Conference in memory of Gena Puninski

University of Manchester

6-9th April 2018

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This is a joint work with Yasser Ibrahim of Cairo University and is dedicated to the memerory of Gena Puninski.

A right R-module M is said to satisfy the (full) exchange property if for any two direct sum decompositions $M \oplus N = \oplus_{i \in I} N_i$, there exist submodules $K_i \subseteq N_i$ such that $M \oplus N = M \oplus (\oplus_{i \in I} K_i)$. If this holds only for $|I| < \infty$, then M is said to satisfy the finite exchange property.

A ring R for which R_R has the finite exchange property is called an exchange ring.

It is an open question due to Crawley and Jónsson whether the finite exchange property always implies the full exchange property.

The exchange property is of importance because it provides a way to build isomorphic refinements of different direct sum decompositions, which is precisely what is needed to prove the famous Krull-Schmidt-Remak-Azumaya Theorem.

This question was provided a positive answer for:

1. Quasi-injective modules by L. Fuchs, where a module M is quasi-injective if it is invariant under endomorphisms of its injective hull.

2. Quasi-continuous modules by Mohamed and Müller & Oshiro and Rizvi, where a module M is quasi-continuous if it is invariant under idempotent-endom of its injective hull.

3. Auto-invariant Modules by P. Guil Assensio and A. Srivastava, where a module M is called auto-invariant if it is invariant under automorphisms of its injective hull.

4. Square-free modules by P. Nielsen, where M is called square-free if it does not contain a submodule isomorphic to a square $A \oplus A$.

A result of Warfield asserts that a module M_R has the finite exchange property iff $End(M_R)$ is an exchange ring. The notion of exchange rings is left-right symmetric, indeed, Nicholson showed that a ring R is an exchange ring iff idempotents lift every right ideal of R, iff idempotents lift every left ideal of R.

Exchange rings are closely related to another interesting class of rings called clean rings that was first introduced by K. Nicholson, where a ring R is called clean if every element is the sum of an idempotent and a unit.

Nicholson proved that every clean ring is an exchange ring, and a ring with central idempotents is clean iff it is an exchange ring. Subsequently, a module M_R is called clean if $End(M_R)$ is a clean ring.

The class of clean rings is quite large and includes, for instance, semiperfect rings, unit-regular rings, strongly-regular rings, rings of linear transformations of vector spaces, endomorphism rings of continuous modules, and endomorphism rings of automorphism-invariant modules.

There is an intimate link between commutative clean rings and topology. A commutative ring is clean if and only if each of its prime ideals is contained in a unique maximal ideal and its maximal ideal space, endowed with the Zariski topology, is zero-dimensional (P. T. Johnstone, Stone spaces. Cambridge Studies in Advanced Mathematics, 1982).

Clean rings naturally arise as rings of continuous functions on zero-dimensional completely regular Hausdorff spaces (F. Azarpanah, 2002).

Clean rings also arise as commutative C^*-algebras of real rank zero (P. Ara, K.R. Goodearl, K.C. O'Meara, E. Pardo, 1998).

Every ring can be embedded in a clean ring as an essential ring extension (W.D. Burgess and R. Raphael, 2013).

For the last ten years, the search has been going on to find other interesting classes of clean rings and clean modules. The existence of such classes is closely related to Crawley and Jónsson's question as I will explain below. Let me first give some definitions.

In his work on continuous rings, almost half a century ago, Utumi identified three conditions on a ring that are satisfied if the ring is self-injective. These conditions were extended to modules by Mohamed & Müller.

Definition 1 *A module M is called a $C1$-module, if every submodule is essential in a direct summand of M.*

M is called a $C2$-module, if whenever A and B are submodules of M with $A \cong B$, and $B \subseteq^{\oplus} M$, then $A \subseteq^{\oplus} M$.

M is called a $C3$-module, if whenever A and B are submodules of M with $A \subseteq^{\oplus} M$, $B \subseteq^{\oplus} M$, and $A \cap B = 0$, then $A \oplus B \subseteq^{\oplus} M$.

M is called continuous if it is both a $C1$- and a $C2$-module,

M is called quasi-continuous if it is both a $C1$- and a $C3$-module.

It was shown by P. Guil Assensio and A. Srivastava that auto-invariant Modules are clean, and

it was also shown by Camillo, Khurana, Lam, Nicholson and Zhou that every continuous module is clean.

The authors asked: Is a CS module M necessarily clean if it has the finite exchange property?.

While their question still remains open, they provided an affirmative answer for some subclasses of CS modules, namely:

1. when M is quasi-continuous, and

2. when M is square-free.

By modifying and combining the above continuity conditions in one single definition and in honor of Y. Utumi, we consider the following new class of modules.

Definition 2 *A right R-module M is called a Utumi-module (U-module) if for any two non-zero submodules A and B of M with $A \cong B$ and $A \cap B = 0$, there exist two summands K and L of M such that $A \subseteq^{ess} K$, $B \subseteq^{ess} L$ and $K \oplus L \subseteq^{\oplus} M$. Moreover, a ring R is called right U-ring if the right R-module R_R is a U-module.*

Example 3 *Every square-free module and every quasi-continuous module is a U-module.*

Example 4 *Every automorphism-invariant module is a U-module.*

Proof. Let M be an automorphism-invariant module and let X and Y be two non-zero submodules of M with $X \cong Y$ and $X \cap Y = 0$. Let

$$\mathcal{F} = \left\{ \begin{array}{c} (A,B,f)\colon\ A,B \subseteq M,\ X \subseteq^{ess} A,\ Y \subseteq^{ess} B, \\ A \cap B = 0,\ \text{and}\ A \overset{f}{\cong} B \end{array} \right\}.$$

Order $\mathcal{F}$ as follows: $(A,B,f) \leq (A_1,B_1,f_1)$ if $A \subseteq A_1$, $B \subseteq B_1$, and f_1 extends f. Clearly, $\mathcal{F}$ is a non-empty inductive set. Let (A,B,f) be a maximal element of $\mathcal{F}$. We were able to show that $(A \oplus B) \subseteq^{\oplus} M$, and hence M is a U-module. ■

Remark 5 *From the above examples, since the classes of square-free, quasi-continuous and auto-invariant modules are not contained in one another, the class of U-modules is a non-trivial simultaneous generalization of each of these classes of modules.*

More Examples of U-Modules

Example 6 *A right R-module M is called distributive if $A \cap (B + C) = (A \cap B) + (A \cap C)$ for all submodules A, B, and C of M. It is well-known that every distributive module is square-free, and hence a U-module.*

Recall that a ring R is called strongly regular if for every $x \in R$, there is $y \in R$ such that $x^2 y = x$. Since strongly regular rings are square-free, the following example is clear.

Example 7 *Every strongly regular ring is a left and right U-ring.*

Remark 8 *By an example of Bergman, regular rings need not be U-rings. For, if it were a U-ring then by one of our results below it would be clean, a contradiction.*

Example 9

1. *Consider the* $\mathbb{Z}$*-module* $M := \mathbb{Q} \oplus N$ *where* $N := \oplus_{i \in I} \mathbb{Z}_{p_i}$ *is an arbitrary (finite or infinite) direct sum, where* $\{p_i : i \in I\}$ *is a set of distinct primes.* M *is a square-free module and hence a* U*-module that is neither* CS *nor auto-invariant.*

2. *If* p *is a prime number and* $N := \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots$ *an arbitrary (finite or infinite) direct sum of at least two copies of* $\mathbb{Z}_p$*, then the* $\mathbb{Z}$*-module* $M := \mathbb{Q} \oplus N$ *is an orthogonal direct sum of a quasi-injective module* N *and a square-free module* $\mathbb{Q}$*. Clearly* M *is not square-free, and is neither auto-invariant nor quasi-continuous (this follows from the fact that* $\mathbb{Q}$ *and* N *are not relatively-injective). In fact* M *is not a* CS*-module.* M *is a* U*-module. To see this, let* $0 \neq A, B \subseteq M$*, with* $A \cap B = 0$ *and* $A \cong B$*. A simple calculation shows that both* A *and* B *must be contained in the semisimple module* N*. Thus* $A \oplus B$ *is a semisimple summand of* M *and hence* M *is a* U*-module.*

U-Modules

Definition 10 *A module M is called pseudo-injective relative to another module N (pseudo-N-injective) if every monomorphism $f : K \to M$, where $K \subseteq N$, can be extended to a homomorphism from N into M.*

Proposition 11 *If $M = A \oplus B$ is a U-module, then A and B are relatively pseudo-injective.*

Proposition 12 *If $A \oplus B$ is a U-module such that A and B are subisomorphic, then $A \cong B$ and $A \oplus B$ is quasi-injective. In particular, $A \oplus A$ is a U-module if and only if A is quasi-injective.*

Example 13 *As indicated before, the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$ is a U-module that is neither CS nor auto-invariant, we claim that the $\mathbb{Z}$-module $M := \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$ is not a U-module. For if it were a U-module then by the above result, the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}_p$ would be quasi-injective, a contradiction.*

Proposition 14 *Let M be a right U-module.*

1. *If $M = A \oplus B$ with $E(A) \cong E(B)$, then M is quasi-injective.*

2. *If A and B are submodules of M such that $A \cap B = 0$ and $A \cong B$, then A and B have isomorphic complements.*

Two right R-modules M and N are called orthogonal to each other, if they don't contain non-zero isomorphic submodules.

Theorem 15 *If M is a U-module, then $M = Q \oplus T$ where:*

1. *Q is a quasi-injective module,*
2. *$Q = A \oplus B \oplus D$, where $A \cong B$ and D is isomorphic to a summand of $A \oplus B$,*
3. *T is a square-free module,*
4. *T is Q-injective, and*
5. *Q and T are orthogonal.*

We use the above decomposition theorem to establish our main result:

Theorem 16 *A right U-module M is clean if and only if it has the finite exchange property, if and only if it has the full exchange property.*

Theorem 17 $\boldsymbol{\Sigma}$*-U-modules are clean and satisfy the full exchange property.*

Example 18 *If $M := \mathbb{Q} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots$ is the $\mathbb{Z}$-module provided above, then M is a U-module which is clean by the above theorem. The module M is not square-free, not pseudo-continuous, not quasi-continuous, and not auto-invariant.*

Special Modules

A result of Singh and Srivastava states that if M is an auto-invariant module and $E(M) = E_1 \oplus E_2 \oplus E_3$ with $E_1 \cong E_2$, then $M = (M \cap E_1) \oplus (M \cap E_2) \oplus (M \cap E_3)$.

This result is also valid if M is a quasi-continuous module. In fact a module M is quasi-continuous if and only if every decomposition $E(M) = \oplus_{i \in I} E_i$ induces $M = \oplus_{i \in I}(M \cap E_i)$.

In the next definition we consider a weaker version of these results and introduce a new class of modules, called special modules. It turns out that this new class of modules coincide with that of U-modules, and has some interesting features which we state below.

Definition 19 *A right R-module M is called special if, for every decomposition $E(M) = E_1 \oplus E_2 \oplus E_3$ with $E_1 \cong E_2$, $M = (M \cap E_1) \oplus (M \cap E_2) \oplus T$ for a submodule $T \subseteq M$.*

Theorem 20 *If M is a right R-module, then the following conditions are equivalent:*

1. *M is a special module.*

2. *M is a U-module.*

The above theorem can be improved as follows:

Theorem 21 *Let M be a U-module. If $E(M) = E_1 \oplus E_2 \oplus E_3$ with $E_1 \cong E_2$, then $M = (M \cap E_1) \oplus (M \cap E_2) \oplus (M \cap E(T))$ where $T \subseteq M$, $E(T) \cong E_3$, $(M \cap E_1) \cong (M \cap E_2)$ and $(M \cap E_1) \oplus (M \cap E_2)$ is quasi-injective.*

U-Rings

Recall that a ring R is strongly regular iff R is an abelian regular ring, iff R is square-free and regular. Also, if R is abelian, then R is exchange iff R is clean. For U-rings, we have:

Theorem 22 *If R is an abelian ring, then the following conditions are equivalent:*

1. *R is a right U-ring.*
2. *R is square-free as a right R-module.*
3. *Every right ideal of R is a U-module.*

Observe that being a right U-ring is not a Morita-invariant property. For example the ring of integers $\mathbb{Z}$ is quasi-continuous and hence a U-ring. However, $\mathbb{M}_2(\mathbb{Z})$ is not a right (or left) U-ring. In fact we have the following:

Theorem 23 *For $n > 1$, the following conditions on a ring R are equivalent:*

1. *R is a right self-injective ring.*

2. *$M_n(R)$ is a right U-ring.*

Theorem 24 *The following are equivalent for a ring R:*

1. *R is a right U^*-ring (every right ideal is a U-module).*

2. *R is a direct product of a square-full semisimple artinian ring and a right square-free ring.*

Theorem 25 *A ring R is (countably) Σ-U-ring iff R is quasi-Frobenius.*

Remark 26 *There are examples of pure-injective modules that are not U-modules. For, if every pure-injective is a U-module, then every right Σ-pure-injective ring R is a Σ-U-module as a right R-module, and hence quasi-Frobenius, a contradiction.*

Remark 27 *Since pure-injective modules are cotorsion, by the above remark cotorsion modules need not be U-modules.*

The next theorem extends a result of Goodearl which asserts that every indecomposable regular right continuous ring is right self-injective. Our result also extends a result by Er, Singh and Srivastava, which states that every prime right non-singular right auto-invariant ring is right self-injective. We should point out that the later result by Er et al. was a positive answer in response to a question raised by Clark and Huynh: whether simple right pseudo-injective (equivalently, right auto-invariant) rings are right self-injective.

Theorem 28 *If R is a regular right U-ring, then R can be decomposed as a direct sum of a strongly regular ring and a regular right self-injective ring. In particular every indecomposable regular right U-ring is right self-injective.*

Corollary 29 *Let R be an indecomposable right non-singular ring. If R is either right continuous or right auto-invariant, then R is right self-injective. In particular, every simple right continuous (or right auto-invariant) ring is right self-injective.*

More on Exchange and Clean U-Modules

1. The module M is said to have the cancellation property if whenever $M \oplus X \cong M \oplus Y$, then $X \cong Y$.

2. The module M is said to have the internal cancellation property if whenever $M = A \oplus X = B \oplus Y$ with $A \cong B$, then $X \cong Y$.

3. A module N is said to have the substitution property if for every module M with decompositions $M = N_1 \oplus H = N_2 \oplus K$ with $N_1 \cong N \cong N_2$, there exists a submodule C of M such that $M = C \oplus H = C \oplus K$.

4. A module M_R is said to be Dedekind-finite (or directly-finite) if $M \cong M \oplus N$ implies $N = 0$.

5. A ring R is said to have stable range 1 if, for any elements $a, b \in R$ with $Ra + Rb = R$, there is an element $y \in R$ such that $a + yb$ is a unit of R.

In general, we have the following implications:

Substitution $\Rightarrow$ Cancellation $\Rightarrow$
Internal Cancellation $\Rightarrow$ Dedekind-finite

By a well-known result, the first three notions coincide if M has the finite exchange property, and it was shown by Asensio and Srivastava that for an auto-invariant module the above four notions are equivalent. For U-modules we have the following:

Proposition 30 *If M is a right U-module with the (finite) exchange property, then the following are equivalent:*

1. *M has the substitution property.*
2. *M has the cancellation property.*
3. *M has the internal cancellation property.*
4. *$End(M_R)$ has stable range 1.*

5. *M is Dedekind-finite.*

Rings Generated by its Idempotents

J. Erdös showed that the linear transformations of a finite-dimensional vector space over a division ring which are products of proper idempotents are precisely the singular ones. Erdös' results were subsequently extended by Reynolds and Sullivan to linear transformations of an arbitrary dimension vector space over a division ring. Subsequently, O'Meara provided a complete characterization of the elements of a prime, regular, right self-injective ring which can be written as a product of idempotents. After that Hannah and O'Meara extended the work of O'Meara to regular right self-injective rings.

In the mean time it has been known that many of the above classes of regular self-injective rings are generated as rings by their idempotents, that is each element is a sum of products of idempotents. Indeed, Wolfson proved that every right full linear ring that is not a division ring is generated as a ring by its idempotents. Right full linear rings R are known to be prime, regular, and right self-injective, and if the ring R is not a division ring then R is also totally non-abelian, where a ring R (not necessarily regular) is called totally non-abelian if every non-zero right ideal of R contains a non-central idempotent. Consequently, Wolfson's result was extended by Utumi, who proved that if R is a regular right self-injective ring that is totally non abelian, then R is generated as a ring by its idempotents. Utumi's result was provided an excellent proof by Goodearl in his book Nonsingular Rings and Modules.

Theorem 31 *Let R be a right U-ring. If every non-zero square-free right ideal of R contains a non-central idempotent, then $R = A \oplus B \oplus K$ with $A \cong B$ and K is a square-free module isomorphic to a submodule of $A \oplus B$. Moreover, R is a right self-injective ring and is generated as a ring by its idempotents.*

Theorem 32 *If R is a totally non-abelian right U-ring, then R is a regular right self-injective ring and is generated as a ring by its idempotents.*

Corollary 33 *If R is a totally non-abelian right quasi-continuou ring, then R is a regular right self-injective ring and is generated as a ring by its idempotents.*

Corollary 34 *If R is a totally non-abelian right automorphism-invariant ring, then R is a regular right self-injective ring and is generated as a ring by its idempotents.*

Recall that a submodule T of a module M is called a square-root in M if $T^2 =: T \oplus T$ embeds in M. A module M is called square-full if every non-zero submodule of M contains a non-zero square-root in M.

Corollary 35 *Every square-full right U-ring is right self-injective and is generated as a ring by its idempotents.*