

MATH10242 Sequences and Series: Exercises for Week 8 Tutorials, Solutions

Question 1: Using L'Hôpital's Rule, or otherwise, find the limit of the sequences

- (i) $\left(\frac{\ln(7n^{\frac{1}{4}} - 2)}{\ln(n+1)} \right)_{n \in \mathbb{N}}$
 (ii) $\left(\frac{e^{e^n}}{e^n} \right)_{n \in \mathbb{N}}$
 (iii) $\left(\frac{1 - e^{-n}}{2 - e^{-2n}} \right)_{n \in \mathbb{N}}$ (iv) $\left(\frac{1 - e^n}{2 - e^{2n}} \right)_{n \in \mathbb{N}}$

Solutions: (i) If $f(x) = \ln(7x^{\frac{1}{4}} - 2)$ and $g(x) = \ln(x+1)$, then

$$a_n = \frac{\ln(7n^{\frac{1}{4}} - 2)}{\ln(n+1)} = \frac{f(n)}{g(n)}.$$

Clearly $\lim_{n \rightarrow \infty} f(n) = \infty = \lim_{n \rightarrow \infty} g(n)$ and $g(x) > 0$ for $x \geq 1$, so the hypotheses of L'Hôpital's rule are satisfied. Thus,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)} = \lim_{n \rightarrow \infty} \frac{(\frac{7}{4}n^{-\frac{3}{4}})(7n^{\frac{1}{4}} - 2)^{-1}}{(n+1)^{-1}} = \lim_{n \rightarrow \infty} \frac{7}{4} \frac{(n+1)}{(7n - 2n^{\frac{3}{4}})}.$$

Now the fastest-growing term is n so by dividing top and bottom by n and using the AoL we get

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{7}{4} \frac{(n+1)}{(7n - 2n^{\frac{3}{4}})} = \frac{7}{4} \cdot \frac{(1+0)}{(7-0)} = \frac{1}{4}.$$

(ii) Again $f(x) = e^{e^x} \rightarrow \infty$ and $g(x) = e^x \rightarrow \infty$ as $x \rightarrow \infty$ and g is positive. So we can apply L'Hôpital's rule to get

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)} = \lim_{n \rightarrow \infty} \frac{e^n e^{e^n}}{e^n} = \lim_{n \rightarrow \infty} e^{e^n} = \infty.$$

(iii) If you're not careful, you might argue as follows: By L'Hôpital's rule, $\lim_{n \rightarrow \infty} \frac{1 - e^{-n}}{2 - e^{-2n}} = \lim_{n \rightarrow \infty} \frac{e^{-n}}{2e^{-2n}} = \lim_{n \rightarrow \infty} \frac{e^n}{2} = \infty$. But that's not correct - what's wrong with this argument? What is the correct value of the limit? See the last page for the answer.

(iv) Here, either by dividing top and bottom through by e^{2n} (or using L'Hôpital's rule, which is valid here) one sees that the sequence has limit zero.

Question 2: (i) Use L'Hôpital's Rule to show that $\frac{(\ln n)^2}{n} \rightarrow 0$ as $n \rightarrow \infty$.

(ii) Show by induction that for any $k \in \mathbb{N}$, $\frac{(\ln n)^k}{n} \rightarrow 0$ as $n \rightarrow \infty$.

Solution: (i) We take $f(x) = (\ln(x))^2$ and $g(x) = x$; thus $a_n = \frac{(\ln(n))^2}{n} = \frac{f(n)}{g(n)}$. Clearly $\lim_{n \rightarrow \infty} f(n) = \infty = \lim_{n \rightarrow \infty} g(n)$ and $g(x) > 0$ for $x > 0$. Thus the hypotheses of L'Hôpital's rule are satisfied. Thus,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)} = \lim_{n \rightarrow \infty} \frac{2 \frac{\ln(n)}{n}}{1} = \lim_{n \rightarrow \infty} 2 \frac{\ln(n)}{n}.$$

Now either by another application of L'Hôpital's rule or the notes (see 8.1.3), we get

$$\lim_{n \rightarrow \infty} a_n = 2 \lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = 0.$$

(ii) We show by induction that $\frac{(\ln(n))^k}{n} = 0$. We have already done the case $k = 1$ in the notes (and $k = 2$ above), so assume that the result holds for some integer $k \geq 2$. Then by L'Hôpital's rule (which does again apply!) with $f(x) = (\ln(x))^{k+1}$ and $g(x) = x$ we get

$$\lim_{n \rightarrow \infty} \frac{(\ln(n))^{k+1}}{n} = \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)} = \lim_{n \rightarrow \infty} \frac{\frac{(k+1)(\ln(n))^k}{n}}{1} = (k+1) \lim_{n \rightarrow \infty} \frac{(\ln(n))^k}{n}.$$

But this final limit is zero by the inductive hypothesis. Hence $\lim_{n \rightarrow \infty} \frac{(\ln(n))^{k+1}}{n} = 0$ and the inductive step is complete. In other words, $\lim_{n \rightarrow \infty} \frac{(\ln(n))^m}{n} = 0$ for all integers m .

The observant reader may have noticed that we did not need to prove part (i) since it is contained in part (ii).

Question 3: (i) Using the formula $(x - y) = \frac{(x - y)(x^2 + xy + y^2)}{(x^2 + xy + y^2)} = \frac{(x^3 - y^3)}{(x^2 + xy + y^2)}$ or otherwise, find

$$\lim_{n \rightarrow \infty} \sqrt[3]{n^3 + n^2} - n.$$

(ii) Show that $\lceil \sqrt[3]{n^3 + n^2} \rceil = n$.

(iii) Using subsequences show that $\lceil \sqrt[3]{n} \rceil - \sqrt[3]{n}$ does not have a limit.

Solution: (i) Substituting $x = \sqrt[3]{n^3 + n^2}$ and $y = n$ into $(x - y) = \frac{(x^3 - y^3)}{(x^2 + xy + y^2)}$ gives

$$\begin{aligned} \sqrt[3]{n^3 + n^2} - n &= \frac{((n^3 + n^2) - n^3)}{((n^3 + n^2)^{2/3} + n(n^3 + n^2)^{1/3} + n^2)} = \frac{n^2}{((n^3 + n^2)^{2/3} + n(n^3 + n^2)^{1/3} + n^2)} \\ &= \frac{1}{((1 + n^{-1})^{2/3} + (1 + n^{-1})^{1/3} + 1)} \rightarrow \frac{1}{3} \quad \text{as } n \rightarrow \infty. \end{aligned}$$

(ii) It suffices to prove that $n \leq \sqrt[3]{n^3 + n^2} < n + 1$. The left hand inequality is obvious while the right hand inequality is equivalent to $(n^3 + n^2) < (n + 1)^3 = n^3 + 3n^2 + 3n + 1$, which is certainly true.

(iii) We use subsequences. One subsequence is to take $k_n = n^3$ in which case

$$a_{k_n} = \lceil \sqrt[3]{n^3} \rceil - \sqrt[3]{n^3} = n - n = 0.$$

Thus this subsequence $(a_{k_n}) = (0)$ has limit 0.

For the other subsequence we use (i) as the hint and try $k_n = n^3 + n^2$. In this case $a_{k_n} = [\sqrt[3]{n^3 + n^2}] - \sqrt[3]{n^3 + n^2} = n - \sqrt[3]{n^3 + n^2}$.

By part (i) this subsequence has limit $-\frac{1}{3}$. Since the two subsequences have different limits the original sequence cannot have a limit.

Question 2(iii) cont. The problem, of course, is that neither $f(x) = 1 - e^{-x}$ nor $g(x) = 2 - e^{-2x}$ tends to zero (or infinity) and so L'Hôpital's rule cannot be applied. One should more simply note that AoL applies to give the correct limit, which is $\frac{1-0}{2-0} = \frac{1}{2}$.