Leavitt path algebras are Bézout

Alberto Tonolo, Università di Padova

Conference in memory of Gena Puninski
Manchester, April 2018

The key dates for Leavitt path algebras

late ’50’s: Leavitt and non-IBN rings

mid ’60’s: Cohn and a class of algebras that “... *may be regarded as pathological rings*”

mid ’70’s: Bergman and the monoid-realization theorem

late ’70’s: Cuntz and graph C*-algebras

2005: Abrams and Aranda Pino   2007: Ara, Moreno and Pardo

Abrams, Ara, Siles Molina: Leavitt path algebras, Lecture Notes in Mathematics Vol. 2191, Springer Verlag, 2017.

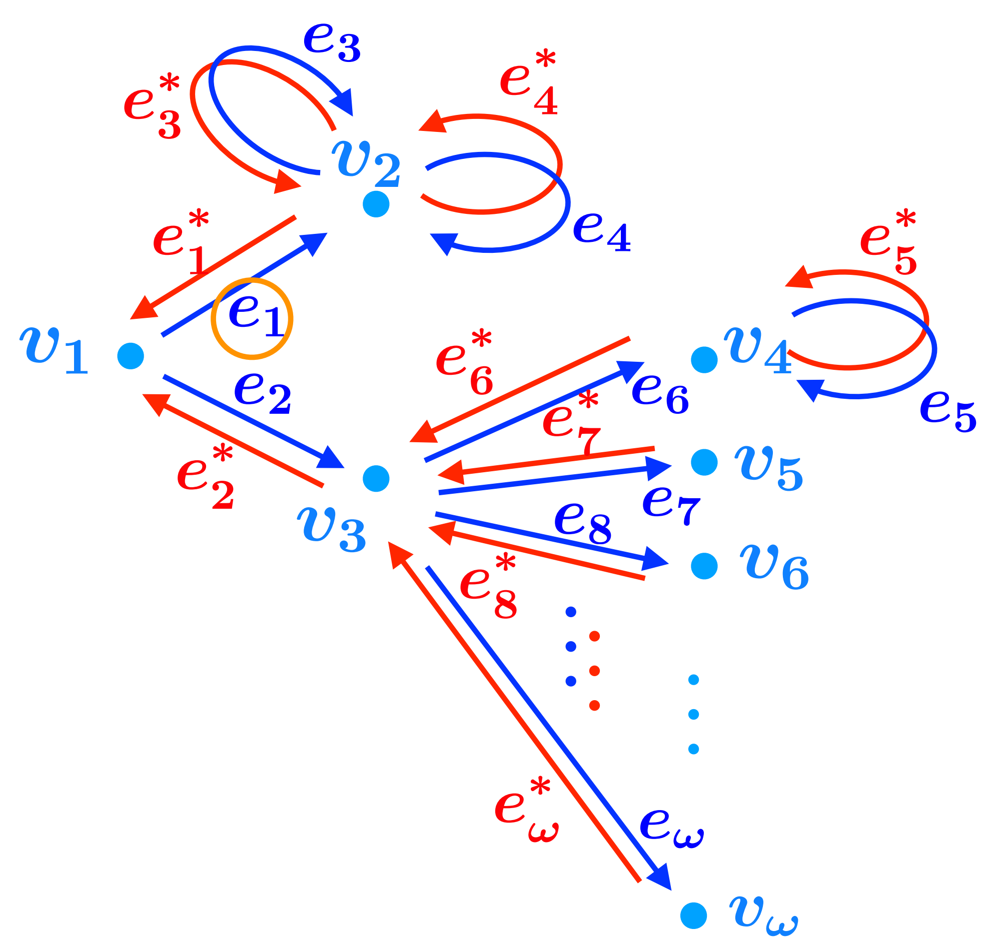
A graph and its extended graph

$$E = (E^0, E^1, s, r)$$

$$\widehat{E} = (E^0, E^1 \cup E^{1*}, \hat{s}, \hat{r})$$

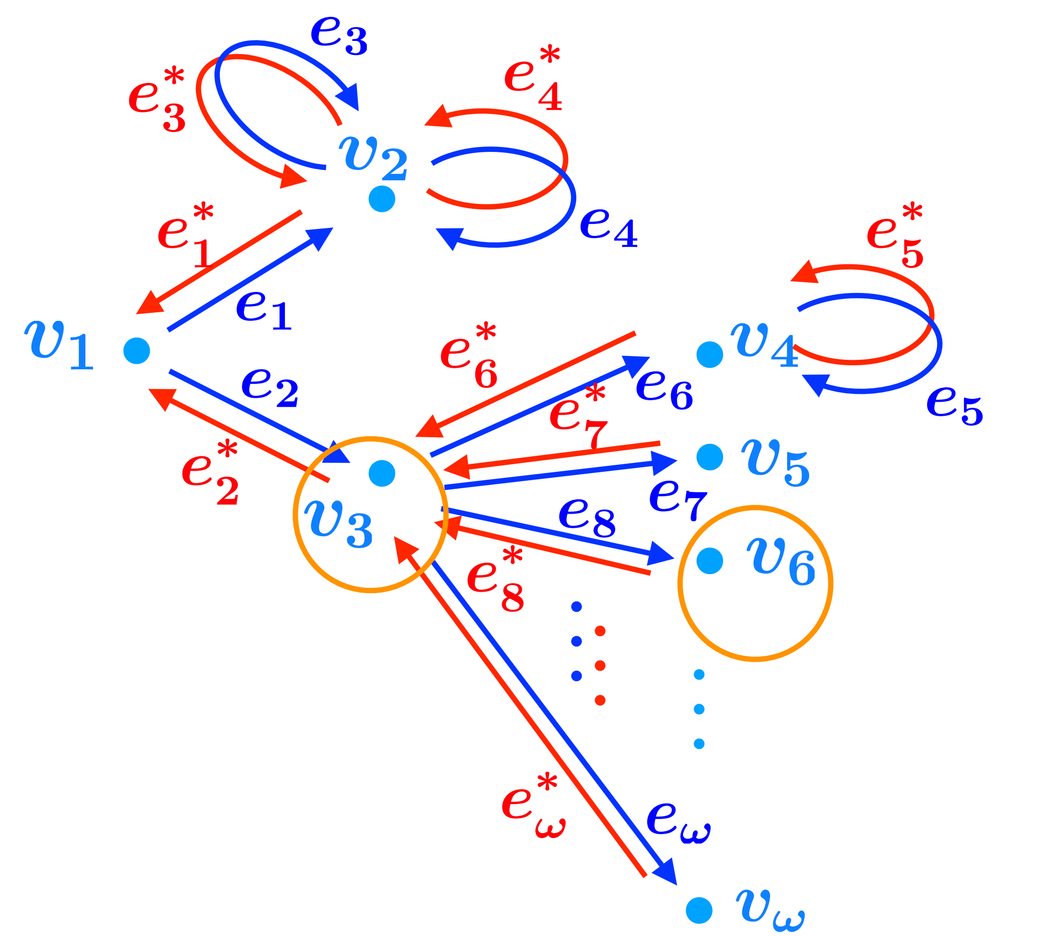
$$s : E^1 \to E^0$$
$$e_1 \mapsto v_1$$

$$r : E^1 \to E^0$$
$$e_1 \mapsto v_2$$



Leavitt path algebras

$L_K(E) =$ free K-algebra on the symbols $E^0 \cup E^1 \cup E^{1*}$ modulo the following five types of relations:



$$vv' = \delta_{v,v'} v$$

$$s(e)e = er(e) = e$$

$$s'(e^*)e^* = e^*r'(e^*) = e^*$$

$$e^*e' = \delta_{e,e'} r(e)$$

$$e_1^* e_1 = v_2$$
$$e_1^* e_2 = 0$$

$$v = \sum_{\{e : s(e) = v\}} ee^* \quad \text{if } 0 < |s^{-1}(v)| < \infty$$

$$e_1 e_1^* + e_2 e_2^* = v_1$$

$$(e_1e_1^* + e_2e_2^*)(e_1 - v_2)e_3^*e_4^*(e_3 + e_4) =$$

$$\boxed{v_1 = e_1e_1^* + e_2e_2^*}$$

$$= v_1(e_1 - v_2)e_3^*e_4^*(e_3 + e_4)$$

$$\boxed{v_1e_1 = e_1}$$

$$\boxed{v_1v_2 = 0}$$

$$= e_1e_3^*e_4^*(e_3 + e_4)$$

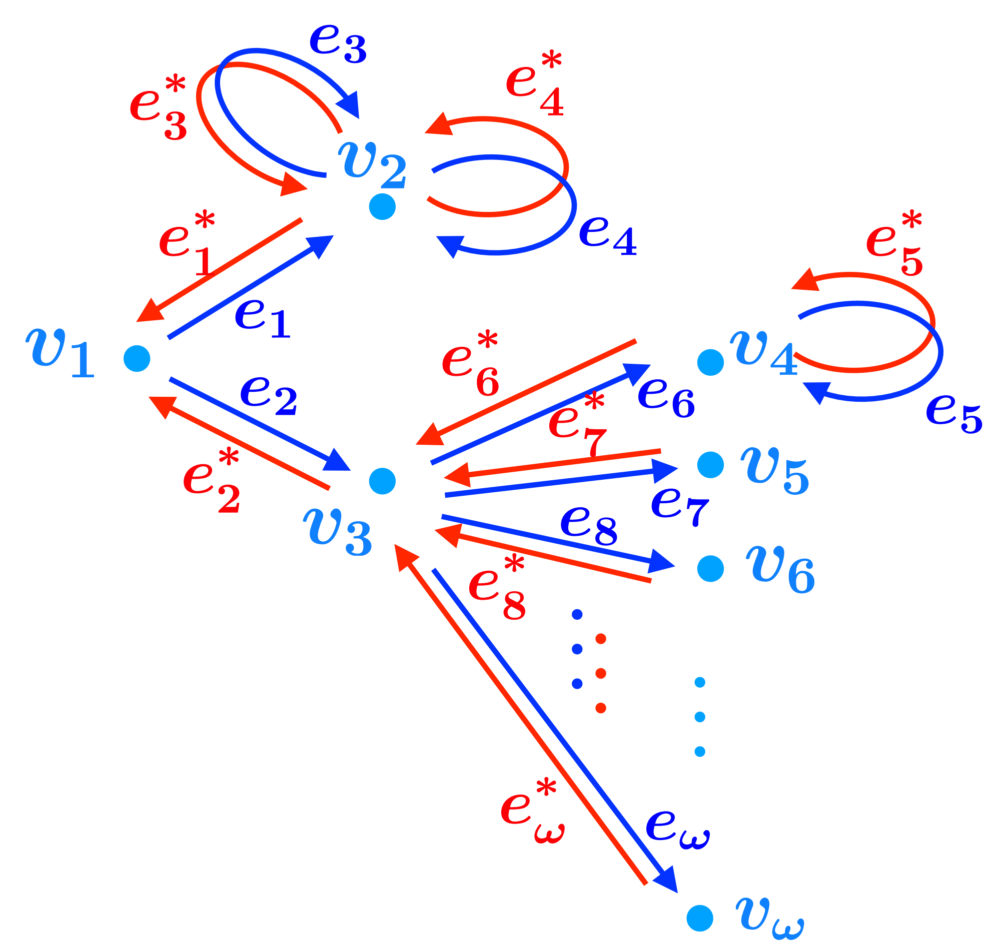
$$\boxed{e_4^*e_3 = 0}$$

$$\boxed{e_4^*e_4 = v_2}$$

$$= e_1e_3^*v_2$$

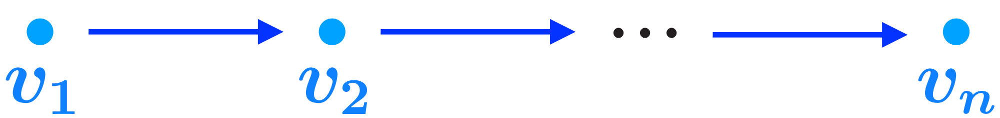
$$\boxed{e_3^*v_2 = e_3^*}$$

$$= e_1e_3^*$$



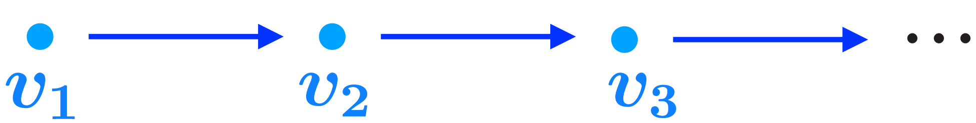
Main Examples

A_n :



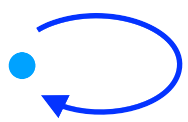
$$L_K(A_n) = M_n(K)$$

$A_{\mathbb{N}}$:



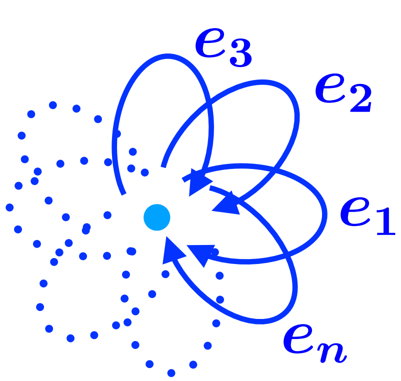
$$L_K(A_{\mathbb{N}}) = FM_{\mathbb{N}}(K)$$

R_1 :



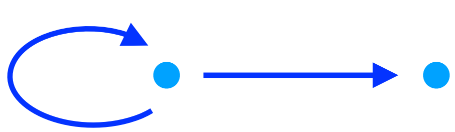
$$L_K(R_1) = K[x, x^{-1}]$$

R_n :



$$L_K(R_n) = L_K(1, n)$$

T :



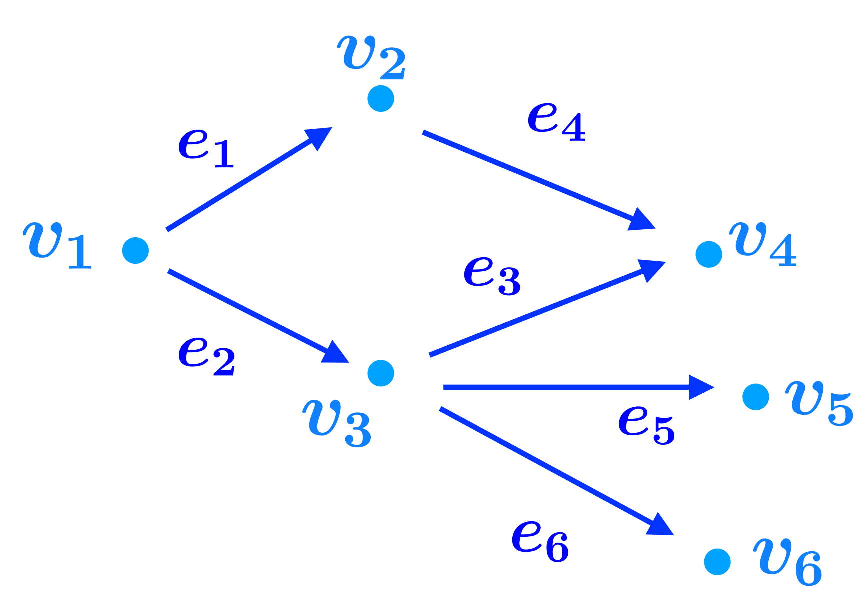
$$L_K(T) = K\langle X, Y | XY = 1\rangle$$

Properties of Leavitt path algebras

$L_K(E)$ has some specified algebraic property $\Leftrightarrow$

E has some specified graph-theoretic property

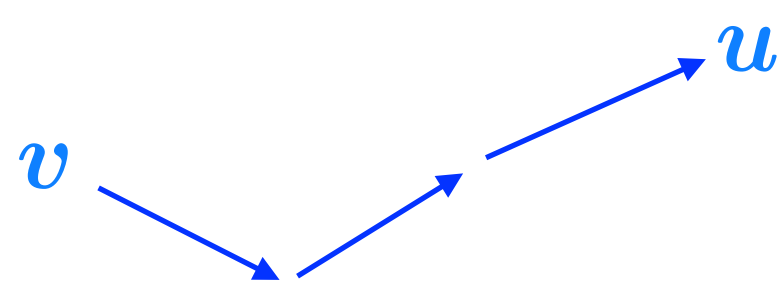
$L_K(E)$	E
finite dimensional	finite and acyclic



Properties of Leavitt path algebras

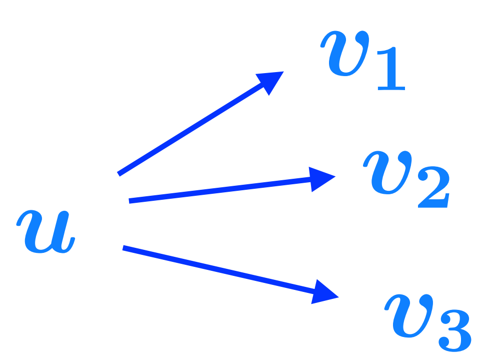
$L_K(E)$	E
simple	hereditary saturated subsets are trivial and every cycle has an exit

$V \subseteq E^0$,

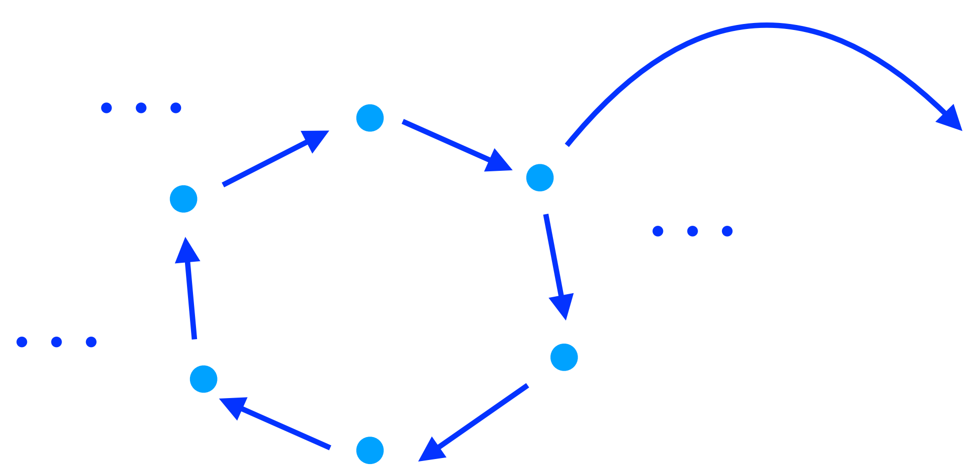


$v \in V \Rightarrow u \in V$

$V \subseteq E^0$,

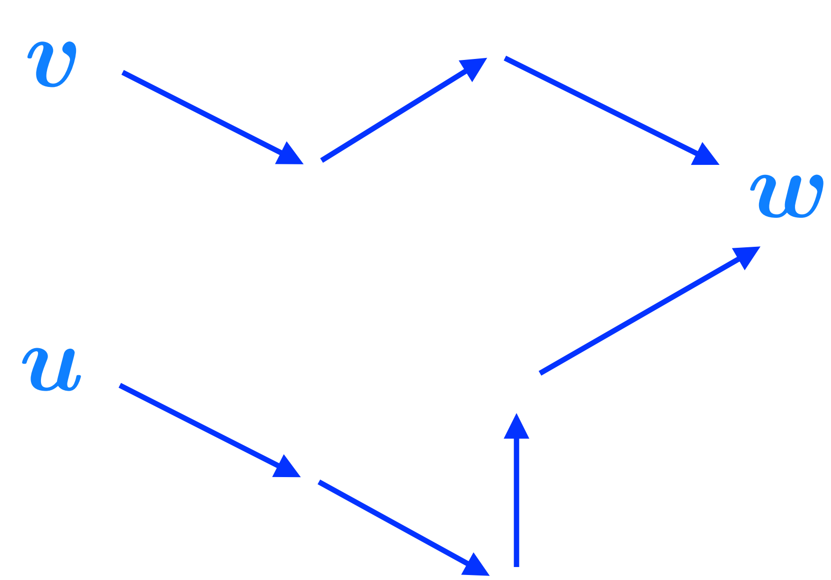


$v_1, v_2, v_3 \in V \Rightarrow u \in V$



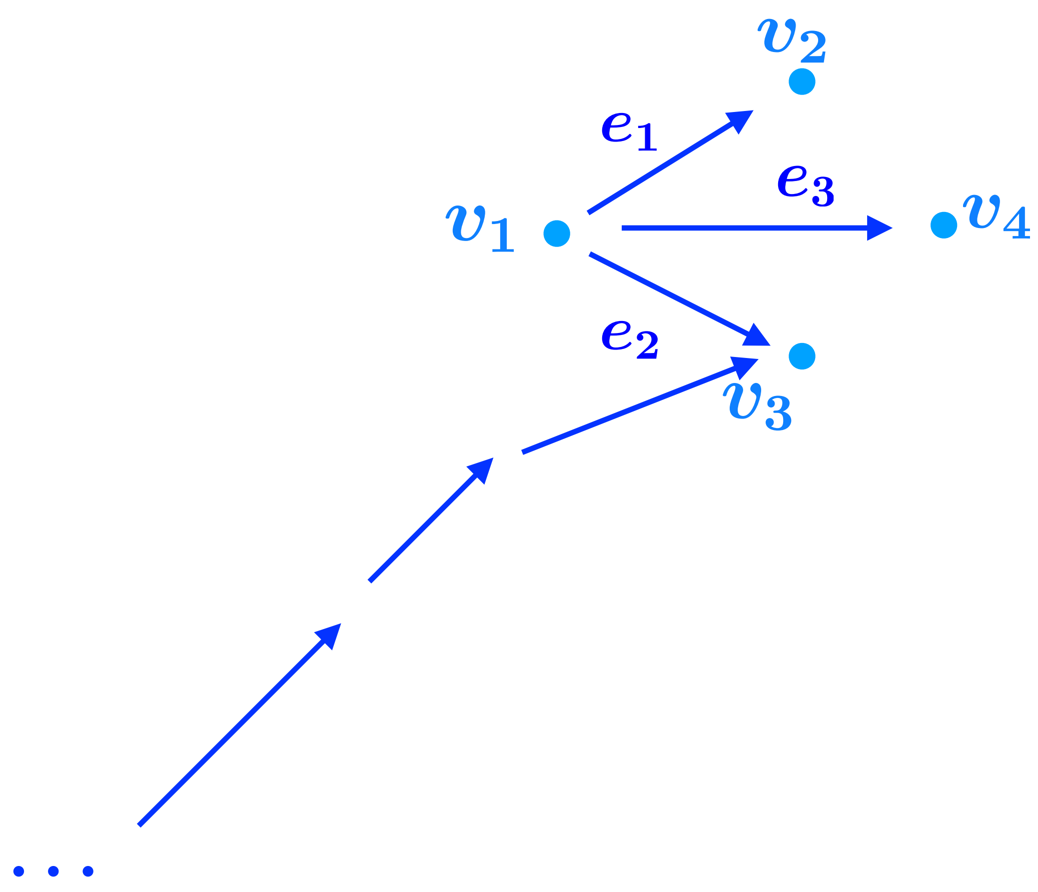
Properties of Leavitt path algebras

$L_K(E)$	E
prime	downward directed



Properties of Leavitt path algebras

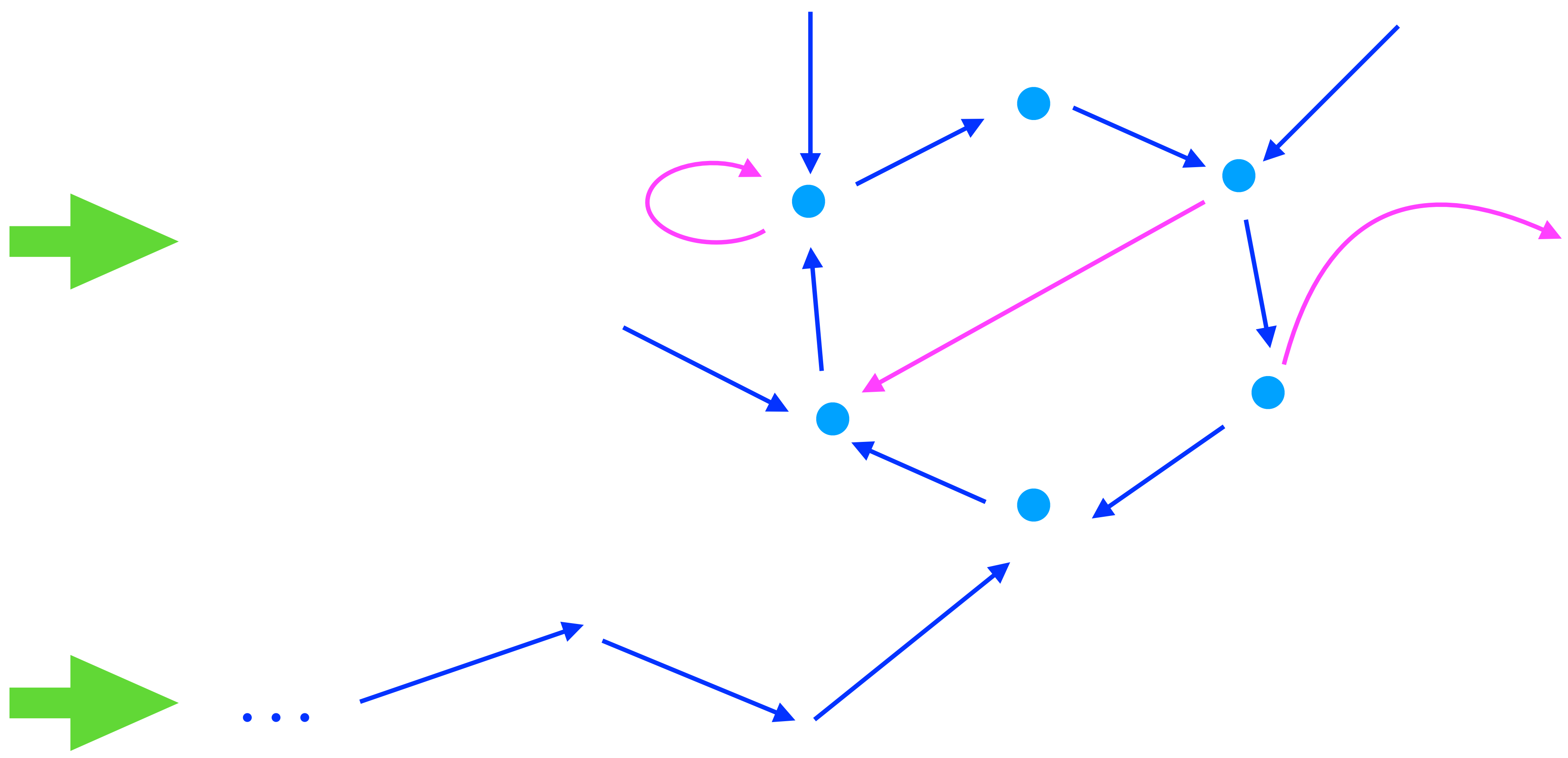
$L_K(E)$	E
one sided artinian	acyclic, row-finite, infinite paths end in a sink



$$v \in E^0 \Rightarrow |s^{-1}(v)| < \infty$$

Properties of Leavitt path algebras

$L_K(E)$	E
one sided notherian	cycles has no exits, row-finite, infinite paths end in a sink or in a cycle



Ring theoretic properties of Leavitt path algebras

- Leavitt path algebras are isomorphic to their opposite.
- [Ara, Goodearl '12] Leavitt path algebras are hereditary.
- [Abrams, Ara, Siles Molina '17] Leavitt path algebras are semiprimitive.

Bézout rings

A ring R is Bézout in case every finitely generated one-sided ideal is principal.

- Warfield showed that if $\boldsymbol{R}$ is Bézout, then so is $\boldsymbol{M_n}(\boldsymbol{R})$
- The Bézout property need not pass to (full) corners. In particular it is not a Morita invariant.
- Intuitively, the Bézout property allows us to do some "Number Theory", since for each $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}$, there exists $\boldsymbol{c} \in \boldsymbol{R}$ for which $\boldsymbol{Ra} + \boldsymbol{Rb} = \boldsymbol{Rc}$.

Bézout rings and Leavitt path algebras

- $M_n(K)$, $FM_{\mathbb{N}}(K)$ and $K[x, x^{-1}]$ are Bézout

- $L_K(1, n) \cong L_K(1, n)^n$ and hence it is Bézout

- $K\langle X, Y | XY = 1\rangle$ is Bézout [Gerritzen, '00]

- [Abrams, Mantese, - '18] Leavitt path algebras are Bézout.

Proof: main ingredients

1) Reduction to unital Leavitt path algebras:

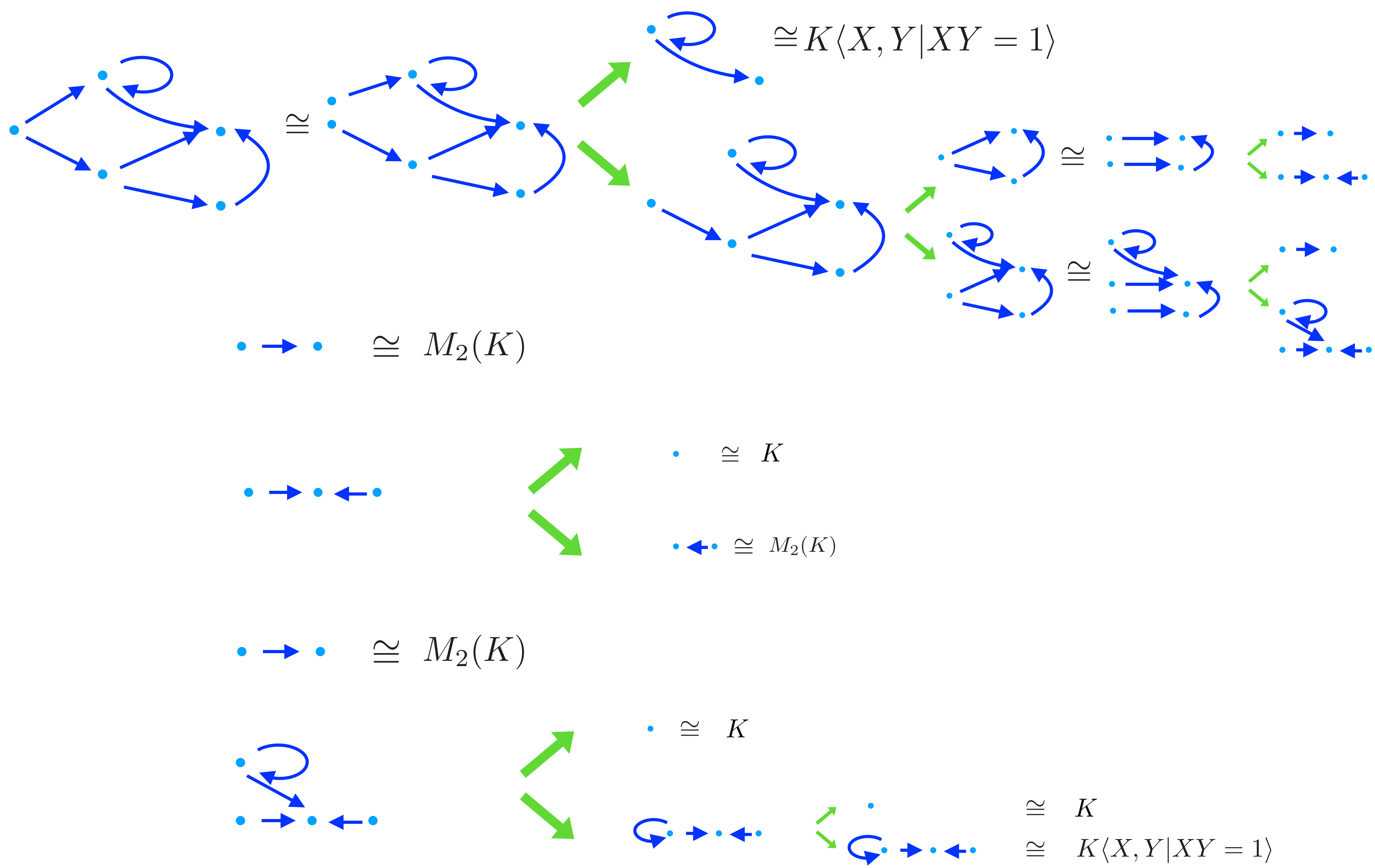
Theorem [Hazrat, Rangaswamy '16]. Any Leavitt path algebra is the direct limit of unital subalgebras, each one isomorphic to the Leavitt path algebra of a finite graph.

2) Reduction to Leavitt path algebras associated to graphs containing source vertices or source cycles:

Theorem [Abrams, Nam, Phus '15]. The Leavitt path algebra associated to a finite graph which contains neither source vertices nor source cycles is Bézout.

Proof: main ingredients

3) Induction on the number of vertices:



Principal ideal ring

- [Abrams, Mantese, - '18]

Let E be any graph. Then the associated Leavitt path algebra is a principal ideal ring if and only if E is finite and no cycle in E has an exit.

Structure of projective modules

Albrecht '61. Each projective left module over a left semi-hereditary unital ring is isomorphic to a direct sum of finitely generated left ideals.

- [Abrams, Mantese, - '18]

Let E be a finite graph and K any field. Any projective left $L_K(E)$-module is isomorphic to a direct sum of principal left ideals.

Thank you for your attention!