MATH10242 Sequences and Series:
Exercises for Week 7 Tutorials, Solutions

Question 1: Do the following sequences converge/diverge/tend to infinity or tend to minus infinity?

(*These also appear in the course notes, at the end of Chapter 5.*)

(a) $\big(\cos(n\pi)\sqrt{n}\big)_{n\in\mathbb{N}}$ (b) $\big(\sin(n\pi)\sqrt{n}\big)_{n\in\mathbb{N}}$

c) $\left(\dfrac{\sqrt{n^2+2}}{\sqrt{n}}\right)_{n\in\mathbb{N}}$ (d) $\left(\dfrac{n^3+3^n}{n^2+2^n}\right)_{n\in\mathbb{N}}$

e) $\left(\dfrac{n^2+2^n}{n^3+3^n}\right)_{n\in\mathbb{N}}$ (f) $\left(\dfrac{1}{\sqrt{n}-\sqrt{2n}}\right)_{n\in\mathbb{N}}$

Solutions:

(a) $a_n = \cos(n\pi)\sqrt{n} = (-1)^n\sqrt{n}$. Hopefully it is clear that this ought to diverge, but it also does not tend to ∞ or to $-\infty$. But we should prove it carefully.

None of our rules really applies directly, but what we do know is that $\sqrt{n} \to \infty$ by 5.1.4. In particular $(|a_n|)$ is not bounded, and hence (a_n) is also not bounded. Thus it cannot converge by 2.3.9.

It also does not tend to ∞ simply because it is negative half the time. In other words, with $K = 0$ or $K = 1$ there cannot exist N such that $a_n > K$ for all $n \geq N$. Similarly it cannot tend to $-\infty$, so it just diverges.

(b) $\sin(n\pi)\sqrt{n} = 0$ for all n so the sequence converges (to 0).

(c) We have $\dfrac{\sqrt{n^2+2}}{\sqrt{n}} = \dfrac{\sqrt{n+2/n}}{\sqrt{1}} \geq \sqrt{n}$ for $n \geq 1$. By 5.1.4, $\sqrt{n} \to \infty$ and so $\dfrac{\sqrt{n^2+2}}{\sqrt{n}} \to \infty$ as $n \to \infty$ by the Sandwich Rule 5.1.8(ii).

(d) Note that $\dfrac{n^2+2^n}{n^3+3^n} = \dfrac{\frac{n^2}{3^n}+(2/3)^n}{\frac{n^3}{3^n}+1} \to \dfrac{0+0}{0+1} = 0$ by the AoL and Chapter 4. Hence, by the Reciprocity Theorem 5.1.6, $\dfrac{n^3+3^n}{n^2+2^n} \to \infty$.

(e) By the work in (d) we see this has limit 0.

(f) Since $\sqrt{n}-\sqrt{2n} = \sqrt{n}(1-\sqrt{2})$ (by the usual trick, see the Week 3 Example Sheet) $\to -\infty$, we deduce, by the $-\infty$ version of 5.1.6(i), that
$\dfrac{1}{\sqrt{n}-\sqrt{2n}} \to 0$.

Question 2: Complete the proof of Theorem 5.1.8, by proving the following result:

Theorem: Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ both are sequences that tend to infinity. Prove:

(i) $a_n + b_n \to \infty$ as $n \to \infty$;

(ii) $a_n \cdot b_n \to \infty$ as $n \to \infty$.

(iii) Let $M \in \mathbb{N}$. Assume that $(c_n)_{n\in\mathbb{N}}$ is a sequence such that $c_n \geq a_n$ for all $n \geq M$. Prove that $c_n \to \infty$ as $n \to \infty$.

Solution: (i) Given $K > 0$ we know, by definition, that there exists N_1 and N_2 such that $a_n > K$ for all $n \geq N_1$ and $b_n > K$ for all $n \geq N_2$. So certainly $a_n + b_n > 2K > K$ for all $n \geq \max\{N_1, N_2\}$.
(ii) Let $K > 0$ and set $K_1 = \max\{K, 1\}$. Then, again, we know that there exists N_1 and N_2 such that $a_n > K_1$ for all $n \geq N_1$ and $b_n > K_1$ for all $n \geq N_2$. So certainly $a_n \cdot b_n > K_1^2 \geq K$ for all $n \geq \max\{N_1, N_2\}$.
(iii) Given $K > 0$ we know that there exists N such that $a_n > K$ for all $n \geq N$. So certainly $c_n \geq a_n > K$ for all $n \geq \max\{N, M\}$.

Question 3: There are many variants on Question 1. Can you think of some? Here is one:

(i) Suppose that $(a_n)_{n\in\mathbb{N}} \to \infty$ as $n \to \infty$ and that $(b_n)_{n\in\mathbb{N}}$ is a sequence of non-zero numbers that converges to $\ell > 0$. Prove that $\dfrac{a_n}{b_n} \to \infty$ as $n \to \infty$.

(ii) What happens if $\ell = 0$ in part (i)?

Solution: (i) (First proof) As $a_n \to \infty$, certainly $a_n > 0$ for large n. Similarly, $b_n > 0$ for large n (for a detailed proof of this step see the next paragraph). Therefore by the Reciprocal Rule 5.1.6, it is enough to prove that $\lim\limits_{n\to\infty} \dfrac{1}{a_n} \cdot b_n = 0$. But, by the Reciprocal Rule, again, $\lim\limits_{n\to\infty} \dfrac{1}{a_n} = 0$ and $\lim\limits_{n\to\infty} b_n = \ell$. Thus, by the Algebra of Limits Theorem $\lim\limits_{n\to\infty} \dfrac{1}{a_n} \cdot b_n = 0 \cdot \ell = 0$.
(Alternative proof) First, there exists N such that $b_n \geq \frac{1}{2}\ell$ and $b_n \leq \ell + \frac{1}{2}\ell \leq 2\ell$ for all $n \geq M$. (We have done this trick before, but if you need an extra hint try using the definition of convergence with $\epsilon = \frac{1}{2}\ell$.)

In particular $b_n > 0$ for $n \geq N$. Thus using $b_n \leq 2\ell$ for $n \geq N$ we get that $\dfrac{a_n}{b_n} \geq \dfrac{a_n}{2\ell}$ for $n \geq N$. Finally $\dfrac{a_n}{2\ell} \to \infty$ by Question 2(ii) and hence $\dfrac{a_n}{b_n} \to \infty$ by Question 2(iii).
(ii) If $\ell = 0$ the sequence $(\frac{a_n}{b_n})$ either goes to infinity (for example if $b_n = \frac{1}{n}$) or to $-\infty$ (for example if $b_n = -\frac{1}{n}$) or it just diverges when the (b_n) oscillate; for example $b_n = (-1)^n \frac{1}{n}$.

Question 4: Use the subsequence test to show that:-

(i) the sequence $\left(\dfrac{n}{8} - [\dfrac{n}{8}]\right)_{n\in\mathbb{N}}$ does not converge;

(ii) the sequence $\left([\sin(\frac{n\pi}{4})] - \sin(\frac{n\pi}{4})\right)_{n\in\mathbb{N}}$ does not converge.

Solution: (i) If we take $k_n = 8n$ then $a_{k_n} = a_{8n} = n - [n] = 0$.

On the other hand if $k_n = 8n + 1$

$$a_{k_n} = a_{8n+1} = \frac{8n+1}{8} - [\frac{8n+1}{8}] = (n + \frac{1}{8}) - [n + \frac{1}{8}] = \frac{1}{8}.$$

Since we have two subsequences with distinct limits, Theorem 6.1.3 says that the original sequence (a_n) cannot have a limit.

(ii) This is similar. For $k_n = 8n + 1$

$$a_{k_n} = a_{8n+1} = \left[\sin(\frac{\pi}{4} + 2n\pi)\right] - \sin(\frac{\pi}{4} + 2n\pi) = [\frac{1}{\sqrt{2}}] - \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}.$$

But if $k_n = 8n$

$$a_{k_n} = a_{8n} = [\sin(2n\pi)] - \sin(2n\pi) = 0.$$

So, again Theorem 6.1.3 says that the original sequence (a_n) cannot have a limit.

Question 5: Assume that $n^{\frac{1}{\sqrt{n}}} \to \ell$ as $n \to \infty$.

Use the subsequence test to show that $\ell = 1$. [Hint: We do know $\lim_{m\to\infty} m^{\frac{1}{m}}$.]

Solution: Since the hint suggests thinking about $m^{\frac{1}{m}}$, we can try the subsequence (a_{k_n}) for $k_n = n^2$. Then

$$a_{k_n} = a_{n^2} = (n^2)^{\frac{1}{n}} = n^{\frac{2}{n}} = \left(n^{\frac{1}{n}}\right)^2.$$

Now 4.1.5 and the AoL says that this sequence has $\lim_{n\to\infty} a_{k_n} = 1^2 = 1$.

Finally, since we are told that the original sequence (a_n) has a limit ℓ, Theorem 6.1.3 gives $\ell = 1$.

What is harder, and not done here, is to show that the sequence does have a limit.

Question 6*: Prove that $(n!)^{-\frac{1}{n}} \to 0$ as $n \to \infty$. [Hint: Use 4.1.4 with $c = \frac{1}{\epsilon}$.]

Solution: Let $\epsilon > 0$ be given and (by the hint) apply 4.1.4. This implies that $\left(\frac{1}{\epsilon}\right)^n \frac{1}{n!} \to 0$ as $n \to \infty$. In particular there exists $N \in \mathbb{N}$ such that $\left|\left(\frac{1}{\epsilon}\right)^n \frac{1}{n!}\right| < 1$ for all $n \geq N$. Equivalently, $\frac{1}{n!} < \epsilon^n$ for all $n \geq N$.

Now we take n^{th} roots to get: For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\left(\frac{1}{n!}\right)^{\frac{1}{n}} < \epsilon$ for all $n \geq N$. Which is exactly what we need to prove. □

Extra Questions for Week 7:

Question 7.
(a) Does every bounded increasing sequence converge?
(b) Does every increasing sequence of negative terms converge?
(c) Does every decreasing sequence of negative terms converge?
(d) Is every bounded sequence convergent?
(e) Is the limit of an increasing, convergent sequence necessarily equal to the supremum of the set of its terms?
(f) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonzero terms. If $\frac{1}{a_n} \to l$ as $n \to \infty$ and $l \neq 0$, does it necessarily follow that the sequence $(a_n)_{n\in\mathbb{N}}$ converges?
(g) Let (a_n) be a convergent sequence and let (b_n) be a bounded sequence. Is $(a_n b_n)_{n\in\mathbb{N}}$ necessarily a convergent sequence?

Question 8.
(a) Suppose that the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converge to a and b respectively. Show that the sequence $(a_n - b_n)_{n\in\mathbb{N}}$ converges to $a - b$.

(b) Suppose that the sequence $(a_n)_{n\in\mathbb{N}}$ converges to a limit ℓ. Suppose also that, for every n, $a_n \leq r$. Prove that $\ell \leq r$.

(c) Suppose that the sequence $(a_n)_{n\in\mathbb{N}}$ converges to a limit ℓ. Suppose also that there is an integer M such that, for every $n \geq M$, $a_n \leq r$. Is $\ell \leq r$? If so, give a proof; if not, give a counterexample.

Solutions to Extra Questions for Week 8:

Question 7.
(a) Does every bounded increasing sequence converge? **Yes**
(b) Does every increasing sequence of negative terms converge? **Yes** [Note that 0, or 1 or... is an upper bound, so this is a special case of (a).]
(c) Does every decreasing sequence of negative terms converge? **No** [For instance $(-n)_n$.]
(d) Is every bounded sequence convergent? **No** [Standard example: $(-1)^n$.]
(e) Is the limit of an increasing, convergent sequence necessarily equal to the supremum of the set of its terms? **Yes** [For example, one of the main points in the proof of the Monotone Convergence Theorem.]
(f) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonzero terms. If $\frac{1}{a_n} \to l$ as $n \to \infty$ and $l \neq 0$, does it necessarily follow that the sequence $(a_n)_{n\in\mathbb{N}}$ converges? **Yes** [Follows from the Algebra of Limits Theorem (vi).]
(g) Let (a_n) be a convergent sequence and let (b_n) be a bounded sequence. Is $(a_nb_n)_{n\in\mathbb{N}}$ necessarily a convergent sequence? **No** [Take $a_n = 1$, $b_n = (-1)^n$.]

Question 8.
(a) Suppose that the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converge to a and b respectively. Show that the sequence $(a_n - b_n)_{n\in\mathbb{N}}$ converges to $a - b$.
Proof: Choose $\epsilon > 0$. There is N_1 such that $\mid a_n - a \mid < \epsilon/2$ for all $n \geq N_1$. Similarly there is N_2 such that $\mid b_n - b \mid < \epsilon/2$ for all $n \geq N_2$. Set $N = \max\{N_1, N_2\}$. Then, for $n \geq N$, we have $\mid (a_n - b_n) - (a - b) \mid = \mid (a_n - a) - (b_n - b) \mid \leq \mid a_n - a \mid + \mid b_n - b \mid$ (by the triangle inequality) $< \epsilon/2 + \epsilon/2 = \epsilon$, as required.

(b) Suppose that the sequence $(a_n)_{n\in\mathbb{N}}$ converges to a limit ℓ. Suppose also that, for every n, $a_n \leq r$. Prove that $\ell \leq r$.
Proof: Argue by contradiction. If not, then $\ell > r$ so $\ell - r > 0$; set $\epsilon = \ell - r$. Since $a_n \to \ell$, there is N such that, for all $n \geq N$, we have $\mid a_n - \ell \mid < \epsilon$. So, for $n \geq N$, $a_n > \ell - \epsilon = \ell - (\ell - r) = r$ - a contradiction, as required [Of course, you don't need to introduce the notation ϵ, and can just write $r - \ell$ all the way through.]
[An alternative way of finishing off (if you don't notice the faster argument above) from the point where we have $\mid a_n - \ell \mid < \epsilon$: $\mid a_n - \ell \mid = \mid \ell - a_n \mid = \mid (\ell - r) + (r - a_n) \mid = (\ell - r) + (r - a_n)$ (both terms are ≥ 0 since $\ell > r$ and $r \geq a_n$). That is, $(\ell - r) + (r - a_n) < \epsilon = \ell - r$, so $r - a_n < 0$, that is $r < a_n$, contradiction, as required.

(c) Suppose that the sequence $(a_n)_{n\in\mathbb{N}}$ converges to a limit ℓ. Suppose also that there is an integer M such that, for every $n \geq M$, $a_n \leq r$. Is $\ell \leq r$? If so, give a proof; if not, give a counterexample.
Yes. Proof: The proof is almost the same as (b). Just replace the last sentence by: "So, for $n \geq N$, $a_n > \ell - \epsilon = \ell - (\ell - r) = r$. So if $n \geq \max\{M, N\}$ then we have both $a_n > r$ and $a_n \leq r$, a contradiction, as required."