A categorical approach to tilting theory

Robert Wisbauer

University of Düsseldorf, Germany

Manchester, April 2018

Overview

- Adjoint pair of functors
- Properties of units and counits
- Adjoints and (co)monads
- n-$*$-pairs of functors

Category theory

Eilenberg - Mac Lane,
General Theory of natural equivalences
Trans. APS 1945

Category $\mathbb{A}$: objects and morphisms $\mathrm{Mor}_{\mathbb{A}}(A, A')$

- functors $F : \mathbb{A} \to \mathbb{B}$:
- object A in $\mathbb{A}$ is sent to object $F(A)$ in $\mathbb{B}$,
- morphism $f : A \to A'$ sent to $F(f) : F(A) \to F(A')$ in $\mathbb{B}$.

Properties of functors - induced map for $A, A' \in \mathbb{A}$

$$\varphi_F : \mathrm{Mor}_{\mathbb{A}}(A, A') \to \mathrm{Mor}_{\mathbb{B}}(F(A), F(A')),$$

- F is *faithful* if φ_F is injective,
- F is *full* if φ_F is surjective,
- F is *separable* if φ_F is a (naturally) split injection.

Eilenberg-Moore: Adjoint functors and triples, 1965

Adjoint pair of functors $F : \mathbb{A} \to \mathbb{B}$, $G : \mathbb{B} \to \mathbb{A}$, bijection

$$\varphi : \mathrm{Mor}_{\mathbb{B}}(F(A), B) \xrightarrow{\simeq} \mathrm{Mor}_{\mathbb{A}}(A, G(B)),$$

$$\text{unit} \quad \eta : 1_{\mathbb{A}} \to GF, \qquad \text{counit} \quad \varepsilon : FG \to 1_{\mathbb{B}}$$

Triangular identities

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & {\scriptstyle 1_F} \searrow & \downarrow {\scriptstyle \varepsilon F} \\ & & F, \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & {\scriptstyle 1_G} \searrow & \downarrow {\scriptstyle G\varepsilon} \\ & & G \end{array}$$

F	preserves epimorphisms and coproducts
G	preserves monomorphisms and products

Adjoint functors

Adjunction $\eta : 1_{\mathbb{A}} \to GF$, counit $\varepsilon : FG \to 1_{\mathbb{B}}$

η and ε isomorphisms	F is equivalence
ε split monomorphism	G is full
ε epimorphism	G is faithful
ε split epimorpism	G is separable
ε isomorphism	G is fully faithful
η split epimorphism	F is full
η monomorphism	F is faithful
η split monomorphism	F is separable
η isomorphism	F is fully faithful
η extremal epimorph, ε extremal monomorph	(F, G) pair of $*$-functors

Module categories: Adjoint functors

Adjoint pair: bimodule ${}_RP_S$

$$ {}_RP\otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}, \quad \mathrm{Hom}_R(P,-) : {}_R\mathbb{M} \to {}_S\mathbb{M} $$

$$ \mathrm{Hom}_R(P\otimes_S Y, X) \xrightarrow{\simeq} \mathrm{Hom}_S(Y, \mathrm{Hom}_R(P,X)) $$

$$ \begin{aligned} \text{counit} \quad & \varepsilon_X : P\otimes_S \mathrm{Hom}_R(P,X) \to X, \\ \text{unit} \quad & \eta_Y : Y \to \mathrm{Hom}_R(P, P\otimes_S Y). \end{aligned} $$

- η and ε isomorphisms: ${}_RP$ progenerator
  ${}_R\mathbb{M} \simeq {}_S\mathbb{M}$, **Morita equivalence**
- η isomorphisms: P_S faithfully flat
  $\mathrm{Pres}({}_RP) \simeq {}_S\mathbb{M}$, **Sato equivalence**
- ε monomorph and η epimorph (tilting theory)
  $\mathrm{Gen}(P) \simeq \mathrm{Cog}({}_SU)$, **Brenner-Butler equivalence**
  $U = \mathrm{Hom}(P,Q)$, Q cogenerator in $\mathrm{Gen}(P)$

Categories: Monads and comonads

Monads: $T : \mathbb{A} \to \mathbb{A}$ endofunctor, nat. transf.

$$m : TT \to T, \qquad \eta : 1_{\mathbb{A}} \to T,$$

with associativity and unitality conditions

T-modules: $\varrho : T(A) \to A$, associative, unital

T-morphisms $(A, \varrho) \xrightarrow{f} (A', \varrho')$,

$$\begin{array}{ccc} T(A) & \xrightarrow{T(f)} & T(A') \\ \varrho \downarrow & & \downarrow \varrho' \\ A & \xrightarrow{f} & A' \end{array}$$

(Eilenberg-Moore) category of T-modules $\mathbb{A}_T$

Free and forgetful functor - adjoint

$$\begin{aligned} \phi_T : \mathbb{A} \to \mathbb{A}_T, \quad & A \mapsto (T(A), m_A : TT(A) \to T(A)), \\ U_T : \mathbb{A}_T \to \mathbb{A}, \quad & (A, \varrho_A) \mapsto A. \end{aligned}$$

$$\mathrm{Mor}_{\mathbb{A}_T}(\phi_T(A), B) \xrightarrow{\simeq} \mathrm{Mor}_{\mathbb{A}}(A, U_T(B)), \quad f \mapsto f \circ \eta_A.$$

Categories: Monads and comonads

Comonads: $S : \mathbb{A} \to \mathbb{A}$ endofunctor, nat. transf.

$$\delta : S \to SS, \qquad \varepsilon : S \to 1_{\mathbb{A}},$$

with coassociativity and counitality conditions

S-comodules: $\omega : A \to S(A)$, coassociative, counital

S-morphisms $(A, \omega) \xrightarrow{g} (A', \omega')$,

$$\begin{array}{ccc} A & \xrightarrow{g} & A' \\ \omega \downarrow & & \downarrow \omega' \\ S(A) & \xrightarrow{S(g)} & S(A') \end{array}$$

(Eilenberg-Moore) category of S-comodules $\mathbb{M}^S$

Forgetful and free functor - adjoint

$$\phi^S : \mathbb{A} \to \mathbb{A}^S, \qquad A \mapsto (S(A), \delta_A),$$
$$U^S : \mathbb{A}^S \to \mathbb{A}, \quad (A, \omega) \mapsto A.$$

$$\mathrm{Mor}^S(A, S(X)) \xrightarrow{\simeq} \mathrm{Mor}_{\mathbb{A}}(U^S(A), X), \quad h \mapsto \varepsilon_X \circ h$$

Adjoints $\eta : 1_{\mathbb{A}} \to GF, \quad \varepsilon : FG \to 1_{\mathbb{B}}$

Related monad on $\mathbb{A}$, comonad on $\mathbb{B}$

$$T = GF : \mathbb{A} \to \mathbb{A}, \qquad \text{product} \quad m : GFGF \xrightarrow{G\varepsilon F} GF,$$
$$\text{unit} \quad \eta : 1_{\mathbb{A}} \to GF.$$

$\mathbf{T} = (GF, G\varepsilon F, \eta)$ is a monad on $\mathbb{A}$.

$$S = FG : \mathbb{B} \to \mathbb{B}, \qquad \text{coproduct} \quad \delta : FG \xrightarrow{F\eta G} FGFG,$$
$$\text{counit} \quad \varepsilon : FG \to 1_{\mathbb{B}}.$$

$\mathbf{S} = (FG, F\eta G, \varepsilon)$ is a comonad on $\mathbb{B}$;

Comparison functors

(1) There is a functor $\overline{G} : \mathbb{B} \to \mathbb{A}_{GF}$, $B \mapsto (G(B), G\varepsilon_B)$.

(2) There is a functor $\overline{F} : \mathbb{A} \to \mathbb{B}^{FG}$, $A \mapsto (F(A), F\eta_A)$.

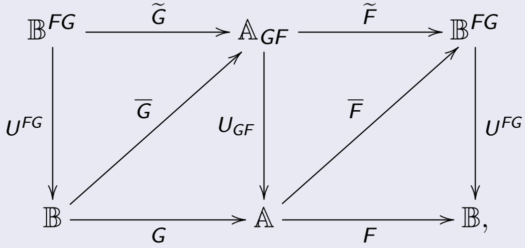
Adjoints $\eta : 1_{\mathbb{A}} \to GF, \quad \varepsilon : FG \to 1_{\mathbb{B}}$

Related functors

(1) For the monad GF on $\mathbb{A}$, composing U_{GF} with $\overline{F}$,

(2) for the comonad FG on $\mathbb{B}$, composing U^{FG} with $\overline{G}$,

lead to the commutative diagram



In general $(\widetilde{F}, \widetilde{G})$ need not be an adjoint pair of functors.

Adjoints $\eta : 1_{\mathbb{A}} \to GF, \quad \varepsilon : FG \to 1_{\mathbb{B}}$

$(\widetilde{F}, \widetilde{G})$ as an adjoint pair: the following are equivalent

(a) *by restriction and corestriction, φ induces an isomorphism* $\mathrm{Mor}^{FG}(\widetilde{F}(A), B) \to \mathrm{Mor}_{GF}(A, \widetilde{G}(B))$, $A \in \mathbb{A}_{GF}$, $B \in \mathbb{B}^{FG}$;

(b) $(\widetilde{F}, \widetilde{G})$ *is an adjoint pair of functors;*

(c) $\eta G : G \to GFG$ *is an isomorphism;*

(d) *the product* $G\varepsilon F : GFGF \to GF$ *is an isomorphism;*

(e) *the coproduct* $F\eta G : FG \to FGFG$ *is an isomorphism.*

In this case, (F, G) is called an *idempotent pair of adjoints*.

Definitions.

$$\mathrm{Fix}(GF, \eta) = \{A \in \mathbb{A} \,|\, \eta_A : A \to GF(A) \text{ is an isomorphism}\},$$
$$\mathrm{Fix}(FG, \varepsilon) = \{B \in \mathbb{B} \,|\, \varepsilon_B : FG(B) \to B \text{ is an isomorphism}\}.$$

Adjoints $\eta : 1_{\mathbb{A}} \to GF, \quad \varepsilon : FG \to 1_{\mathbb{B}}$

Definition

An adjoint pair (F, G) is said to be a *pair of $\star$-functors* provided
$\eta_A : A \to GF(A)$ is an extremal epimorphism for all $A \in \mathbb{A}$ and
$\varepsilon_B : FG(B) \to B$ is an extremal monomorphism for all $B \in \mathbb{B}$.

Theorem

For a pair of $\star$-functors (F, G), the related functors

$$\widetilde{F} : \mathbb{A}_{GF} \to \mathbb{B}^{FG}, \quad \widetilde{G} : \mathbb{B}^{FG} \to \mathbb{A}_{GF},$$

induce an equivalence where

(i) $\mathbb{A}_{GF} = \mathrm{Fix}(GF, \eta)$ is a reflective subcategory of $\mathbb{A}$, closed under subobjects in $\mathbb{A}$ and

(ii) $\mathbb{B}^{FG} = \mathrm{Fix}(FG, \varepsilon)$ is a coreflective subcategory of $\mathbb{B}$, closed under factor objects in $\mathbb{B}$.

Module categories: Adjoints and (co)monads

Bimodule - ${}_R P_S$

$$P \otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}, \qquad \mathrm{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_S\mathbb{M}$$

$$\eta : (-) \to \mathrm{Hom}_R(P, P \otimes_S -),$$
$$\varepsilon : P \otimes_S \mathrm{Hom}_R(P, -) \to (-),$$

monad $\quad \mathrm{Hom}_A(P, P \otimes_S -) : {}_S\mathbb{M} \to {}_S\mathbb{M},$

comonad $\quad P \otimes_S \mathrm{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_R\mathbb{M}$

${}_R P$ finitely generated and projective: comonad

$$P \otimes_S \mathrm{Hom}_R(P, -) \simeq P \otimes_S P^* \otimes_R - : {}_R\mathbb{M} \xrightarrow{\simeq} {}_R\mathbb{M}$$

$P \otimes_S P^* \quad R$-coring

Category theory: Simplicial objects

Simplicial objects

Let Δ be category

- *objects*: finite ordered sets $[n] = \{0 < 1 < \ldots < n\}$, $n \geq 0$,
- *morphisms*: nondecreasing monotone functions.

If $\mathbb{A}$ is any category,

- *simplicial object* $A \in \mathbb{A}$: contravariant functor $A : \Delta^{op} \to \mathbb{A}$, write A_n for $A([n])$.
- *cosimplicial object* $C \in \mathbb{A}$: covariant functor $C : \Delta \to \mathbb{A}$, write C^n for $C([n])$.
- *morphism of simplicial objects*: natural transformations, category $\mathcal{S}\mathbb{A}$ of simplicial objects in $\mathbb{A}$: functor category $\mathbb{A}^{\Delta^{op}}$.

Category theory: Simplicial objects

Simplicial objects in $\mathbb{A}$

To give a simpicial object A in $\mathbb{A}$ is to give a sequence of objects $A_0, A_1, \ldots$ together with *face* and *degeneracy* operators

$$\partial_i : A_n \to A_{n-1}, \quad \sigma_i : A_n \to A_{n+1} \quad (i = 0, 1, \ldots, n),$$

which satisfy the "simplicial" identities

$$\begin{array}{rll} \partial_i \partial_j = & \partial_{j-1}\partial_i & \text{if } i < j \\ \sigma_i \sigma_j = & \sigma_{j+1}\sigma_i & \text{if } i \leq j \\ & \sigma_{j-1}\partial_i & \text{if } i < j \\ \partial_i \sigma_j = & \text{identity} & \text{if } i = j \text{ or } i = j+1 \\ & \sigma_j \partial_{i-1} & \text{if } i > j+1 \end{array}$$

Category theory: Simplicial objects

Associated chain complexes: A simplicial object in *abelian* $\mathbb{A}$

- The *associated*, or *unnormalized, chain complex* $C = C(A)$ has $C_n = A_n$, and boundary morphisms (with $\partial_i : C_n \to C_{n-1}$),

$$d = \partial_0 - \partial_1 + \partial_2 + \ldots (-1)^n \partial_n : C_n \to C_{n-1}.$$

- The *normalized*, or *Moore*, chain complex $N(A)$ is the chain complex with

$$N_n(A) = \bigcap_{i=0}^{n-1} \ker(\partial_i : A_n \to A_{n-1})$$

and differential $d = (-1)^n \partial_n$.

Category theory: Simplicial objects - $\mathbb{A}$ abelian

Homotopy group - A simplicial object in $\mathbb{A}$

$N(A)$ is a subcomplex of $C(A)$ and we define

$$\pi_n(A) = H_n(N(A)).$$

Theorem - A simplicial object in $\mathbb{A}$

The homotopy $\pi_*(A)$ is naturally isomorphic to the homology $H_*(C)$ of the unnormalized chain complex $C = C(A)$,

$$\pi_*(A) = H_*(N(A)) \simeq H_*(C(A)).$$

Dold-Kan correspondence.

The normalized chain complex functor N is equivalence between

- $\mathcal{S}\mathbb{A}$, the category of simplicial objects in $\mathbb{A}$, and
- $Ch_{\geq 0}(\mathbb{A})$, the chain complexes C in $\mathbb{A}$ with $C_n = 0$ for $n < 0$.

Category theory: Simplicial objects

Simplicial object of a comonad.

Given a cotriple (S, δ, ε) on $\mathbb{A}$ and an object $A \in \mathbb{A}$, set $S_n A = S^{n+1} A$ and define face and degeneracy operators by

$$\partial_i = S^i \varepsilon S^{n-i} : S^{n+1} \to S^n,$$
$$\sigma_i = S^i \delta S^{n-i} : S^{n+1} \to S^{n+2},$$

$S^* A$ is a simplicial object in $\mathbb{A}$.

$\varepsilon_A : SA \to A$ satisfies $\varepsilon \partial_0 = \varepsilon \partial_1$, so $S^* A \to A$ is *augmented*.

$$A \xleftarrow{\ \varepsilon\ } SA \underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}} S^2 A \underset{\partial_1}{\overset{\partial_0}{\Lleftarrow}} S^3 A \underset{\partial_3}{\overset{\partial_0}{\Lleftarrow}} S^4 A \ldots\ldots.$$

$$SA \xrightarrow{\ \delta\ } S^2 A \underset{\sigma_1}{\overset{\sigma_0}{\rightrightarrows}} S^3 A \underset{\sigma_2}{\overset{\sigma_0}{\Rrightarrow}} S^4 A \ldots\ldots$$

Categories: $\star$-functors

(F, G) $\star$-pair: ε mono, η epi

then $\partial_0 = \partial_1$, FG is idempotent. If $\mathbb{A}$ abelian, for the associated unnormalized chain complex (see above),

$$0 = d^{(1)} = \partial_0 - \partial_1 : S^2 A \to SA$$

for n-$\star$-functors - to be defined for $\mathbb{A}$ abelian

in the unnormalized chain complex, assume

$$d^{(n-1)} = \partial_0 - \partial_1 + \partial_2 + \dots (-1)^n \partial_n : S^n A \to S^{n-1} A$$

to be an (extremal) mono (plus symmetric condition for η)
Then, in particular,

$$0 = d^{(n)} = \partial_0 - \partial_1 + \partial_2 + \dots (-1)^{n+1} \partial_{n+1} : S^{n+1} A \to S^n A$$

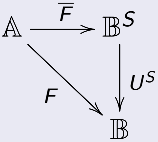
(What would this mean for the simplicial object if $\mathbb{A}$ is not abelian?)

Categories: Adjoint monads and comonads

Monadic decomosition (see Adámek et al.)

For $F \dashv G$ and $S = FG$, (Eilenberg-Moore) comparison functor

$$\overline{F} : \mathbb{A} \to \mathbb{B}^S, \quad A \mapsto (F(A), F\eta_A : F(A) \to FGF(A)),$$



$\overline{F}$ has a right adjoint $\overline{G} : \mathbb{B}^S \to \mathbb{A}$, the equalizer, for $(B, \omega) \in \mathbb{B}^S$,

$$\overline{G}(B,\omega) \xrightarrow{j_{(B,\omega)}} G(B) \underset{\eta G(B)}{\overset{G\omega}{\rightrightarrows}} GFG(B).$$

Categories: Adjoint monads and comonads

Monadic decomosition

Assume $\overline{F} \dashv \overline{G}$, then

$$\mathbb{A} \xrightarrow{\overline{F}} (\mathbb{B}^{FG}) \xrightarrow{\overline{G}} \mathbb{A}, \qquad \mathbb{A} \xrightarrow{\overline{\overline{F}}} (\mathbb{B}^{FG})^{\overline{FG}} \xrightarrow{\overline{\overline{G}}} \mathbb{A}, \ldots$$

If $\mathbb{A}$ has equalisers, then by iterating the above construction we can build a tower of adjoint pairs indexed by the non-negative integers having the given one at stage 0. For limit ordinals one needs more technical details.
This defines the *monadic length* of the adjunction (the step when the adjunctions become idempotent).
How is monadic length n related to n-$\star$-functors?

Theorem

If $\mathbb{A}$ has coequalisers and colimits of (perhaps large) chains of strong epimorphisms, then every adjunction (F, G) has a canonical monadic decomposition.

Thank you !

Categories: Adjoint monads and comonads

Adjoint endofunctors $F : \mathbb{A} \to \mathbb{A}$, $G : \mathbb{A} \to \mathbb{A}$

$$\mathrm{Mor}_{\mathbb{A}}(F(X), Y) \xrightarrow{\varphi} \mathrm{Mor}_{\mathbb{A}}(X, G(Y)), \qquad \begin{array}{l}\eta : 1_{\mathbb{A}} \to GF,\\ \varepsilon : FG \to 1_{\mathbb{A}}.\end{array}$$

F **monad,** $m : FF \to F$, $e : I_{\mathbb{A}} \to F$

$$\begin{array}{ccc}
\mathrm{Mor}_{\mathbb{A}}(F(X), Y) & \xrightarrow{\varphi_{X,Y}} & \mathrm{Mor}_{\mathbb{A}}(X, G(Y)) \\
\Big\downarrow {\scriptstyle \mathrm{Mor}(m_X, Y)} & & \vdots\ {\scriptstyle \mathrm{Mor}(X, ?)} \\
\mathrm{Mor}_{\mathbb{A}}(FF(X), Y) & \xrightarrow{\simeq} & \mathrm{Mor}_{\mathbb{A}}(X, GG(Y))
\end{array}$$

implies G **comonad,** $\underline{\delta} : G \to GG$, $\underline{\varepsilon} : G \to I_{\mathbb{A}}$

$$\underline{\delta} : G \xrightarrow{\eta G} GFG \xrightarrow{G\eta FG} GGFFG \xrightarrow{GGmG} GGFG \xrightarrow{GG\varepsilon} GG,$$

$$\underline{\varepsilon} : G \xrightarrow{eG} FG \xrightarrow{\varepsilon} 1_{\mathbb{A}}.$$

Categories: Adjoints and (co)monads

Adjoint pair: $F \dashv G : \mathbb{B} \to \mathbb{A}$, $\eta : 1_{\mathbb{A}} \to GF$, $\varepsilon : FG \to 1_{\mathbb{B}}$

ε split epi (G separable) : $\varepsilon^{-1} : 1_{\mathbb{B}} \to FG$

coproduct $\delta' : GF \xrightarrow{G\varepsilon^{-1}F} GFGF$, $\quad m \circ \delta' = 1_{GF}$

monad $T = GF$: (T, m, δ') with Frobenius condition (no counit)

$$\begin{array}{ccc} TT & \xrightarrow{\delta T} & TTT \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle Tm} \\ T & \xrightarrow{\delta} & TT, \end{array} \qquad \begin{array}{ccc} TT & \xrightarrow{T\delta} & TTT \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle mT} \\ T & \xrightarrow{\delta} & TT, \end{array}$$

T separable monad

η split mono (F separable) : $\eta^{-1} : GF \to 1_{\mathbb{A}}$

product $m' :=: FGFG \xrightarrow{F\eta^{-1}G} FG$, $\quad m' \circ \delta = 1_{FG}$

comonad $S = FG$: (S, δ, m') with Frobenius condition (no unit)

S coseparable comonad

Adjoint endofunctors

(F, m, e) monad, $F \dashv G$

$(G, \underline{\delta}, \underline{\varepsilon})$ is comonad, equivalence of categories $\mathbb{A}_F \simeq \mathbb{A}^G$

$$F(A) \xrightarrow{h} A \quad \mapsto \quad A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(h)} G(A)$$

$F \dashv F$ (Frobenius monad) $\Rightarrow$ $\mathbb{A}_F \simeq \mathbb{A}^F$

(G, δ, ε) comonad, $G \dashv F$

$(F, \underline{m}, \underline{e})$ is monad, equival. of Kleisli categories $\widetilde{\mathbb{A}}^G \simeq \widetilde{\mathbb{A}}_F$

$$\begin{aligned} \mathrm{Mor}_{\mathbb{A}^G}(\phi^G(A), \phi^G(A')) &\simeq \mathrm{Mor}_{\mathbb{A}}(G(A), A') \\ &\simeq \mathrm{Mor}_{\mathbb{A}}(A, F(A')) \\ &\simeq \mathrm{Mor}_{\mathbb{A}_F}(\phi_F(A), \phi_F(A')) \end{aligned}$$

Module categories

Adjoint endofunctors $A \otimes_R -,\ \mathrm{Hom}_R(A, -) : {}_R\mathbb{M} \to {}_R\mathbb{M}$

$$\mathrm{Hom}_R(A \otimes_R X, Y) \xrightarrow{\simeq} \mathrm{Hom}_R(X, \mathrm{Hom}_R(A, Y))$$

$$A \otimes_R - \text{ monad} \Leftrightarrow \mathrm{Hom}_R(A, -) \text{ comonad},\ \ \mathbb{M}_A \simeq \mathbb{M}^{\mathrm{Hom}(A,-)}$$

$$\text{Frobenius monad} \quad A \otimes_R - \simeq \mathrm{Hom}_R(A, -), \quad \mathbb{M}_A \simeq \mathbb{M}^A$$

$$A \otimes_R - \text{ comonad } (R\text{-coring}) \ \Leftrightarrow\ \mathrm{Hom}_R(A, -) \text{ monad}$$

$$\widetilde{\mathbb{M}}^A \simeq \widetilde{\mathbb{M}}_{\mathrm{Hom}_R(A,-)},\ \ A \otimes_R X \mapsto \mathrm{Hom}_R(A, X)$$

${}_RA$ fin. gen., projective $\mathrm{Hom}_R(A, -) \simeq A^* \otimes_R -$

$$A \otimes_R - \text{ monad} \quad \Leftrightarrow \quad A^* \otimes_R - \text{ comonad}$$

$$A \otimes_R - \text{ comonad} \quad \Leftrightarrow \quad A^* \otimes_R - \text{ monad}$$

$$\text{Frobenius} \quad A \simeq A^*, \quad \mathbb{M}_A \simeq \mathbb{M}^A$$

Thank you !

Composition of monads and comonads

Tensorproduct of R-algebras (A, m, e), (B, m', e')

$$A \otimes B \otimes A \otimes B \xrightarrow{A \otimes \tau \otimes B} A \otimes A \otimes B \otimes B \xrightarrow{m \otimes m'} A \otimes B$$

Distributive law: $\tau : B \otimes_R A \to A \otimes_R B$

$$\begin{array}{ccc} B \otimes B \otimes A & \xrightarrow{\quad m' \otimes A \quad} & B \otimes A \\ {\scriptstyle B \otimes \tau} \downarrow & & \downarrow {\scriptstyle \tau} \\ B \otimes A \otimes B \xrightarrow{\tau \otimes B} A \otimes B \otimes B & \xrightarrow{A \otimes m'} & A \otimes B, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{e' \otimes A} & B \otimes A \\ & {\scriptstyle A \otimes e'} \searrow & \downarrow {\scriptstyle \tau} \\ & & A \otimes B. \end{array}$$

Lifting of endofunctors

$$\begin{array}{ccc} {}_B\mathbb{M} & \xrightarrow{\ ?\ } & {}_B\mathbb{M} \\ {\scriptstyle U_B} \downarrow & & \downarrow {\scriptstyle U_B} \\ \mathbb{M} & \xrightarrow{A \otimes_R -} & \mathbb{M}, \end{array} \qquad ? = A \otimes B \otimes -$$

Composition of monads and comonads

Liftings of endofunctors

$F, G : \mathbb{A} \to \mathbb{A}$, (F, m, e) monad, consider the diagram

$$\begin{array}{ccc} \mathbb{A}_F & \xrightarrow{\overline{G}} & \mathbb{A}_F \\ {\scriptstyle U_F}\downarrow & & \downarrow{\scriptstyle U_F} \\ \mathbb{A} & \xrightarrow{G} & \mathbb{A}, \end{array}$$

Questions

- does a lifting $\overline{G}$ exist ?
- F and G monads, when is $\overline{G}$ a monad ?
- F monad, G comonad, when is $\overline{G}$ a comonad ?

Beck, J., *Distributive laws*, 1969

$F, G : \mathbb{A} \to \mathbb{A}$ natural transformations $\lambda : FG \to GF$

Mixed distributive law (entwining): (F, m, e), (G, δ, ε)

lifting $\overline{G}$ comonad $\Leftrightarrow$ $\lambda : FG \to GF$ with comm. diagrams

$$\begin{array}{ccccc} FFG & \xrightarrow{\quad m_G \quad} & & & FG \\ {\scriptstyle F\lambda}\downarrow & & & & \downarrow{\scriptstyle \lambda} \\ FGF & \xrightarrow{\lambda_F} & GFF & \xrightarrow{Gm} & GF, \end{array} \qquad \begin{array}{ccccc} FG & \xrightarrow{F\delta} & FGG & \xrightarrow{\lambda_G} & GFG \\ {\scriptstyle \lambda}\downarrow & & & & \downarrow{\scriptstyle G\lambda} \\ GF & & \xrightarrow{\quad \delta_F \quad} & & GGF, \end{array}$$

$$\begin{array}{ccc} G & \xrightarrow{e_G} & FG \\ & {\scriptstyle Ge}\searrow & \downarrow{\scriptstyle \lambda} \\ & & GF, \end{array} \qquad \begin{array}{ccc} FG & \xrightarrow{F\varepsilon} & F \\ {\scriptstyle \lambda}\downarrow & \nearrow{\scriptstyle \varepsilon_F} & \\ GF. & & \end{array}$$

Mixed modules: $\varrho_A : F(A) \to A$, $\varrho : A \to G(A)$, category $\mathbb{A}_F^G$

$$\begin{array}{ccccc} F(A) & \xrightarrow{\varrho_A} & A & \xrightarrow{\varrho^A} & G(A) \\ {\scriptstyle F(\varrho^A)}\downarrow & & & & \uparrow{\scriptstyle (\varrho_A)} \\ FG(A) & & \xrightarrow{\quad \lambda_A \quad} & & GF(A). \end{array}$$

Bimonad $B : \mathbb{A} \to \mathbb{A}$, (B, m, e), (B, δ, ε)

mixed distributive law $\lambda : BB \to BB$, compatibility

$$\begin{array}{ccccc} BB & \xrightarrow{m} & B & \xrightarrow{\delta} & BB \\ {\scriptstyle B\delta}\downarrow & & & & \uparrow{\scriptstyle Bm} \\ BBB & & \xrightarrow{\lambda_B} & & BBB. \end{array} \qquad B(A) \in \mathbb{A}^B_B$$

$\eta : 1_{\mathbb{A}} \to B$ monad morphism, $\quad \varepsilon : B \to 1_{\mathbb{A}}$ comonad morphism

Category of (mixed) B-bimodules $\mathbb{A}^B_B$ - free functor

$$\phi^B_B : \mathbb{A} \to \mathbb{A}^B_B, \quad A \;\mapsto\; BB(A) \xrightarrow{m_A} B(A) \xrightarrow{\delta_A} BB(A).$$

full and faithful by

$$\mathrm{Mor}^B_B(B(A), B(A')) \simeq \mathrm{Mor}_B(B(A), A') \simeq \mathrm{Mor}_{\mathbb{A}}(A, A')$$

Hopf monads (antipode $S : B \to B$)

$$\phi^B_B \text{ equivalence} \quad \Leftrightarrow \quad BB \xrightarrow{B\delta} BBB \xrightarrow{mB} BB$$

References

- Adámek, J., Herrlich, H. and Tholen, W., *Monadic decomposition*, J. Pure Appl. Algebra **59**, 111-159 (1989)
- Bazzoni, S., *The t-structure induced by an n-tilting module*, preprint arxiv.org/abs/1604.00797
- Clark, J. and Wisbauer, R., *Idempotent monads and $\star$-functors*, J. Pure Appl. Algebra 215(2), 145-153 (2011).
- Fiorot, L., Mattiello, F. and Saorin, M, *Derived equivalences induced by nonclassical tilting objects*, Proc. AMS **145**(4), 1505-1514 (2017).
- Kowalzig, N., Krähmer, U. and Slevin, P., *Cyclic homology arising from adjunctions*, TAC **30**, 1067-1095 (2015).
- Tonolo, A., *Tilting modules of finite projective dimension: Sequentially static and costatic modules*, J. Algebra Appl. **1**(3), 295-305 (2002).

References

Wei, Jiaqun, *n-Star modules and n-tilting modules*, J. Algebra **283**, 711-722 (2005).

Weibel, Ch., *An introduction to homological algebra*, Cambridge University Press (1994).

Wisbauer, R., *Tilting in module categories*, Abelian groups, module theory, and topology (Padua), LN Pure Appl. Math. **201**, 421-444, Dekker, New York (1998).

Wisbauer, R., *Static modules and equivalences*, Interactions between ring theory and representations of algebras (Murcia), LN Pure Appl. Math. **210**, 423-449, Dekker, New York (2000).

Wisbauer, R., *Separability in algebra and category theory*, Proc. Aligarh (2016).