Faith's problem on R-projectivity is independent of ZFC

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In memory of Gena Puninski

Overview

I. Baer's Criterion for injectivity, and Faith's Problem on its dual.

II. The Dual Baer Criterion for small modules.

1st set-theoretic interlude:
III. Shelah's Uniformization and the vanishing of Ext.

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I. Baer's Criterion for injectivity, and Faith's Problem on its dual.

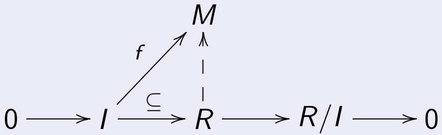
Baer's Criterion for Injectivity

[Baer 1940]

The injectivity of a module M is equivalent to its R-injectivity,
for any ring R and any module $M \in \mathrm{Mod}–R$.

Definition

M is R-injective, if for each right ideal I, all $f \in \mathrm{Hom}_R(I, M)$ extend to R:



Corollaries for the stucture theory

Definition

Let R be an integral domain. A module M is divisible, if $M.r = M$ for each $0 \neq r \in R$.
Equivalently, $\mathrm{Ext}^1_R(R/rR, M) = 0$ for each $0 \neq r \in R$.

Corollaries of Baer's Criterion

- injectivity $=$ divisibility for R a Dedekind domain.
- Let R be a right noetherian ring. Then each injective module is uniquely a direct sum of modules isomorphic to $E(R/I)$ for some ideals I of R such that R/I uniform.
- (Matlis) Let R be a commutative noetherian ring. Then each injective module is uniquely a direct sum of modules isomorphic to $E(R/p)$ for some prime ideals p of R.

Faith's Problem

Original formulation

Algebra II - Ring Theory, Springer GMW 191, 1976.
Notes for Chapter 22 on p.175:

Sandomierski [64] showed that over a perfect ring R, that R is a "test module" for projectivity in a sense dual to the requirement for injectivity of a module M that maps of submodules of R into M can be lifted to maps of $R \to M$ (Baer's Criterion for Injectivity 3.41 (I, p. 157)). The characterization of all such rings is still an open problem.

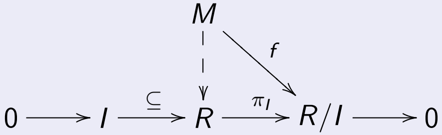
Faith's problem in short

For what rings R does the Dual Baer Criterion hold, i.e.,
when is projectivity equivalent to R-projectivity for all modules?

Notation

Definition

M is R-projective, if for each right ideal I, all $f \in \mathrm{Hom}_R(M, R/I)$ factorize through π_I:



Equivalently, $\mathrm{Hom}_R(M, \pi_I)$ is surjective for each right ideal I of R.

Definition

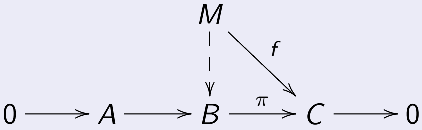
The rings R such that projectivity of a module $M \in \mathrm{Mod}\text{–}R$ is equivalent to its R-projectivity are called right testing.

II. The Dual Baer Criterion for small modules.

Projectivity relative to a module

Definition

Let M and B be modules. Then M is projective relative to B, or B-projective, if for each short exact sequence $0 \longrightarrow A \longrightarrow B \stackrel{\pi}{\longrightarrow} C \longrightarrow 0$, all $f \in \mathrm{Hom}_R(M, C)$ factorize through π:



Basic properties of relative projectivity

Lemma

- Assume that M is B_i-projective for each $i < n$. Then M is B-projective, where $B = \bigoplus_{i<n} B_i$.
- Assume M is N-projective and $P \subseteq N$. Then M is both P-projective and N/P-projective.
- M is R-projective, iff M is F-projective for each finitely generated module F.
- The class of all right testing rings is closed under Morita equivalence and finite ring direct products.

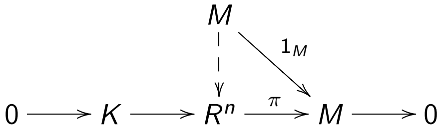
The Dual Baer Criterion holds for all finitely generated modules

Corollary

Assume $M \in \mathrm{Mod}\text{–}R$ is finitely generated. Then M is R-projective, iff M is projective.

Proof: By the above, the R-projectivity implies R^n-projectivity for each $n < \omega$.

Assume M is n-generated. Then the identity map $1_M : M \to M$ factorizes through π in the free presentation of M:



i.e., the free presentation splits. □

R-projectivity of divisible modules

Lemma

Let R be an integral domain and M be a divisible module. Then M is R-projective.

Proof: Assume M is divisible and let I be a non-zero ideal of R such that $0 \neq \mathrm{Hom}_R(M, R/I)$. Then R/I contains a non-zero divisible submodule of the form J/I for an ideal $I \subsetneq J \subseteq R$. Let $0 \neq r \in I$. The r-divisibility of J/I yields $Jr + I = J$, but $Jr \subseteq I$, a contradiction. So $\mathrm{Hom}_R(M, R/I) = 0$ for each non-zero ideal I of R, and M is R-projective. □

Corollary

$\mathbb{Q}$ is a countable $\mathbb{Z}$-projective, but not projective, $\mathbb{Z}$-module.
So the Dual Baer Criterion fails for countably generated modules in general.

Perfect versus non-perfect rings

Definition

A ring R is right perfect, if R contains no infinite strictly decreasing chain of principal left ideals. E.g., each right artinian ring is right perfect.

The positive case [Sandomierski 1964]

Each right perfect ring is right testing. That is, the Dual Baer Criterion holds for all modules.

Some negative cases

- [Hamsher 1966] If R is commutative and noetherian, then R is testing, iff R is artinian.
- If R is an integral domain, then R is testing, iff R is a field.
- [Puninski et. al. 2017] Let R be a semilocal right noetherian ring. Then R is right testing, iff R is right artinian.

Warning: there is no a priori size bound for testing

An example in ZFC

Let K be a skew-field, κ an infinite cardinal, and R the endomorphism ring of a κ-dimensional left vector space over K. Then the Dual Baer Criterion holds for all $\leq \kappa$-generated modules.

Proof: Since κ is infinite, R contains a right R-independent set of elements $\{r_\alpha \mid \alpha < \kappa\}$ such that $\text{r-ann}(r_\alpha) = 0$ for each $\alpha < \kappa$.
Then the right ideal $I = \sum_{\alpha<\kappa} r_\alpha R$ is free of rank κ.
Assume M is R-projective and $\leq \kappa$-generated. Then M is $R^{(\kappa)}$-projective, and hence M is projective. □

III. Shelah's Uniformization and the vanishing of Ext.

1st set-theoretic interlude

Ladders

Let κ be an uncountable cardinal of cofinality ω and $E \subseteq E_\omega$, where $E_\omega = \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \omega\}$.
A sequence $(n_\alpha \mid \alpha \in E)$ is a ladder system, if for each $\alpha \in E$, n_α is a ladder, i.e., a strictly increasing countable sequence $(n_\alpha(i) \mid i < \omega)$ consisting of non-limit ordinals such that $\sup_{i<\omega} n_\alpha(i) = \alpha$.

Stationary sets

Let κ be a regular uncountable cardinal.

- A subset $C \subseteq \kappa$ is called a club provided that C is closed in κ (i.e., $\sup(D) \in C$ for each subset $D \subseteq C$ such that $\sup(D) < \kappa$) and C is unbounded (i.e., $\sup(C) = \kappa$).
- $E \subseteq \kappa$ is stationary provided that $E \cap C \neq \emptyset$ for each club $C \subseteq \kappa$.

Example: E_ω is stationary in κ^+.

Shelah's Uniformization Principle (UP)

Uniformization of colorings

(UP_κ) There exist a stationary set $E \subseteq E_\omega$ and a ladder system $(n_\alpha \mid \alpha \in E)$, such that for each cardinal $\lambda < \kappa$ and each sequence $(h_\alpha \mid \alpha \in E)$ of maps (local λ-colorings) from ω to λ there exists a map (global λ-coloring) $f : \kappa^+ \to \lambda$, such that for each $\alpha \in E$, $f(n_\alpha(i)) = h_\alpha(i)$ for almost all $i < \omega$.

(UP) UP_κ holds for each uncountable cardinal κ of cofinality ω.

Theorem (Eklof-Shelah 1991)

UP is consistent with ZFC + GCH.

Faith's problem under Shelah's uniformization

[T. 1996]

Let R be a non-right perfect ring and κ an uncountable cardinal of cofinality ω, such that $\mathrm{card}(R) < \kappa$ and UP_κ holds. Then there exists a κ^+-generated module M_κ of projective dimension 1 such that $\mathrm{Ext}^1_R(M_\kappa, I) = 0$ for each right ideal I of R.

[Puninski et al. 2017]

The module M_κ is R-projective, but not projective.

Proof: $\mathrm{Hom}_R(M_\kappa, R) \overset{\mathrm{Hom}_R(M_\kappa,\pi_I)}{\longrightarrow} \mathrm{Hom}_R(M_\kappa, R/I) \to \mathrm{Ext}^1_R(M_\kappa, I) = 0$ is an exact sequence. So $\mathrm{Hom}_R(M_\kappa, \pi_I)$ is surjective for each right ideal I of R, and M_κ is R-projective. □

Corollary

Assume UP. Then right testing rings coincide with the right perfect ones.

The construction of the module M_κ

M_κ is defined by a free presentation

$$(*) \qquad 0 \to G \xrightarrow{\nu} F \to M_\kappa \to 0,$$

where $F = \bigoplus_{\alpha<\kappa^+} F_\alpha$, $F_\alpha = R^{(\omega)}$ for $\alpha \in E$, and $F_\alpha = R$ otherwise.

Let 1_α be the canonical free generator of F_α for $\alpha \notin E$, and $\{1_{\alpha,i} \mid i < \omega\}$ the canonical free basis of F_α for $\alpha \in E$.

Let $R \supsetneq Ra_0 \supsetneq Ra_1a_0 \supsetneq \dots \supsetneq Ra_n \dots a_0 \supsetneq Ra_{n+1}a_n \dots a_0 \supsetneq \dots$ be a strictly decreasing chain of principal left ideals of R.

For $\alpha \in E$ and $i < \omega$, we define $g_{\alpha,i} = 1_{\nu_{\alpha(i)}} - 1_{\alpha,i} + 1_{\alpha,i+1}.a_i$, and $G = \bigoplus_{\alpha \in E, i<\omega} g_{\alpha,i} R$.

Lemma

The presentation $(*)$ above is free, but non-split, whence the projective dimension of $M_\kappa = F/G$ equals 1.

The vanishing of $\mathbf{Ext}_R^1(M_\kappa, I)$

Recall that $\text{Ext}_R^1(M, I) = 0$, iff $\text{Hom}_R(G, I) = \text{Im}(\text{Hom}_R(\nu, I))$, iff each homomorphism $\varphi \in \text{Hom}_R(G, I)$ extends to some $\psi \in \text{Hom}_R(F, I)$.

Let $\lambda = \text{card}(I)$. Then $\lambda < \kappa$, and h defines a local λ-coloring from ω to λ by $h_\alpha(i) = \varphi(g_{\alpha,i})$.

The global λ-coloring $f : \kappa^+ \to \lambda$ provided by (UP_κ) can be used to define $\psi \in \text{Hom}_R(F, I)$ so that $\varphi = \psi \restriction G$, i.e., prove that $\text{Ext}_R^1(M_\kappa, I) = 0$. □

Remark: The global coloring f coincides with each of the local colorings h_α **almost everywhere**, *while we need ψ to restrict to φ* **everywhere**. *This can be fixed using the extra space provided by F_α (recall that for $\alpha \in E$, F_α has rank $\aleph_0$ rather than 1).*

IV. The algebra of eventually constant sequences.

The algebra of eventually constant sequences

Let K be a field. Denote by $E(K)$ the unital K-subalgebra of K^ω generated by $K^{(\omega)}$. In other words, $E(K)$ is the subalgebra of K^ω consisting of all eventually constant sequences in K^ω.

Basic properties

Let $R = E(K)$.

- R is a commutative von Neumann regular hereditary semiartinian ring of Loewy length 2 with $\mathrm{Soc}(R) = K^{(\omega)}$.
- R is not perfect.
- A module M is R-projective, if each $f \in \mathrm{Hom}_R(M, \mathrm{Soc}(R))$ factors through the canonical projection $\pi : R \to R/\mathrm{Soc}(R)$.
- If $M \in \mathrm{Mod}\text{–}R$ is countably generated, then M is R-projective, iff M is projective.

V. Jensen's Diamond, and the independence of Faith's Problem of ZFC.

2nd set-theoretic interlude

Jensen's functions

Let κ be a regular uncountable cardinal.

- Let A be a set of cardinality $\leq \kappa$. An increasing continuous chain, $\mathcal{A} = (A_\alpha \mid \alpha < \kappa)$, consisting of subsets of A of cardinality $< \kappa$, such that $A_0 = 0$ and $A = \bigcup_{\alpha<\kappa} A_\alpha$, is called a κ-filtration of the set A.
- Let E be a stationary subset of κ. Let A and B be sets of cardinality $\leq \kappa$. Let $\mathcal{A}$ and $\mathcal{B}$ be κ-filtrations of A and B, respectively. For each $\alpha < \kappa$, let $c_\alpha : A_\alpha \to B_\alpha$ be a map. Then $(c_\alpha \mid \alpha < \kappa)$ are Jensen-functions provided that for each map $c : A \to B$, the set $E(c) = \{\alpha \in E \mid c \restriction A_\alpha = c_\alpha\}$ is stationary in κ.

Theorem (Jensen 1972)

Assume Gödel's Axiom of Constructibility ($V = L$). Let κ be a regular uncountable cardinal, $E \subseteq \kappa$ a stationary subset of κ, and A and B sets of cardinality $\leq \kappa$. Let $\mathcal{A}$ and $\mathcal{B}$ be κ-filtrations of A and B, respectively. Then there exist Jensen-functions $(c_\alpha \mid \alpha < \kappa)$.

Faith's problem under V = L

Theorem (T. 2017)

Assume $V = L$. Let K be a field of cardinality $\leq 2^\omega$, and $R = E(K)$. Then $M_n(R)$ is right testing for each $n > 0$.

Sketch of proof: Let M be an R-projective module and κ be the minimal number of R-generators of M. The proof is by induction on κ:

I. For κ **countable**, use the basic properties of $E(K)$ mentioned above.

II. For κ **regular and uncountable**, we express M as the union of a continuous chain of its $< \kappa$-generated submodules $\mathcal{M} = (M_\alpha \mid \alpha < \kappa)$. W.l.o.g., we can assume that if M_β / M_α is not R-projective, then $M_{\alpha+1}/M_\alpha$ is not R-projective, too.

Using Jensen-functions, one proves that the set $E = \{\alpha < \kappa \mid M_{\alpha+1}/M_\alpha \text{ is not } R\text{-projective } \}$ is not stationary in κ.

Then we can select a continuous subchain $\mathcal{M}'$ of $\mathcal{M}$ such that $M'_{\alpha+1}/M'_\alpha$ is R-projective for each $\alpha < \kappa$. By the inductive premise, $M'_{\alpha+1}/M'_\alpha$ is projective, and hence $M'_{\alpha+1} = M'_\alpha \oplus P_\alpha$ for a $< \kappa$-generated projective module P_α. Then $M = M'_0 \oplus \bigoplus_{\alpha<\kappa} P_\alpha$ is projective.

III. For κ **singular**, we use a version of Shelah's Compactness Theorem pro projective modules. □

Faith's problem is independent of ZFC + GCH

The statement 'There exists a right testing, but non-right perfect ring' is independent of ZFC + GCH.

Proof: Assuming UP, we get that each right testing ring is right perfect, but V = L implies that the non-right perfect ring of all eventually constant sequences $E(K)$ is right testing. □

Chronology of references

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