

$e$  as the limit of  $(1 + 1/n)^n$   
 Math 122 Calculus III  
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This is a small note to show that the number  $e$  is equal to a limit, specifically

$$\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e.$$

Sometimes this is taken to be the definition of  $e$ , but I'll take  $e$  to be the base of the natural logarithms.

For a positive number  $x$  the natural logarithm of  $x$  is defined as the integral

$$\ln x = \int_1^x \frac{1}{t} dt.$$

Then  $e$  is the unique number such that  $\ln e = 1$ , that is,

$$1 = \int_1^e \frac{1}{t} dt.$$

The natural exponential function  $e^x$  is the function inverse to  $\ln x$ , and all the usual properties of logarithms and exponential functions follow.

Here's a synthetic proof that  $e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$ . A synthetic proof is one that begins with statements that are already proved and progresses one step at a time until the goal is achieved. A defect of synthetic proofs is that they don't explain why any step is made.

*Proof.* Let  $t$  be any number in an interval  $[1, 1 + \frac{1}{n}]$ . Then

$$\frac{1}{1 + \frac{1}{n}} \leq \frac{1}{t} \leq 1.$$

Therefore

$$\int_1^{1 + \frac{1}{n}} \frac{1}{1 + \frac{1}{n}} dt \leq \int_1^{1 + \frac{1}{n}} \frac{1}{t} dt \leq \int_1^{1 + \frac{1}{n}} 1 dt.$$

The first integral equals  $\frac{1}{n+1}$ , the second equals  $\ln(1 + \frac{1}{n})$ , and the third equals  $\frac{1}{n}$ . Therefore,

$$\frac{1}{n+1} \leq \ln(1 + \frac{1}{n}) \leq \frac{1}{n}.$$

Exponentiating, we find that

$$e^{\frac{1}{n+1}} \leq 1 + \frac{1}{n} \leq e^{\frac{1}{n}}.$$

Taking the  $(n + 1)^{\text{st}}$  power of the left inequality gives us

$$e \leq (1 + \frac{1}{n})^{n+1}$$

while taking the  $n^{\text{th}}$  power of the right inequality gives us

$$(1 + \frac{1}{n})^n \leq e.$$

Together, they give us these important bounds on the value of  $e$ :

$$(1 + \frac{1}{n})^n \leq e \leq (1 + \frac{1}{n})^{n+1}.$$

Divide the right inequality by  $1 + \frac{1}{n}$  to get

$$\frac{e}{1 + \frac{1}{n}} \leq (1 + \frac{1}{n})^n$$

which we combine with the left inequality to get

$$\frac{e}{1 + \frac{1}{n}} \leq (1 + \frac{1}{n})^n \leq e.$$

But both  $\frac{e}{1 + \frac{1}{n}} \rightarrow e$  and  $e \rightarrow e$ , so by the pinching theorem

$$(1 + \frac{1}{n})^n \rightarrow e,$$

also.

Q.E.D.