $A \in \bigcap_g g\mathcal{U}$. On the other hand, if A is infinite-coinfinite, we can choose an infinite-coinfinite set $B \in \mathcal{U}$ and a bijection $g : X \to X$ such that $gB = A$. So $A \in g\mathcal{U}$. This implies that $\mathcal{F} \subseteq \bigcup_g g\mathcal{U}$ but $\mathcal{F} \nsubseteq g\mathcal{U}$ for any $g \in G$. $\square$

Corollary 4.3.17. *Let $G = \mathrm{Sym}(X)$ for a countable set X. Then there is a pure-injective G-set, none of whose indecomposable components is pure-injective.*

Proof. The pure-injective hull $H(X) = X \coprod \left(\coprod_{H\in \mathbb{IM}_X} G/H\right)$ has this property by the above proposition. $\square$

4.4 Definable subcategories and the Ziegler Spectrum

4.4.1 Definable subcategories of G-Set

Recall that definable subcategories are those full subcategories whose objects are models of a coherent theory.

Lemma 4.4.1. *The collection of definable subcategories is closed under finite unions and arbitrary intersections.*

Proof. Let $\mathbf{D}_1 = \operatorname{Mod} T_1$ and $\mathbf{D}_2 = \operatorname{Mod} T_2$ be two definable subcategories where T_1 and T_2 are sets of coherent sentences. Then $\mathbf{D}_1 \cup \mathbf{D}_2 = \operatorname{Mod} T$ for

$$T = \{\varphi_1 \vee \varphi_2 : \varphi_1 \in T_1,\ \varphi_2 \in T_2\}$$

Since coherent sentences are closed under disjunctions, we have that $\mathbf{D}_1 \cup \mathbf{D}_2$ is definable.

Now suppose that $\mathbf{D}_i = \operatorname{Mod} T_i$ for $i \in I$. Then

$$\bigcap_i \mathbf{D}_i = \operatorname{Mod}\left(\bigcup_i T_i\right)$$

and so $\bigcap_i \mathbf{D}_i$ is definable. □

If the class of all G-sets was a *set*, then the definable subcategories would be closed sets for a topology on G-**Set**. In the case of modules, definable subcategories are determined by (nonzero) indecomposable pure-injectives (Proposition 2.3.10) and there is only a set of these (up to isomorphism). We therefore have a nice (and in some cases completely classifiable [30, Chpts. 5 & 8]) set of modules with a topology whose closed sets correspond to coherent theories. This is the *Ziegler spectrum* for modules. It is natural to ask whether the same is true of G-sets. The first natural set to consider, in view of what is true about modules, is the set of indecomposable pure-injective G-sets.

4.4.2 A Ziegler spectrum for G-sets

Proposition 4.4.2. *Let G be a finitely generated free abelian group of rank ≥ 2. Then there is a definable subcategory $\mathbf{D}$ containing a non-trivial G-set such that $\mathbf{D}$ contains no indecomposable G-set other than $\{*\}$*

Proof. Let $\{g_1, \ldots, g_n\}$ be a set of free generators for G. Let $H_1 = \langle g_1\rangle$ and $H_2 = \langle g_2, \ldots, g_n\rangle$ and put

$$T = \{\exists z_1(g_1 z_1 = z_1),\ \exists z_2(z_2 \in \operatorname{fix}(H_2))\}$$

and let $\mathbf{D} = \operatorname{Mod} T$. Then

$$\begin{aligned} G/Q \models T \quad &\text{iff} \quad \exists h_1, h_2 \in G \text{ s.t. } H_1^{h_1} \leq Q \text{ and } H_2^{h_2} \leq Q \\ &\text{iff} \quad H_1 \leq Q \text{ and } H_2 \leq Q \text{ (since } G \text{ is abelian)} \\ &\text{iff} \quad Q = G \end{aligned}$$

□

It is clear from this example that definable subcategories are not determined by the indecomposable pure-injectives they contain. For example, suppose the G above has rank ≥ 3. If $H_1' = \langle g_1, \ldots, g_{n-1} \rangle$, $H_2' = \langle g_n \rangle$ and $\mathbf{D}' = \operatorname{Mod}(\operatorname{fix}(H_1') \neq \emptyset \wedge \operatorname{fix}(H_2') \neq \emptyset)$, then $\mathbf{D}'$ and $\mathbf{D}$ both contain $\{*\}$ as the only indecomposable pure-injective but $\mathbf{D} \neq \mathbf{D}'$ since $G/H_1 \coprod G/H_2 \in \mathbf{D} \setminus \mathbf{D}'$.

So we have seen that the set of indecomposable pure-injectives (or even the set of indecomposables) is too small to function as the underlying set for a "Ziegler topology". Let $\{H_i : i \in I\}$ be the set of all subgroups of a group G and consider the following set:

$$\mathcal{X} = \left\{ \coprod_{i \in I'} G/H_i : I' \subseteq I \text{ s.t. } H_i \text{ not conjugate to } H_j \text{ for } i \neq j \right\}$$

So $\mathcal{X}$ is the set of G-sets such that no two indecomposable components are isomorphic.

Lemma 4.4.3. *Definable subcategories are determined by their intersection with the set $\mathcal{X}$.*

Proof. Let X be any G-set. Let X' be the G-set whose decomposition consists of exactly one isomorphic copy for each indecomposable G-set occuring in the decomposition of X. For any subgroup $S \leq G$, $\operatorname{fix}_X(S) \neq \emptyset \Leftrightarrow \operatorname{fix}_{X'}(S) \neq \emptyset$. It is now easy to see that X and X' satisfy exactly the same coherent sentences. So $X \in \mathbf{D}$ if and only if $X' \in \mathbf{D}$. □

Note that with X and X' as in the above proof, we have that X' is pure-injective if X is. Let $\operatorname{pinj}(\mathcal{X})$ be the set of pure-injectives in $\mathcal{X}$.

Proposition 4.4.4. *Definable subcategories are determined by their intersection with* $\mathrm{pinj}(\mathcal{X})$.

Proof. The crucial fact is that if $X \in \mathcal{X}$, then also $H(X) \in \mathcal{X}$ which can be seen by considering Banaschewski's construction of $H(X)$ described earlier.

Let $\mathbf{D}$ and $\mathbf{D}'$ be two definable subcategories such that $\mathbf{D} \cap \mathrm{pinj}(\mathcal{X}) = \mathbf{D}' \cap \mathrm{pinj}(\mathcal{X})$ and suppose that $\mathbf{D} \neq \mathbf{D}'$. By the previous lemma this means that $\mathbf{D} \cap \mathcal{X} \neq \mathbf{D}' \cap \mathcal{X}$. Let $X \in \mathbf{D} \cap \mathcal{X}$ such that $X \notin \mathbf{D}' \cap \mathcal{X}$. Then $H(X) \in \mathbf{D} \cap \mathrm{pinj}(\mathcal{X})$. Since X is elementary equivalent to $H(X)$ (by Lemma 4.3.6) and $X \notin \mathbf{D}'$, we must have $H(X) \notin \mathbf{D}'$ which is a contradiction. $\square$

In light of the above proposition, we can define for a group G, a topological space Zg_G (the *Ziegler spectrum of* G) which has underlying set $\mathrm{pinj}(\mathcal{X}) \setminus \{*\}$ and such that the closed sets are of the form

$$C(T) = (\mathrm{pinj}(\mathcal{X}) \setminus \{*\}) \cap \mathrm{Mod}\, T$$

where T ranges over the coherent theories of G-sets.

Proposition 4.4.5. *The Ziegler topology for G-sets has a basis of compact open sets.*

Proof. Consider the closed set $C(T)$ where $T = \{\varphi_\lambda\}_\lambda$ for coherent sentences φ_λ. So

$$C(T) = \bigcap_\lambda C(\varphi_\lambda)$$

In other words, the open sets of the form $O(\varphi) = \mathrm{Zg}_G \setminus C(\varphi)$ give a basis of open sets for Zg_G. We claim that each $O(\varphi)$ is compact.

Suppose that $O(\varphi) \subseteq \bigcup_\lambda O(\varphi_\lambda)$. So

$$\mathrm{Mod}(\neg\varphi) \cap \mathrm{pinj}(\mathcal{X}) \subseteq \bigcup_\lambda \mathrm{Mod}(\neg\varphi_\lambda) \cap \mathrm{pinj}(\mathcal{X})$$

But this implies (by Proposition 4.4.4)

$$\mathrm{Mod}(\neg\varphi) \subseteq \bigcup_\lambda \mathrm{Mod}(\neg\varphi_\lambda)$$

Or, equivalently

$$\bigcap_\lambda \mathrm{Mod}(\varphi_\lambda) \subseteq \mathrm{Mod}(\varphi)$$

That is,

$$\{\varphi_\lambda\}_\lambda \vdash \varphi$$

By the compactness theorem for first-order logic, there are $\lambda_1, \ldots, \lambda_n$ such that

$$\varphi_{\lambda_1}, \ldots, \varphi_{\lambda_n} \vdash \varphi$$

which implies that

$$O(\varphi) \subseteq \bigcup_{i=1}^{n} O(\varphi_{\lambda_i})$$

by retracing the above steps. □

Proposition 4.4.6. *Let G be a finite group. Then the points of Zg_G are the G-sets in $\mathcal{X} \setminus \{*\}$ and each point is locally closed.*

Proof. The first statement is clear since for a finite group G every G-set is pure-injective. For the second statement, let $X \in \mathcal{X}$ be the G-set with indecomposable decomposition

$$G/H_1 \coprod \cdots \coprod G/H_n$$

For any subgroup $H \leq G$, the sentence $\exists z\, \mathrm{Stab}(z) = H$ is definable by the negation of a coherent sentence, namely

$$\exists z \left(z \in \mathrm{fix}(H) \wedge \neg \bigvee_{h \notin H} hz = z \right)$$

Let $O(H)$ be the basic open set defined by this sentence. Then we can write

$$\{X\} = \left(\bigcap_{i=1}^{n} O(H_i) \right) \cap \bigcap \{O(H)^c : H \text{ not conjugate to any } H_i\}$$

So the point X is locally closed. □

It would be nice to explicitly describe the Ziegler topology for some specific groups. However it seems that it is a hard problem to classify even the PIP subgroups in the simplest nontrivial examples. Consider for example the locally finite group $\mathrm{FSym}(\mathbb{N})$ of finitary permutations on $\mathbb{N}$ (these are the permutations with finite support). This group has a very simple normal structure, with a unique normal subgroup (which has index 2), namely, the group of even permutations A_∞. It is possible that there are no PIP subgroups other than the finite subgroups and A_∞. However, we have not been able to decide this question (even after discussions with group-theorists).

Chapter 5

Imaginaries as coherent affine functors?

5.1 Overview

In chapters 2 and 3 we saw that we could characterise coherent functors syntactically as pairs of formulas. In Chapter 2 our functors went from **mod**-R, a category *enriched* in abelian groups (meaning the hom-sets had an abelian group structure), to the category **Ab** of abelian groups, and had to preserve the abelian group structure on hom-sets. The resulting category of additive functors was a well behaved abelian category and we could write the coherent functors as quotients of pp-formulas (Proposition 2.2.7) in the langauge of R-modules.

In Chapter 3 we saw that by removing the additive structure completely we get another well defined functor category (a topos) whose coherent objects can be characterised in terms of pairs of pe-formulas (Theorem 3.2.12). This is an example of the close parallels between abelian category theory and topos theory. It would be interesting if there was a situation in between the two, where there is some additive structure around but no full-blown abelian group structure, and in which we can prove a similar result regarding coherent functors.

In fact, there is a natural context for such a "quasi-abelian" category of functors. If we consider abelian groups without a distinguished identity element, we enter the domain of *affine spaces*. These are typically either $\emptyset$ or cosets of subgroups in abelian groups. These objects can be made into a category **Aff** by taking the maps to be those that preserve the *affine structure* (this will be made precise). Let us expand the language of left R-modules by adding a finite tuple of γ constants. Models of the theory of modules in this expanded language are simply left R-modules with a distinguished γ-tuple. If we make these *pointed modules* into a category $R^\gamma/R\text{-}\mathbf{mod}$ by taking the maps to be those R-linear maps which take one tuple to another (pointwise), then the resulting category is enriched in affine spaces in the sense that the hom-sets have the structure of an affine space. Let us now consider the category of *affine functors* $(R^\gamma/R\text{-}\mathbf{mod}, \mathbf{Aff})$. This is the category of functors which preserve the affine structure on hom-sets. A pp-formula φ with γ constants defines such a functor. Moreover, any pair φ/ψ of pp-formulas with constants such that $\psi \to \varphi$ will define an affine functor by associating to a pointed module $(M, \boldsymbol{a})$ the quotient $\varphi(M, \boldsymbol{a})/\psi(M, \boldsymbol{a}) \in \mathbf{Aff}$ (this is a coset of $\varphi(M, \mathbf{0})/\psi(M, \mathbf{0})$ in $M^n/\psi(M, \mathbf{0})$ where n is the number of free variables of φ and ψ). In view of the results of chapters 2 and 3 one might expect to be able to characterise the coherent affine functors as quotients of pp-formulas with γ constants. In this chapter we will show that this is not the case.

5.2 Affine spaces

5.2.1 Definition and notation

An *affine space* is a set X endowed with a ternary operation $t : X^3 \to X$ subject to the following equations:

A1 $t(x, z, z) = x$

A2 $t(x, y, z) = t(z, y, x)$

A3 $t(t(x, u, y), w, z) = t(x, t(u, y, w), z) = t(x, u, t(y, w, z))$

For our purposes it is necessary to include $\emptyset$ as an affine space. This implies that the category **Aff** of all affine spaces and homomorphisms is not an affine category in the sense of [36].

If we fix an element 0 of the affine space X, and define $x + y = t(x, 0, y)$, then X becomes an abelian group with the addition operation $+$ and zero element 0. Moreover, we have

$$t(x, y, z) = x - y + z$$

See [28] for the details.

Given an affine space X we can *define* the expression $x - y + z$ to mean $t(x, y, z)$. Note that we are not implying that the expression $x + y$ should mean anything. We can recursively define

$$\begin{aligned} 0x - 0y + z &= z \\ nx - ny + z &= x - y + ((n-1)x - (n-1)y + z) \quad \text{for } n > 0 \end{aligned}$$

And hence we can define

$$nx + (1-n)y = \begin{cases} nx - ny + y & \text{for } n \geq 0 \\ (-n)y - (-n)x + y & \text{for } n < 0 \end{cases}$$

Now, if $n_1, \ldots, n_k$ are integers such that $n_1 + n_2 + \ldots n_k = 1$, we can define (by induction on k)

$$\begin{aligned} n_1x_1 + \cdots + n_kx_k &= (n_1x_1 + \cdots + n_{k-2}x_{k-2} + (n_{k-1} + n_k)x_{k-1}) \\ &\quad -(n_kx_{k-1} + (1-n_k)x_k) + x_k \end{aligned}$$

We shall call homomorphisms of affine spaces *affine maps*. It is easy to check that an affine map f satisfies

$$f(n_1x_1 + \cdots + n_kx_k) = n_1f(x_1) + \cdots + n_kf(x_k)$$

whenever $n_1 + \cdots + n_k = 1$.

If $A, B \in$ **Aff** then $\mathrm{Hom}(A, B) \in$ **Aff** for whenever we have three affine maps $f, g, h : A \to B$, then $f - g + h$ (defined pointwise) is another affine map $A \to B$. Note that any constant map $A \to B$ is an affine map.

Let $x_1, \dots x_k$ be elements of an affine space X. If $n_1, \dots, n_k$ are integers such that $n_1 + \cdots + n_k = 1$ then we say that $n_1x_1 + \cdots + n_kx_k$ is an *affine combination* of $x_1, \dots, x_k$. The subset $S \subseteq X$ generates X as an affine space (in the sense that X is the smallest affine subspace containing S) if and only if every element of X can be written as an affine combination of finitely many elements from S. One can easily check this using the definitions above and the axioms A1-A3. Note also that an affine subspace of an affine space is a subset which is closed under all affine combinations.

An abelian group $(G, +)$ is naturally an affine space with $t(x, y, z) = x - y + z$ so that the above definition of the expression "$x - y + z$" coincides with its usual interpretation in G.

Proposition 5.2.1. *Any affine subspace of an abelian group G is a coset of a subgroup of G.*

Proof. Let A be an affine subspace of G. Fix an arbitrary element $a_0 \in A$ and consider the set

$$A_0 = \{x \in G : x + a_0 \in A\}$$

Then, since $a = a_0 + (a - a_0)$, we have

$$A = a_0 + A_0$$

We have $0 \in A_0$ since $a_0 \in A$. If $x, y \in A_0$ then $x + a_0, y + a_0 \in A$. So $(x - y) + a_0 = (x + a_0) - (y + a_0) + a_0 \in A$ which implies $x - y \in A_0$. This means A_0 is a subgroup of G and the result is proved. □

Corollary 5.2.2. *Any affine subspace A of a finitely generated abelian group G is finitely generated as an affine space.*

Proof. By the Proposition we can write $A = a_0 + A_0$ where A_0 is a subgroup of G. Since G is a finitely generated abelian group, so is A_0. Suppose A_0 is generated as a group by $x_1, \dots, x_k$. Then A is generated as an affine space by $a_0, a_0 + x_1, \dots, a_0 + x_k$ since if $a \in A$ then there are integers $\lambda_1, \dots, \lambda_k$ such that

$$\begin{aligned} a &= a_0 + (\lambda_1 x_1 + \cdots + \lambda_k x_k) \\ &= \lambda_1(a_0 + x_1) + \cdots + \lambda_k(a_0 + x_k) + (1 - \lambda_1 - \cdots - \lambda_k)a_0 \end{aligned}$$

□

5.2.2 Free objects and coproducts

Proposition 5.2.3. *The free object on k generators is the affine subspace of $\mathbb{Z}^k$ generated by $\mathbf{e}_1, \ldots, \mathbf{e}_k$. In other words, it is the set*

$$C_k = \{(n_1, \ldots, n_k) \in \mathbb{Z}^k : n_1 + \cdots + n_k = 1\}$$

Proof. Let A be an affine space and suppose that $a_1, \ldots, a_k \in A$. Define the map $f : C_k \to A$ by

$$f(n_1, \ldots, n_k) = n_1 a_1 + \cdots + n_k a_k$$

It is easy to see that f is an affine map and clearly $f(\mathbf{e}_i) = a_i$ for $i = 1, \ldots, k$. □

Note that $\mathbb{Z}^k$ itself is generated as an affine space by the elements $\mathbf{e}_1, \ldots, \mathbf{e}_k$ and $\mathbf{0}$.

Let $A_1, \ldots, A_n$ be nonempty affine spaces and choose for each i a fixed element $0^i \in A_i$. Let A be the set

$$A_1 \times \cdots \times A_n \times C_n$$

with the induced pointwise structure as an affine space. Let $v_i : A_i \to A$ be the map

$$x \longmapsto (0^1, \ldots, 0^{i-1}, x, 0^{i+1}, \ldots, 0^n, \mathbf{e}_i)$$

Proposition 5.2.4. *$\{v_i : A_i \to A\}_i$ is a coproduct in* $\mathbf{Aff}$.

Proof. Let $\{f_i : A_i \to B\}_i$ be a family of affine maps. We need to show that there is a unique map $f : A \to B$ such that $f v_i = f_i$ for each i. Define the map $f : A \to B$ by

$$f(a_1, \ldots, a_n, l_1, \ldots, l_n) = \sum_{i=1}^{n} \Big(f_i(a_i) + (l_i - 1) f_i(0^i) \Big)$$

Note that the coefficients in the right hand summation sum to 1 so it is an affine combination. It is easy to check that f is affine. Clearly $f v_i = f_i$. So we only need to show uniqueness. We can write an arbitrary element $(a_1, \ldots, a_n, l_1, \ldots, l_n) \in A$ as an affine combination

$$\sum_{i=1}^{n} \Big(v_i(a_i) + (l_i - 1)v_i(0^i) \Big)$$

So if $g : A \to B$ is any affine map such that $gv_i = f_i$, then clearly $g = f$. □

5.2.3 Remarks

The category **Aff** is a finitary variety in the universal algebra sense. It follows that it is locally finitely presented and the forgetful functor to **Set** creates products (see [1, 3.6, 3.7]). So if $\{A_i\}$ is a family of affine spaces then their product is simply the set $\prod_i A_i$ with the induced pointwise affine structure.

5.3 Pointed modules

5.3.1 Definition

Let γ be a fixed ordinal number. The category of γ-pointed modules is the comma category $R^{(\gamma)}/R$-**Mod**. In other words, the objects of this category are left R-modules M with a distinguished tuple $\boldsymbol{a} \in M^\gamma$, denoted $(M, \boldsymbol{a})$ or simply $\mathcal{M}$ when there is no need to specify the tuple. A map $f : (M, \boldsymbol{a}) \to (N, \boldsymbol{b})$ is a an R-linear map $f : M \to N$ such that $f(\boldsymbol{a}) = \boldsymbol{b}$.

If $\mathcal{M}, \mathcal{N}$ are two pointed modules, then $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ is an affine space with $t(f, g, h) = f - g + h$, where the right hand side is defined pointwise. For if f, g, h all take $\boldsymbol{a}$ to $\boldsymbol{b}$, then $f - g + h$ takes $\boldsymbol{a}$ to $\boldsymbol{b} - \boldsymbol{b} + \boldsymbol{b} = \boldsymbol{b}$.

5.3.2 Coproducts

Let $(A_i, \boldsymbol{a}_i)$ be γ-pointed modules where $i \in I$. Here $\boldsymbol{a}_i = (a_{ik})_{k<\gamma}$. Let $H \leq \bigoplus_{i \in I} A_i$ be the submodule generated by the tuples $\boldsymbol{a}_{mnk} \in \bigoplus_i A_i$, for $m, n \in I$ and $k < \gamma$, where

$$a_{mnk,i} = \begin{cases} a_{mk} & \text{when } i = m \\ -a_{nk} & \text{when } i = n \\ 0 & \text{otherwise} \end{cases}$$

Let $\iota_i : A_i \to \bigoplus_i A_i$ be the canonical ith injection. Define v_i to be the composition of ι_i with the projection

$$\bigoplus_i A_i \to \bigoplus_i A_i/H$$

Then $\forall i, j \in I$, $v_i(\boldsymbol{a}_i) = v_j(\boldsymbol{a}_j)$ since $\iota_i(a_{ik}) - \iota_j(a_{jk}) = \boldsymbol{a}_{ijk}$. So if we put $A = \bigoplus_i A_i/H$ and $\boldsymbol{a} = v_{i_0}(\boldsymbol{a}_{i_0})$ for a fixed $i_0 \in I$, then the maps $v_i : (A_i, \boldsymbol{a}_i) \to (A, \boldsymbol{a})$ are morphisms of pointed modules.

Proposition 5.3.1. *In the above notation,* $\{v_i : (A_i, \boldsymbol{a}_i) \to (A, \boldsymbol{a})\}_i$ *is a coproduct of the family* $\{(A_i, \boldsymbol{a}_i)\}_i$ *of pointed modules.*

Proof. For each $i \in I$, let $f_i : (A_i, \boldsymbol{a}_i) \to (B, \boldsymbol{b})$ be maps of pointed modules. Let $f = \bigoplus_i f_i : \bigoplus_i A_i \to B$ and let $\boldsymbol{x} \in H$. So

$$\boldsymbol{x} = r_1 \boldsymbol{a}_{i_1 j_1 k_1} + \cdots + r_n \boldsymbol{a}_{i_n j_n k_n}$$

and

$$\begin{aligned} f(\boldsymbol{x}) &= r_1 f(\boldsymbol{a}_{i_1 j_1 k_1}) + \cdots + r_n f(\boldsymbol{a}_{i_n j_n k_n}) \\ &= r_1(f_{i_1}(a_{i_1 k_1}) - f_{j_1}(a_{j_1 k_1})) + \cdots + r_n(f_{i_n}(a_{i_n k_n}) - f_{j_n}(a_{j_n k_n})) \\ &= r_1(b_{k_1} - b_{k_1}) + \cdots + r_n(b_{k_n} - b_{k_n}) \\ &= 0 \end{aligned}$$

So f factors uniquely through A as $\overline{f}$ say. We have $\overline{f}(v_i(x)) = f\iota_i(x) = f_i(x)$ and $\overline{f}(\boldsymbol{a}) = \overline{f}(v_0(\boldsymbol{a}_0)) = f_0(\boldsymbol{a}_0) = \boldsymbol{b}$. Moreover, $\overline{f}$ is the unique such map $A \to B$. For suppose $\overline{g}$ is another such map. Then $\overline{g}$ corresponds to a map $g : \bigoplus_i A_i \to B$ such that $H \leq \ker g$ and such that $g\iota_i = f_i$. but this means $g = f$ so that $\overline{g} = \overline{f}$. $\square$

5.3.3 Finite presentation

Henceforth $\gamma < \omega$.

Proposition 5.3.2. *The finitely presented objects in R^γ/R-**Mod** are precisely the objects in R^γ/R-**mod**.*

Proof. Suppose $(A, \boldsymbol{a})$ is f.p. in R^γ/R-**Mod**. Let $f : A \to \varinjlim_{i\in I} M_i$ be a map in R-**Mod** where I is a directed poset and the colimit maps $M_j \to \varinjlim_i M_i$ are denoted by m_j and the diagram maps $M_i \to M_j$ by m_{ij} when $i \leq j$. Since $f(\boldsymbol{a})$ is a finite tuple, there is an $i_0 \in I$ such that $f(\boldsymbol{a})$ is represented in M_{i_0}. Let $I' = \{i \in I : i_0 \leq i\}$. Then $\varinjlim_{i\in I} M_i \cong \varinjlim_{i\in I'} M_i$. So f induces a map

$$f : (A, \boldsymbol{a}) \longrightarrow \varinjlim_{i\in I'}(M_i, m_{i_0 i}(f(\boldsymbol{a})))$$

which, by assumption, factors essentially uniquely through some $i_1 \in I'$. This induces an essentially unique factorisation

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & \varinjlim_{i\in I} M_i \\ & M_{i_1} \arrow[ur, "m_{i_1}"'] & \end{tikzcd}$$

Hence A is a f.p. module.

For the converse, suppose A is f.p. in R-**Mod**. Let $\boldsymbol{a}$ be a γ-tuple in A and

$$f : (A, \boldsymbol{a}) \longrightarrow \varinjlim(M_i, \boldsymbol{x}_i)$$

a map to a directed colimit in R^γ/R-**Mod**. Then $f : A \to \varinjlim M_i$ is a map in R-**Mod** which must factor essentially uniquely through some M_i. Let $k \geq i$ be an index such that the tuple $f(\boldsymbol{a})$ is represented by $\boldsymbol{x}_k$ in M_k. Then f also factors through M_k which clearly induces an essentially unique factorisation

$$\begin{tikzcd} (A, \boldsymbol{a}) \arrow[rr, "f"] \arrow[dr] & & \varinjlim_i(M_i, \boldsymbol{x}_i) \\ & (M_k, \boldsymbol{x}_k) \arrow[ur] & \end{tikzcd}$$

So $(A, \boldsymbol{a})$ is f.p. as required. $\square$

Proposition 5.3.3. *The category R^γ/R-**Mod** is locally finitely presented.*

Proof. Using Theorem 1.11 from [1], it is sufficient to show that R^γ/R-**Mod** is cocomplete and has a strong generator consisting of f.p. objects. It is easy to see that the pointed module

$$(R^{\gamma+1}, (\mathbf{e}_1, \ldots, \mathbf{e}_\gamma))$$

is a strong generator. For cocompleteness we need to show that the category has an initial object, coequalisers and arbitrary coproducts. We have already shown the existence of arbitrary coproducts. So consider two arrows

$$(M, \boldsymbol{a}) \underset{g}{\overset{f}{\rightrightarrows}} (N, \boldsymbol{b})$$

and let π be the projection $N \to N/\mathrm{im}(f-g)$. The coequaliser of f and g is simply $(N/\mathrm{im}(f-g), \pi(\boldsymbol{b}))$. The initial object of the category is easily seen to be

$$(R^\gamma, (\mathbf{e}_1, \ldots, \mathbf{e}_\gamma))$$

which proves the result. □

5.4 Affine functors

If $\mathcal{M}$ and $\mathcal{N}$ are two pointed modules, then $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ is an affine space. So we can consider *affine functors* R^γ/R-**mod** $\to$ **Aff**. These are functors F such that for any f.p. pointed modules $\mathcal{M}, \mathcal{N}$ the induced map

$$\mathrm{Hom}(\mathcal{M}, \mathcal{N}) \xrightarrow{F} \mathrm{Hom}(F\mathcal{M}, F\mathcal{N})$$

is a map of affine spaces. The category of these functors will be denoted (R^γ/R-**mod**, **Aff**). Note that for two affine functors F, G, the set $\mathrm{Nat}(F, G)$ has the structure of an affine space via

$$(\theta - \theta' + \theta'')_{\mathcal{M}}(x) = \theta_{\mathcal{M}}(x) - \theta'_{\mathcal{M}}(x) + \theta''_{\mathcal{M}}(x)$$

for $\theta, \theta', \theta'' \in \mathrm{Nat}(F, G)$ and $\mathcal{M} \in R^\gamma/R$-**mod**. We have an affine version of the Yoneda lemma (which is easily proved).

Lemma 5.4.1. *The map* $\Phi : \mathrm{Nat}(\mathrm{Hom}(\mathcal{M}, -), F) \to F\mathcal{M}$ *defined by*

$$\theta \longmapsto \theta_{\mathcal{M}}(\mathrm{id})$$

induces an isomorphism of affine spaces.

The category $(R^\gamma/R\textbf{-mod}, \textbf{Aff})$ is a many sorted variety with sorts $\{s_{\mathcal{M}} : \mathcal{M} \in R^\gamma/R\textbf{-mod}\}$. Its operations and equations are listed below.

- For each sort $s_{\mathcal{M}}$ a ternary operation $t_{\mathcal{M}} : s_{\mathcal{M}} \times s_{\mathcal{M}} \times s_{\mathcal{M}} \to s_{\mathcal{M}}$ satisfying A1-A3.

- For each arrow $f : \mathcal{M} \to \mathcal{N}$ in $R^\gamma/R\textbf{-mod}$, a unary operation $f_* : s_{\mathcal{M}} \to s_{\mathcal{N}}$ such that whenever $f, f', f'' \in \mathrm{Hom}(\mathcal{M}, \mathcal{N})$ and $g : \mathcal{N} \to \mathcal{N}'$ the corresponding operations satisfy the following equations.

$$f_*(t_{\mathcal{M}}(x, , y, z)) = t_{\mathcal{N}}(f_*(x), f_*(y), f_*(z))$$

$$\mathrm{id}_*(x) = x$$

$$(gf)_*(x) = g_* f_*(x)$$

$$(f - f' + f'')_*(x) = t_{\mathcal{N}}(f_*(x), f'_*(x), f''_*(x))$$

As a finitary variety, the category $(R^\gamma/R\textbf{-mod}, \textbf{Aff})$ is locally finitely presented. The representable functors form a strong generating set of f.p. objects. It is natural to ask whether every coherent functor is defined by a quotient of pp-formulas in the language of left R-modules with γ additional constants. We will show that this is not the case.

In order to understand coherent functors, we must first understand finitely generated functors. It is easy to see that these are precisely the quotients of finite coproducts of representable functors either by Proposition 2.2.1 (noting that a map of affine functors which is both epic and monic is necessarily invertible) or by considering functors as many-sorted algebras as above and using the algebraic notion of finite generation together with the Yoneda lemma. So what do finite coproducts of representables look like?

Fix f.p. pointed modules $\mathcal{A}_1, \dots, \mathcal{A}_n$. For each f.p. pointed module $\mathcal{X}$ define $I(\mathcal{X}) = \{i : \mathrm{Hom}(\mathcal{A}_i, \mathcal{X}) \neq \emptyset\}$. Choose for each $\mathcal{X}$ and each $i \in I(\mathcal{X})$ a fixed map $0^i_{\mathcal{X}} : \mathcal{A}_i \to \mathcal{X}$. We shall now define a functor F by specifying that

$$F(\mathcal{X}) = \left(\prod_{i \in I(\mathcal{X})} \mathrm{Hom}(\mathcal{A}_i, \mathcal{X}) \right) \times T(I(\mathcal{X}))$$

where for an arbitrary subset $I \subseteq \{1, \dots, n\}$, $T(I)$ is the affine subspace of $\mathbb{Z}^n$ defined by

$$T(I) = \{(l_1, \dots, l_n) \in C_n : l_i = 0 \ \ \forall i \notin I\}$$

Furthermore, if $f : \mathcal{X} \to \mathcal{Y}$ is a map in R^γ/R-**mod**, then $I(\mathcal{X}) \subseteq I(\mathcal{Y})$ and we define the map f_* induced by F (or $F(f)$) as that which sends the tuple $((g_i)_{i \in I(\mathcal{X})}, l_1, \dots, l_n)$ to the tuple $((g'_i)_{i \in I(\mathcal{Y})}, l_1, \dots, l_n)$ where

$$g'_i = \begin{cases} f g_i + (l_i - 1) f 0^i_{\mathcal{X}} + (1 - l_i) 0^i_{\mathcal{Y}} & \text{when } i \in I(\mathcal{X}) \\ 0^i_{\mathcal{Y}} & \text{when } i \in I(\mathcal{Y}) \setminus I(\mathcal{X}) \end{cases}$$

The need for f_* to be defined in this way will become apparent in the proof of Proposition 5.4.2.

Now define, for each $i = 1, \dots, n$, the element $\iota_i = ((a_j)_{j \in I(\mathcal{A}_i)}, \mathbf{e}_i) \in F(\mathcal{A}_i)$ by putting $a_i = \mathrm{id}_{\mathcal{A}_i}$ and $a_j = 0^j_{\mathcal{A}_i}$ for $j \neq i$. Then, by the Yoneda lemma, ι_i defines a natural transformation $\mathrm{Hom}(\mathcal{A}_i, -) \to F$.

Proposition 5.4.2. *In the above notation, $\{\iota_i : \mathrm{Hom}(\mathcal{A}_i, -) \to F\}_{i=1,\dots,n}$ is a coproduct diagram in the functor category.*

Proof. Let G be an affine functor and fix elements $\eta_i \in G(\mathcal{A}_i)$ (corresponding to natural transformations $\mathrm{Hom}(\mathcal{A}_i, -) \to G$). Define for each $\mathcal{X}$ the map $\Phi_{\mathcal{X}} : F(\mathcal{X}) \to G(\mathcal{X})$ of affine spaces which acts thus

$$((g_i)_{i \in I(\mathcal{X})}, l_1, \dots, l_n) \longmapsto \sum_{i \in I(\mathcal{X})} \Big(G(g_i)(\eta_i) + (l_i - 1) G(0^i_{\mathcal{X}})(\eta_i) \Big)$$

(It is easily checked that this is indeed an affine map since G is an affine functor and $\sum_{i \in I(\mathcal{X})} l_i = 1$.) We claim that the family of maps $\{\Phi_{\mathcal{X}}\}_{\mathcal{X}}$ defines a natural transformation of functors. To see this, let $f : \mathcal{X} \to \mathcal{Y}$ be a map in R^γ/R-**mod**.

$$
\begin{aligned}
G(f)\Phi_{\mathcal{X}}((g_i)_{i\in I(\mathcal{X})}, l_1, \ldots, l_n) &= G(f)\left(\sum_{i\in I(\mathcal{X})}\Big(G(g_i)(\eta_i) + (l_i - 1)G(0^i_{\mathcal{X}})(\eta_i)\Big)\right) \\
&= \sum_{i\in I(\mathcal{X})}\Big(G(fg_i)(\eta_i) + (l_i - 1)G(f0^i_{\mathcal{X}})(\eta_i)\Big) \\
&= \sum_{i\in I(\mathcal{X})}\Big(G(fg_i + (l_i - 1)f0^i_{\mathcal{X}} + (1 - l_i)0^i_{\mathcal{Y}})(\eta_i) + (l_i - 1)G(0^i_{\mathcal{Y}})(\eta_i)\Big) \\
&\quad + \sum_{i\in I(\mathcal{Y})\setminus I(\mathcal{X})}\Big(G(0^i_{\mathcal{Y}})(\eta_i) + (0 - 1)G(0^i_{\mathcal{Y}})(\eta_i)\Big) \\
&= \Phi_{\mathcal{Y}}F(f)((g_i)_{i\in I(\mathcal{X})}, l_1, \ldots, l_n)
\end{aligned}
$$

So we indeed have a natural transformation $\Phi : F \to G$. Moreover, we have that $\Phi_{\mathcal{A}_i}(\iota_i) = \eta_i$ for each i. So we now need only to prove that Φ is unique with this property.

Let Ψ be any other natural transformation $F \to G$ such that $\Psi_{\mathcal{A}_i}(\iota_i) = \eta_i$ for each i. Let $((g_i)_{i\in I(\mathcal{X})}, l_1, \ldots, l_n) \in F(\mathcal{X})$. Then we can write this element as the following affine combination.

$$\sum_{i\in I(\mathcal{X})}\Big(F(g_i)(\iota_i) + (l_i - 1)F(0^i_{\mathcal{X}})(\iota_i)\Big)$$

So, since $\Psi_{\mathcal{X}}$ is an affine map, we have

$$
\begin{aligned}
\Psi_{\mathcal{X}}((g_i)_{i\in I(\mathcal{X})}, l_1, \ldots, l_n) &= \sum_{i\in I(\mathcal{X})}\Big(\Psi_{\mathcal{X}}F(g_i)(\iota_i) + (l_i - 1)\Psi_{\mathcal{X}}F(0^i_{\mathcal{X}})(\iota_i)\Big) \\
&= \sum_{i\in I(\mathcal{X})}\Big(G(g_i)\Psi_{\mathcal{A}_i}(\iota_i) + (l_i - 1)G(0^i_{\mathcal{X}})\Psi_{\mathcal{A}_i}(\iota_i)\Big) \\
&= \sum_{i\in I(\mathcal{X})}\Big(G(g_i)(\eta_i) + (l_i - 1)G(0^i_{\mathcal{X}})(\eta_i)\Big)
\end{aligned}
$$

In other words $\Psi = \Phi$ and the result is proved. □

Note that a different choice of "constants" $0^i_{\mathcal{X}}$ would give a functor naturally isomorphic to F (by the uniqueness of the coproduct).

Consider now the functor

$$G = \mathrm{Hom}\Big((R^\gamma, (\mathbf{e}_1, \ldots, \mathbf{e}_\gamma)), -\Big) \coprod \mathrm{Hom}\Big((R^\gamma, (\mathbf{e}_1, \ldots, \mathbf{e}_\gamma)), -\Big)$$

Since $\mathrm{Hom}((R^\gamma, (\mathbf{e}_1, \ldots, \mathbf{e}_\gamma)), \mathcal{X})$ is a singleton for each $\mathcal{X}$, it is easy to see that $G(\mathcal{X}) \cong C_2 \cong \mathbb{Z}$ for every $\mathcal{X}$ and that for any map of pointed modules $f : \mathcal{X} \to \mathcal{Y}$,

$f_* = \mathrm{id}$. If we extend this functor to a functor on all pointed modules (not just the f.p. ones) then G will still define $\mathbb{Z}$ on all modules (since it will always define a directed colimit of a system with just one object $\mathbb{Z}$ and all identity maps, and this colimit is again $\mathbb{Z}$; see Lemma 3.3.1 for the definition this extension). Hence G can't be defined by a quotient of pp-formulas since if that were the case we would be able to get $|G(\mathcal{X})| > \aleph_0$ for a suitably large $\mathcal{X}$ by the upward Lowenheim-Skolem theorem.

We claim that G is a coherent functor. So we must show that whenever F is a f.g. functor and $\theta : F \to G$ is a map, then the pullback $F \times_G F$ is also f.g. It will be sufficient to show this whenever F is a finite coproduct of representables as above. This is because, if $F = \coprod_{i=1}^n \mathrm{Hom}(\mathcal{A}_i, -)$ and there is an epimorphism $F \to F'$, then for any $\theta : F' \to G$, the induced map $F \times_G F \to F' \times_G F'$ will also be an epimorphism (as we shall see).

So let $F = \coprod_{i=1}^n \mathrm{Hom}(\mathcal{A}_i, -)$ and let $\{0^i_{\mathcal{X}} : \mathcal{X} \in R^\gamma/R\text{-}\mathbf{mod},\ i \in I(\mathcal{X})\}$ be a choice of constants. Let $\theta : F \to G$ be a natural transformation. Then $\theta = \coprod_{i=1}^n \theta_i$ where $\theta_i \in \mathrm{Nat}(\mathrm{Hom}(\mathcal{A}_i, -), G) \cong G(\mathcal{A}_i) \cong \mathbb{Z}$.

Proposition 5.4.3. *We can write the set $(F \times_G F)(\mathcal{X})$ as*

$$\left(\prod_{i \in I(\mathcal{X})} \mathrm{Hom}(\mathcal{A}_i, \mathcal{X})\right)^2 \times S(I(\mathcal{X}))$$

where for an arbitrary subset $I \subseteq \{1, \dots, n\}$,

$$S(I) = \left\{(l_1, \dots, l_n, l'_1, \dots, l'_n) \in T(I)^2 : \sum_{i=1}^n l_i\theta_i = \sum_{i=1}^n l'_i\theta_i\right\}$$

and we can describe the action of $F \times_G F$ on a map $f : \mathcal{X} \to \mathcal{Y}$ as follows. The tuple $((g_i, g'_i)_{i \in I(\mathcal{X})}, l_1, \dots, l_n, l'_1, \dots, l'_n)$ gets taken to the tuple $((h_i, h'_i)_{i \in I(\mathcal{Y})}, l_1, \dots, l_n, l'_1, \dots, l'_n)$ where

$$h_i = \begin{cases} fg_i + (l_i - 1)f0^i_{\mathcal{X}} + (1 - l_i)0^i_{\mathcal{Y}} & \text{when } i \in I(\mathcal{X}) \\ 0^i_{\mathcal{Y}} & \text{when } i \in I(\mathcal{Y}) \setminus I(\mathcal{X}) \end{cases}$$

$$h'_i = \begin{cases} fg'_i + (l'_i - 1)f0^i_{\mathcal{X}} + (1 - l'_i)0^i_{\mathcal{Y}} & \text{when } i \in I(\mathcal{X}) \\ 0^i_{\mathcal{Y}} & \text{when } i \in I(\mathcal{Y}) \setminus I(\mathcal{X}) \end{cases}$$

The canonical projections $\pi^1, \pi^2 : F \times_G F \to F$ are given by pointwise projection of the cartesian product.

Proof. We first need to check that $\theta\pi^1 = \theta\pi^2$. For convenience, we write a typical element of $(F \times_G F)(\mathcal{X})$ as $(\boldsymbol{g}, \boldsymbol{l}, \boldsymbol{g}', \boldsymbol{l}')$ where $(\boldsymbol{g}, \boldsymbol{l}), (\boldsymbol{g}', \boldsymbol{l}') \in F(\mathcal{X})$ and the inner product $\langle \boldsymbol{l} - \boldsymbol{l}', \boldsymbol{\theta} \rangle = 0$.

For $(\boldsymbol{g}, \boldsymbol{l}) \in F(\mathcal{X})$ we have

$$\begin{aligned}
\theta_{\mathcal{X}}(\boldsymbol{g}, \boldsymbol{l}) &= \theta_{\mathcal{X}} \left(\sum_{i \in I(\mathcal{X})} \Big(F(g_i)(\iota_i) + (l_i - 1) F(0^i_{\mathcal{X}})(\iota_i) \Big) \right) \quad \text{see proof of Prop. 5.4.2} \\
&= \sum_{i \in I(\mathcal{X})} (\theta_{\mathcal{A}_i}(\iota_i) + (l_i - 1)\theta_{\mathcal{A}_i}(\iota_i)) \\
&= \sum_{i \in I(\mathcal{X})} l_i \theta_i \\
&= \langle \boldsymbol{l}, \boldsymbol{\theta} \rangle
\end{aligned}$$

So for $(\boldsymbol{g}, \boldsymbol{l}, \boldsymbol{g}', \boldsymbol{l}') \in (F \times_G F)(\mathcal{X})$,

$$\begin{aligned}
\theta_{\mathcal{X}} \pi^1_{\mathcal{X}}(\boldsymbol{g}, \boldsymbol{l}, \boldsymbol{g}', \boldsymbol{l}') &= \theta_{\mathcal{X}}(\boldsymbol{g}, \boldsymbol{l}) \\
&= \langle \boldsymbol{l}, \boldsymbol{\theta} \rangle \\
&= \langle \boldsymbol{l}', \boldsymbol{\theta} \rangle \\
&= \theta_{\mathcal{X}}(\boldsymbol{g}', \boldsymbol{l}') \\
&= \theta_{\mathcal{X}} \pi^2_{\mathcal{X}}(\boldsymbol{g}, \boldsymbol{l}, \boldsymbol{g}', \boldsymbol{l}')
\end{aligned}$$

Now suppose that $\eta^1, \eta^2 : H \to F$ are natural transformations from a functor H such that $\theta\eta^1 = \theta\eta^2$. Define the map $\Phi : H \to F \times_G F$ by

$$\Phi_{\mathcal{X}}(h) = (\eta^1_{\mathcal{X}}(h), \eta^2_{\mathcal{X}}(h))$$

This is well defined since if $\eta^1_{\mathcal{X}}(h) = (\boldsymbol{g}, \boldsymbol{l})$ and $\eta^2_{\mathcal{X}}(h) = (\boldsymbol{g}', \boldsymbol{l}')$, then

$$\langle \boldsymbol{l}, \boldsymbol{\theta} \rangle = \theta_{\mathcal{X}}(\boldsymbol{g}, \boldsymbol{l}) = \theta_{\mathcal{X}}(\boldsymbol{g}', \boldsymbol{l}') = \langle \boldsymbol{l}', \boldsymbol{\theta} \rangle$$

It is obvious that Φ is natural and clearly Φ is the unique natural transformation $H \to F \times_G F$ such that $\pi^1\Phi = \eta^1$ and $\pi^2\Phi = \eta^2$. This establishes the result. $\square$

In order to show that the functor $F \times_G F$ is finitely generated, we must first make the following definitions.

For each $I \subseteq \{1, \ldots, n\}$, let

$$\mathcal{A}_I = \coprod_{i \in I} \mathcal{A}_i$$

and let $w_i^I : \mathcal{A}_i \to \mathcal{A}_I$ be the canonical injections. Furthermore, define for each i,

$$\mathcal{A}_{ii} = \mathcal{A}_i \coprod \mathcal{A}_i$$

with canonical injections $v_i, v_i' : \mathcal{A}_i \to \mathcal{A}_{ii}$.

For each $I \subseteq \{1, \ldots, n\}$, the set $S(I)$ is an affine subspace of $\mathbb{Z}^n \times \mathbb{Z}^n$ and so is finitely generated by Corollary 5.2.2. Let $X_1^I = (\boldsymbol{x}_{I,1}, \boldsymbol{x}'_{I,1}), \ldots, X_{k_I}^I = (\boldsymbol{x}_{I,k_I}, \boldsymbol{x}'_{I,k_I})$ be affine generators for $S(I)$. We fix the notation

$$\boldsymbol{x}_{I,r} = (x_{I,r1}, \ldots, x_{I,rn})$$
$$\boldsymbol{x}'_{I,r} = (x'_{I,r1}, \ldots, x'_{I,rn})$$

For each $i \in I$, let $\boldsymbol{\alpha}_i$ be the element $((g_j, g_j')_{j \in I(\mathcal{A}_{ii})}, \boldsymbol{e}_i, \boldsymbol{e}_i)$ of $(F \times_G F)(\mathcal{A}_{ii})$ defined by

$$g_j = \begin{cases} 0^j_{\mathcal{A}_{ii}} & \text{when } j \neq i \\ v_j & \text{when } j = i \end{cases}$$
$$g_j' = \begin{cases} 0^j_{\mathcal{A}_{ii}} & \text{when } j \neq i \\ v_j' & \text{when } j = i \end{cases}$$

For each $I \subseteq \{1, \ldots, n\}$ such that $I = I(\mathcal{A}_I)$ and generator X_r^I for $S(I)$, let $\boldsymbol{\beta}_{X_r^I}$ be the element $((h_i, h_i')_{i \in I}, \boldsymbol{x}_{I,r}, \boldsymbol{x}'_{I,r})$ of $(F \times_G F)(\mathcal{A}_I)$ defined by

$$h_i = x_{I,ri} w_i^I + (1 - x_{I,ri}) 0^i_{\mathcal{A}_I}$$
$$h_i' = x'_{I,ri} w_i^I + (1 - x'_{I,ri}) 0^i_{\mathcal{A}_I}$$

We claim that the elements $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_{X_r^I}$ generate the functor $F \times_G F$, regarded as a many-sorted algebra. Equivalently, the coproduct of these elements, regarded as maps from representables, yields an epimorphism to $F \times_G F$.

To see this, let $\mathcal{Y}$ be a f.p. pointed module and $((f_i, f_i')_{i\in I(\mathcal{Y})}, l_1, \ldots, l_n, l_1', \ldots, l_n')$ be an element of $(F \times_G F)(\mathcal{Y})$. Let $I = I(\mathcal{Y})$. Note that in this case we have $I(\mathcal{A}_I) = I$. The tuple $(l_1, \ldots, l_n, l_1', \ldots, l_n')$ is an element of $S(I)$ and so can be expressed as an affine combination

$$\lambda_1 X_1^I + \cdots + \lambda_{k_I} X_{k_I}^I$$

where

$$\lambda_1 + \cdots + \lambda_{k_I} = 1$$

An easy calculation shows the following.

$$(0_{\mathcal{Y}}^i \coprod 0_{\mathcal{Y}}^i)_*(\boldsymbol{\alpha}_i) = ((0_{\mathcal{Y}}^j, 0_{\mathcal{Y}}^j)_{j\in I}, \mathbf{e}_i, \mathbf{e}_i)$$

(by using the explicit description of $F \times_G F$ in Proposition 5.4.3) and

$$(f_i \coprod f_i')_*(\boldsymbol{\alpha}_i) = ((p_j, p_j')_{j\in I}, \mathbf{e}_i, \mathbf{e}_i)$$

where $p_i = f_i$, $p_i' = f_i'$ and $p_j = p_j' = 0_{\mathcal{Y}}^j$ when $j \neq i$. Furthermore,

$$\left(\coprod_{j\in I} 0_{\mathcal{Y}}^j\right)_* \left(\boldsymbol{\beta}_{X_r^I}\right) = ((0_{\mathcal{Y}}^j, 0_{\mathcal{Y}}^j)_{j\in I}, \boldsymbol{x}_{I,r}, \boldsymbol{x}_{I,r}')$$

So we can write

$$((f_i, f_i')_{i\in I}, l_1, \ldots, l_n, l_1', \ldots, l_n') = \left(\sum_{i\in I}(f_i \coprod f_i')_*(\boldsymbol{\alpha}_i)\right) - \left(\sum_{i\in I}(0_{\mathcal{Y}}^i \coprod 0_{\mathcal{Y}}^i)_*(\boldsymbol{\alpha}_i)\right)$$
$$+\lambda_1 \left(\coprod_{i\in I} 0_{\mathcal{Y}}^i\right)_* \left(\boldsymbol{\beta}_{X_1^I}\right) + \cdots + \lambda_{k_I} \left(\coprod_{i\in I} 0_{\mathcal{Y}}^i\right)_* \left(\boldsymbol{\beta}_{X_{k_I}^I}\right)$$

This is an affine combination since the λ_j's sum to 1 and the result is proved. We state this in a proposition.

Proposition 5.4.4. *Let G be the functor which is constantly $\mathbb{Z}$ and let F be $\coprod_{i=1}^n \operatorname{Hom}(\mathcal{A}_i, -)$. For any map $\theta : F \to G$ the pullback $F \times_G F$ is finitely generated.*

We must now show that for *any* f.g. functor F' and map $\theta' : F' \to G$, the functor $F' \times_G F'$ is finitely generated. Since F' is f.g. there is a functor $F = \coprod_{i=1}^n \operatorname{Hom}(\mathcal{A}_i, -)$ and an epimorphism

$$\eta = \coprod_{i=1}^n \eta_i : F \twoheadrightarrow F'$$

Let $\theta_i = \theta'(\eta_i(\mathrm{id}_{\mathcal{A}_i})) \in \mathbb{Z}$. Then it is easy to deduce that

$$\theta'(\eta_{\mathcal{X}}((g_i)_{i\in I(\mathcal{X})}, l_1, \ldots, l_n)) = l_1\theta_1 + \cdots + l_n\theta_n$$

because

$$\eta_{\mathcal{X}}((g_i)_{i\in I(\mathcal{X})}, l_1, \ldots, l_n) = \sum_{i\in I(\mathcal{X})} \Big((g_i)_*(\eta_i(\mathrm{id}_{\mathcal{A}_i})) + (l_i - 1)(0^i_{\mathcal{X}})_*(\eta_i(\mathrm{id}_{\mathcal{A}_i}))\Big)$$

and for each $i \in I(\mathcal{X})$,

$$\theta'((g_i)_*(\eta_i(\mathrm{id}_{\mathcal{A}_i}))) = \theta'((0^i_{\mathcal{X}})_*(\eta_i(\mathrm{id}_{\mathcal{A}_i}))) = \theta'(\eta_i(\mathrm{id}_{\mathcal{A}_i}))$$

Let Φ be the induced map $F \times_G F \to F' \times_G F'$. Then the action of $\Phi_{\mathcal{X}}$ is defined by

$$\begin{array}{c} ((g_i, g'_i)_{i\in I(\mathcal{X})}, l_1, \ldots, l_n, l'_1, \ldots, l'_n) \\ \big\downarrow \\ (\eta_{\mathcal{X}}((g_i)_{i\in I(\mathcal{X})}, l_1, \ldots, l_n), \eta_{\mathcal{X}}((g'_i)_{i\in I(\mathcal{X})}, l'_1, \ldots, l'_n)) \end{array}$$

Clearly Φ is an epimorphism since (using the fact that η is epic)

$$\begin{array}{c} (\eta_{\mathcal{X}}((g_i)_{i\in I(\mathcal{X})}, l_1, \ldots, l_n), \eta_{\mathcal{X}}((g'_i)_{i\in I(\mathcal{X})}, l'_1, \ldots, l'_n)) \in F' \times_G F' \\ \Downarrow \\ \sum_i l_i\theta_i = \sum_i l'_i\theta_i \\ \Downarrow \\ ((g_i, g'_i)_{i\in I(\mathcal{X})}, l_1, \ldots, l_n, l'_1, \ldots, l'_n) \in F \times_G F \end{array}$$

This implies that $F' \times_G F'$ is f.g. since $F \times_G F$ is. So G is a coherent functor and we have the following result.

Theorem 5.4.5. *G is a coherent functor R^γ/R-**mod** $\to$ **Aff** which is not definable by a quotient of pp-formulas in the language of left R-modules with γ additional constants.*

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