The torsion free part of the Ziegler spectrum of orders over Dedekind domains

Carlo Toffalori (Camerino)

Manchester, April 6-9, 2018

Work in progress with Lorna Gregory and Sonia L'Innocente
Dedicated to the memory of Gena

Some history and motivations

A group H is *Abelian-by-finite* if H admits a normal subgroup N of finite index.

Let G denote the (finite) quotient group H/N. Most model theory of H is given by that of N, viewed as a module over the group $\mathbb{Z}[G]$. Recall

- $\mathbb{Z}[G] = \{\sum_{g \in G} a_g g \mid a_g \in \mathbb{Z}\}$,
- in $\mathbb{Z}[G]$ addition is calculated componentwise, multiplication extends the corresponding operations of $\mathbb{Z}$ and G.

For every finite group G, let us pass from Abelian-by-G groups to $\mathbb{Z}[G]$-modules.

Representation theory

- Have a finite group G acting, for instance, on a module over a commutative ring R.
- Form the group ring $R[G]$ (same definition as over $\mathbb{Z}$).
- $R[G]$-modules capture several crucial features of the action of G.

Let R be Dedekind domain (of characteristic 0), Q the field of quotients of R.

Modules are assumed to be right.

Representation theory devotes a particular attention to $R[G]$-lattices.

An $R[G]$-*lattice* is a finitely generated R-projective (equivalently R-torsion free) R[G]-module.

For instance, when $R = \mathbb{Z}$, a $\mathbb{Z}[G]$-lattice is over $\mathbb{Z}$ a finite direct sum of copies of $\mathbb{Z}$ (whence the name *lattice*).

$R[G]$-lattices are not an elementary class. An elementary class extending in a natural way $R[G]$-lattices

R-torsion free $R[G]$-modules.

Indeed they are the smallest definable subcategory of the category of all $R[G]$-modules containing $R[G]$-lattices.

The main model theoretic tools in the analysis of modules over a ring S

- pp-formulae over S,
- the Ziegler spectrum of S.

Let L_S denote the first order language of (right) S-modules,

pp-formulae of L_S

A pp-formula $\varphi(\vec{x})$ says that a linear system $\vec{x}A = \vec{y}B$ admits solution in $\vec{y}$. Here

- $\vec{x}$, $\vec{y}$ are tuples of variables of possibly different length,
- A, B are matrices with entries in S and suitable sizes,
- $\vec{x}$ can be viewed as the string of parameters and $\vec{y}$ as that of indeterminates of the system.

Two familiar examples: for $a, b \in S$,

- a divisibility condition $b \mid x := \exists y(x = yb)$,
- a torsion condition $xa = 0$.

To recall: the Baur-Monk theorem (pp-elimination of quantifiers)

The Ziegler spectrum of S, Zg_S

- Points are (isomorphism types of) indecomposable pure injective S-modules.
- Basic open sets are $(\varphi(x)/\psi(x)) := \{N \in Zg_S : \varphi(N) \supset \psi(N) \cap \varphi(N)\}$ where $\varphi(x)$, $\psi(x)$ range over pp-formulae of L_S in one free variable.

This is a compact space, but is seldom Hausdorff. To recall:

- Any S-module admits a unique minimal pure injective elementary extension, its *pure injective hull*.
- Any pure injective S-module is (up to isomorphism) the pure injective hull of a direct sum of indecomposable pure injective modules, possibly plus *superdecomposable* summands.

pp-formulae over S (in one free variable x, say) are a lattice pp_S up to logical equivalence (**in all S-modules**) with respect to $+$ and $\wedge$. Here $+$ is defined by putting, for $\varphi(x)$, $\psi(x)$ pp-formulae,

$$(\varphi + \psi)(x) \doteq \exists y \exists z \, (x = y + z \wedge \varphi(y) \wedge \psi(z)).$$

For every ordinal α define a lattice pp_S^α collapsing, at each (successor) step, intervals of finite length. For instance, in the basic step, two pp-formulae φ and ψ are identified if and only if in pp_S the closed interval $[\varphi \wedge \psi, \, \varphi + \psi]$ is of finite length.

The m-dimension of pp_S, mdim(pp_S), is

- the smallest α such that pp_S^α trivializes, if such an α exists,
- ∞ (or undefined) otherwise.

Let us come back to group rings $R[G]$. The torsion free part of $\mathrm{Zg}_{R[G]}$ consists of R-torsion free indecomposable pure injective $R[G]$-modules (**a closed subset of** $\mathrm{Zg}_{R[G]}$) with the relative topology.

- Some general analysis of the R-torsion free part of the spectrum, Marcja-Prest-T., JSL 2002.
- A specific investigation was done only in a few cases, the most advanced being the integral group ring $\mathbb{Z}G$ where $G \simeq C(2)^2$ is the Klein four group, Puninski-T., JPAA 2011.

Notation: for every integer $q > 1$, $C(q)$ is the multiplicative cyclic group of order q.

The role of lattices

- $R[G]$-lattices may not be pure injective over $R[G]$
- After localizing at a maximal ideal P of R and completing in the P-adic topology, so over $\widehat{R_P}[G]$, lattices are pure injective over $R[G]$.

A wider perspective

Let R, Q be as before, A a finite dimensional Q-algebra. An *R-order* in A is a subring Λ of A such that:

1. the center of Λ contains R,
2. Λ is finitely generated as a module over R,
3. $Q \cdot \Lambda = A$.

Λ-lattices are still introduced as finitely generated R-torsionfree Λ-modules.

Some examples of orders

- a group ring $R[G]$ with G a finite group and $A = Q[G]$;
- the ring of algebraic integers in an algebraic number field L (that is, a finite dimensional field extension L of $\mathbb{Q}$), where $R = \mathbb{Z}$, so $Q = \mathbb{Q}$, and $A = L$;
- a matrix ring $\Lambda = \left(\begin{smallmatrix} R & R \\ I & R \end{smallmatrix}\right)$ where I is an ideal of R and $A = M_2(Q)$.

Our aim: to extend the analysis of the torsion free part of the spectrum from group rings to arbitrary orders Λ.

To recall: Gena's analysis of the Cohen-Macaulay part of the Ziegler spectrum of a Cohen-Macaulay ring.

Notation

- Zg_Λ^t is the torsion free part of Zg_Λ (still a closed subset of Zg_Λ),
- pp_Λ^t is the lattice of pp-formulae in one free variable over Λ, up to logical equivalence **in R-torsion free Λ-modules** (and m-dimension is defined accordingly, as in pp_Λ).

Our setting from now on: R is a Dedekind domain with field of fractions Q and Λ is an R-order in a finite dimensional Q-algebra A.

Theorem

Assume A semisimple (true when $A = Q[G]$ in characteristic 0). If $N \in \mathrm{Zg}^t_\Lambda$, then either

- *N is a simple A-module, or*
- *for some maximal ideal P of R, $N \in \mathrm{Zg}^t_{\widehat{\Lambda_P}}$, moreover N is R_P-reduced (that is, $\cap_{i=0}^{\infty} NP^i = 0$).*

The steps of the proof.

1. For some maximal ideal P or R, N is a module over Λ_P.
2. Any Λ_P-module torsion free and divisible over R_P is injective over Λ_P.
3. N decomposes over Λ_P as $N' \oplus N''$ where N' is R_P-divisible (so an A-module) and N'' is R_P-reduced. Thus, when N is indecomposable, either N is an A-module (hence simple) or N is R_P-reduced.
4. Every R_P-reduced pure injective Λ_P-module M can be equipped with a strucure of $\widehat{\Lambda_P}$-module, and remains pure injective over $\widehat{\Lambda_P}$.
5. If N is an R_P-reduced indecomposable pure injective $\widehat{\Lambda_P}$-module, then N is indecomposable as a Λ-module.

Moreover, for every P, the space of indecomposable pure injective $\widehat{\Lambda_P}$-modules torsion free and reduced over R_P has the same topology whether viewed as a subspace of $\mathrm{Zg}^t_{\widehat{\Lambda_P}}$ or of Zg^t_Λ.

Theorem

Assume A semisimple. Then Zg^t_Λ has Cantor-Bendixson rank if and only if Zg_A and (for every maximal ideal P of R) the R_P-reduced part of $\mathrm{Zg}^t_{\widehat{\Lambda_P}}$ have Cantor-Bendixson.

Theorem

pp^t_Λ has m-dimension if and only if, for all maximal ideals P of R, the interval $[x \in P, x = x]$ of pp^t_Λ has m-dimension and pp_A has m-dimension.

Recall that R is Noetherian, whence each P is finitely generated and the conditions $x \in P$ can be written as a pp-formula.

The proof uses that, under the left assumption, Zg^t_Λ satisfies the *isolation condition*, that is, for every closed subset $\mathcal{C}$ and every isolated point $N \in \mathcal{C}$ (in the relative topology) there is a pp-pair φ, ψ which is minimal such that $(\varphi/\psi) \cap \mathcal{C} = \{N\}$.
This implies that the m-dimension of pp^t_Λ coincides with the Cantor-Bendixson rank of Zg^t_Λ.

Theorem

Indecomposable lattices over $\widehat{\Lambda_P}$ are isolated points in $\mathrm{Zg}^t_{\widehat{\Lambda_P}}$, and such points are dense in $\mathrm{Zg}^t_{\widehat{\Lambda_P}}$.

Actually this is true over any complete Noetherian valuation domain (like $\widehat{R_P}$) and every order over this domain (like $\widehat{\Lambda_P}$).

The key step in the proof.
As the underlying ring is complete, the category of lattices has almost split sequences, and then standard arguments imply that every indecomposable lattice is isolated.

Also, in the local + complete framework,

Theorem

Let R be a complete noetherian valuation domain and Λ an order over R. If N is an indecomposable Λ-lattice and M is in the Ziegler closure of N but is not (isomorphic to) N, then the lattice of pp-formulae over Λ (up to logical equivalence **in** *M) is of finite length, and hence M is a closed point.*

A brief parenthesis, again in the local setting (see Lorna's talk). So R is a discrete valuation domain, π is a generator of its unique maximal ideal.

Then there is a (minimal) non negative integer k_0 such that $\pi^{k_0}\operatorname{Ext}^1(L, L') = 0$ for all Λ-lattices L and L'. For instance, when $\Lambda = R[G]$ for some finite group G, then k_0 is the largest integer such that $|G| \in \pi^{k_0} R$.

Theorem

(Maranda's theorem for pure injective modules) Let N, N' be R-reduced R-torsion free pure injective Λ-modules. If $N/N\pi^k \simeq N'/N'\pi^k$ for some integer $k > k_0$, then $N \simeq N'$. Moreover, if N is indecomposable over Λ, the same is true of $N/N\pi^k$ over $\Lambda/\Lambda\pi^k$ for every $k > k_0$.

On the other hand

Theorem

There exists a module N over $\widehat{\mathbb{Z}_2}[C(2)^2]$ such that N is torsion free and reduced over $\mathbb{Z}_2$, $N/N\pi^k$ is pure injective for every positive integer k but N is not pure injective.

Let R be again an arbitrary Dedekind domain.

Λ is *of finite lattice representation type* if it has only finitely many (pairwise non isomorphic) indecomposable lattices.

Example: $\Lambda = \mathbb{Z}[C(p)]$ or $\mathbb{Z}[C(p^2]$ with p any prime (thus classification of lattices can be done – but it is not easy!).

Warning

- $\mathbb{Z}[C(2)^2]$ is not of finite lattice representation type, but is of *tame* representation type (lattices over $\widehat{\mathbb{Z}_2}[C(2)^2]$ can be classified up to isomorphism).
- When lattices over Λ are of *wild* representation type cannot expect to classify them, and to describe Zg^t_Λ. This is true of most integral group rings $\mathbb{Z}[G]$.

The *support* of a Λ-module M, $\mathrm{Supp}(M)$, is the set of indecomposable pure injective Λ-modules N such that for all pp-pairs φ/ψ, $\varphi(M) = \psi(M)$ implies $\varphi(N) = \psi(N)$. If $M \in \mathrm{Zg}_\Lambda$ then $\mathrm{Supp}(M)$ is the closure of M in Zg_Λ.

Lemma

- *If $N \in \mathrm{Zg}^t_\Lambda$ and $S \in \mathrm{Supp}(QN)$ then S is in the closure of N. In particular, if $N \in \mathrm{Zg}^t_\Lambda$ and $S \in \mathrm{Zg}_A$ is a direct summand of QN, then S is in the closure of N.*
- *If $N \in \mathrm{Zg}^t_\Lambda$ is a closed point then $N \in \mathrm{Zg}_A$ and N is a closed point in Zg_A.*

Lemma

(The local + complete case) Let R be a complete discrete valuation domain, A be semisimple. Assume Λ of finite lattice representation type. Then Zg^t_Λ consists exactly of

- *finitely many indecomposable lattices over Λ,*
- *finitely many simple A-modules.*

The indecomposable lattices are isolated points, and are dense in the whole space, while modules over A are closed points. If N is an indecomposable lattice, then a simple A-module M is in the closure of N if and only if M is a direct summand of QN.

Corollary

Assume R and Λ as before, hence in particular Λ of finite lattice representation type. Let p be a non finitely generated indecomposable pp-type in the theory of R-torsionfree Λ-modules. Then p contains all divisibility formulas $\pi^k \mid x$ for k a positive integer.

In 1996 Herzog and Puninskaya verified a similar result for torsionfree modules over 1-dimensional commutative noetherian local complete domains.

Theorem

(The global case) Let R be a Dedekind domain, A be separable. Assume Λ of finite lattice representation type. Then the Cantor-Bendixson rank of Zg_Λ^t is 1, and

- *the isolated points are the indecomposable $\widehat{\Lambda_P}$-lattices, where P ranges over maximal ideals of R,*
- *the points of Cantor-Bendixson rank 1 are the simple A-modules.*

This solves a conjecture of Gena Puninski and applies, as said, to $\Lambda = \mathbb{Z}[C(p)], \mathbb{Z}[C(p^2)]$ when p is a prime.

A related result (also implying the isolation property in the described setting and hence a less direct proof of the theorem): *Let Λ be an order over a Dedekind domain R and Λ is of finite lattice representation type. Then the m-dimension of pp^t_Λ is 1.*

A consequence of the main theorem.

Corollary

Let $\Lambda = \mathbb{Z}[C(p)], \mathbb{Z}[C(p^2)]$ with p a prime. Then the theory of $\mathbb{Z}$-torsionfree $\mathbb{Z}[C(p)]$-modules is decidable.

This answers old expectations of Marcja, Prest and T. in 1993.

The proof: give an explicit description of points and topology of Zg^t_Λ.

Not completely straightforward: over $C(p^2)$, can use Butler's functor providing a representation equivalence between

- most indecomposable Λ-lattices and
- indecomposable finite dimensional $\mathbb{Z}/p\mathbb{Z}$-representations of the Dynkin diagram D_{2p}.

A more complicated example suggested by Gena (*"In our opinion one should be guided by examples to create theories."*)

Let R be a complete noetherian valuation domain, π be a generator of its maximal ideal.

Let $\Lambda = \begin{pmatrix} R & R \\ I & R \end{pmatrix}$ with $I = \pi^2 R$.

Put $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

It is known that Λ has finite lattice representation type.

In fact Λ is Gorenstein, i.e. projective modules

- $P_1 = e_1\Lambda = \begin{pmatrix} R & R \\ 0 & 0 \end{pmatrix}$ (so basically (R, R)),
- $P_2 = e_2\Lambda = \begin{pmatrix} 0 & 0 \\ I & R \end{pmatrix}$ (hence (I, R))

are injective (in the category of lattices).

The only remaining indecomposable lattice is

- $P = (\pi R, R)$

(note that $\begin{pmatrix} \pi R & R \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ \pi R & R \end{pmatrix}$ are isomorphic as Λ-modules).

Being Gorestein, Λ admits a unique overorder $\Lambda' = \begin{pmatrix} R & R \\ \pi R & R \end{pmatrix}$ which is hereditary, and P is defined over Λ', i.e. Λ' is the ring of definable scalars of P.

The following is the AR-quiver of Λ

$$\begin{array}{ccccc} & & P_1 & & \\ & \overset{\subset}{\nearrow} & & \overset{\pi}{\searrow} & \\ P & & & & P \\ & \underset{\pi}{\searrow} & & \underset{\subset}{\nearrow} & \\ & & P_2 & & \end{array}$$

where π denotes the multiplication by π.

From that we can see irreducible morphisms in the category of lattices and the unique almost split sequence:

$$0 \to P \xrightarrow{(i,\pi)} P_1 \oplus P_2 \xrightarrow{\binom{\pi}{-i}} P \to 0\,,$$

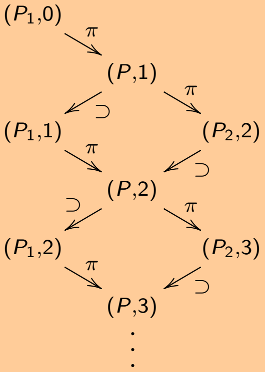
where i denotes inclusion. In detail the two intermediate morphisms act as follows:

- for all $a, b \in R$, (i, π) maps $(\pi a, b)$ to $((\pi a, b), (\pi^2 a, \pi b))$,
- for all $a', b', c', d' \in R$, $\binom{\pi}{-i}$ sends $((a', b'), (\pi^2 c', d'))$ to $(\pi a' - \pi^2 c', \pi b' - d')$.

Let N be an indecomposable R-torsionfree pure injective Λ-module. First suppose that there exists $0 \neq n \in Ne_1$. Look at pointed indecomposable lattices (M, m) such that $m \in Me_1$. Up to equivalence (of types realized by m) here is a complete list of them

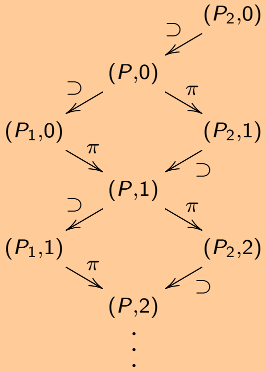
- $(P_1, (\pi^k, 0)), \ \ k \geq 0$,
- $(P_2, (\pi^l, 0))$, $l \geq 2$,
- $(P, (\pi^m, 0))$, $m \geq 1$.

Furthermore the following is the pattern of the module $(P_1, 0)$, i.e., the poset of morphisms from P_1 to indecomposable lattices . Here we use an "exponential" notation: for instance (P_1, k) means for $(P_1, (\pi^k, 0))$.



We easily derive that there is a unique (critical over zero) indecomposable non finitely generated type p in the interval $[x = 0, \; e_1 \mid x]$ in pp_Λ^t. Furthermore p is realized by $(1, 0) \in (Q, Q) = S$, the simple $A = M_2(Q)$-module. Thus in this case $N \cong P_i, P$ or $N \cong S$.

It remains to consider the case when there exists $0 \neq n \in Ne_2$. Again look at indecomposable pointed lattices (M, m) where $m \in Me_2$. They form the following pattern, where (P_1, k) now abbreviates $(P_1, (0, \pi^k))$, and so on.



Then there is a unique indecomposable non finitely generated type q in the interval $[x = 0, e_2 \mid x]$ which is realized as $(0, 1) \in (Q, Q) = S$. Again we conclude that $N \cong P_i, P$ or $N \cong S$.

References

- C. Curtis, I. Reiner, Methods of Representation Theory with Applications to Finite Groups and Orders, Vol. 1, Pure and Applied Mathematics, John Wiley and Sons, 1981.
- M. Prest, Model Theory and Modules, London Mathematical Society Lecture Notes Series, Vol. 130, Cambridge University Press, 1990.
- M. Prest, Purity, Spectra and Localization, Encyclopedia of Mathematics and its Applications, Vol. 121, Cambridge University Press, 2009.

- A. Marcja, M. Prest, C. T., On the undecidability of some classes of Abelian-by-finite groups, Ann. Pure Appl. Logic 62 (1993), 167–173.
- A. Marcja, M. Prest, C. T., The torsionfree part of the Ziegler spectrum of RG where R is a Dedekind domain and G is a finite group, J. Symbolic Logic 67 (2002), 1126–1140.
- G. Puninski, C. T., The torsion-free part of the Ziegler spectrum of the Klein four group, J. Pure Appl. Algebra 215 (2011), 1791–1804.