MATH10242 Sequences and Series: Solutions to Exercises for Week 3 Tutorials

Question 1: Here you should justify a couple of formulas that we will often use. Prove:

(a) $\forall x, y, \quad |x-y| \geq |\,|x|-|y|\,|$;

(b) $\forall x, \ell$ and $\forall \epsilon > 0$, $|x-\ell| < \epsilon \iff \ell - \epsilon < x < \ell + \epsilon$.

Solution: (a) *First proof:* Try to use the triangle inequality. So, $|x| = |x-y+y| \leq |x-y| + |y|$. Rearranging gives $|x-y| \geq |x| - |y|$.

In just the same way, $|x-y| = |y-x| \geq -(\,|x|-|y|\,)$. Combining these two results gives

$$|x-y| \geq \max\{\,|x|-|y|,\ -(|x|-|y|\,)\} = ||x|-|y||\,.$$

Second proof: Perhaps easier, though less elegant is to prove it directly, as I suggested in some of the tutorials. In other words, look at four separate cases. If both $x \geq 0$ and $y \geq 0$ or if both $x \leq 0$ and $y \leq 0$ then it is clear that $|x-y| = ||x|-|y||$. On the other hand, if x and y have different signs then you get $|x-y| > ||x|-|y||$.

(b) $\Rightarrow$ Suppose that $|x-\ell| < \epsilon$. Then $(x-\ell) < \epsilon$ and $(\ell - x) = -(x-\ell) < \epsilon$. The first inequality gives $x < \epsilon + \ell$ while the second gives $-x < \epsilon - \ell$ or $x > \ell - \epsilon$.

$\Leftarrow$ Here, as $x < \epsilon + \ell$, we get $(x-\ell) < \epsilon$. Similarly $x > \ell - \epsilon$ is the same as $-(x-\ell) = (\ell - x)\} < \epsilon$. As usual, these combine to give $|x-\ell| < \epsilon$.

Question 2: Let $\epsilon > 0$ be given. For each of the following sequences (a_n), find a natural number N such that $\forall n \geq N$, one has $|a_n| < \epsilon$ (thereby showing that $a_n \to 0$ as $n \to \infty$).

(a) $a_n = \frac{1}{n^2}$.

(b) $a_n = \frac{n+\sqrt{n}}{n^2+1}$.

(c) $a_n = \frac{\cos(n)}{n}$;

(d) $a_n = \sqrt{n+2} - \sqrt{n}$

(e)* $a_n = \frac{n}{2^n}$.

Hints: In parts (b) and (c) find a nicer function $f(n)$ with $|a_n| < f(n)$. Most of part (d) has already been seen on the Week 2 sheet.

Solution: In parts (a,b,d,e) as $a_n > 0$ for all n, we can drop the absolute value signs—which does make things clearer. Note that (for some x) we often want an integer n with $n \geq x$. Clearly $[x]+1$ works.

(a) Now, $|\frac{1}{n^2}| < \epsilon \iff n > \sqrt{\frac{1}{\epsilon}}$. So, take $N = \left[\sqrt{\frac{1}{\epsilon}}\right] + 1$.

(b) Getting a good bound would be really messy here, so we first want to get an upper bound for a_n that is easier to manipulate. With complicated expressions do always try to do this. So,

$$a_n \;=\; \frac{n+\sqrt{n}}{n^2+1} \;\leq\; \frac{n+n}{n^2+1} \;\leq\; \frac{n+n}{n^2} \;=\; \frac{2}{n}.$$

Now it is easy—we need $\frac{2}{n} < \epsilon$ or $n > \frac{2}{\epsilon}$. So, take $N = 1 + \left[\frac{2}{\epsilon}\right]$.Then, going back through the steps, if $n \geq N$ then

$$0 \;\leq \frac{2}{n} \leq \frac{2}{N} < \epsilon.$$

(c) Here $|a_n| = |\frac{\cos(n)}{n}| \leq 1/n$, so simply take $N = [\frac{1}{\epsilon}] + 1$.
(d) Here the trick is to try replacing differences by sums:

$$\begin{aligned}
\sqrt{n+2} - \sqrt{n} &= \frac{(\sqrt{n+2} - \sqrt{n})(\sqrt{n+2} + \sqrt{n})}{(\sqrt{n+2} + \sqrt{n})} \\
&= \frac{(\sqrt{n+2}^2 - \sqrt{n}^2)}{\sqrt{n+2} + \sqrt{n}} = \frac{(n+2) - n}{\sqrt{n+2} + \sqrt{n}} = \frac{2}{\sqrt{n+2} + \sqrt{n}}
\end{aligned}$$

Thus $\sqrt{n+2} - \sqrt{n} = \dfrac{2}{\sqrt{n+2} + \sqrt{n}} \leq \dfrac{2}{2\sqrt{n}}$. So, we need $\dfrac{1}{\sqrt{n}} < \epsilon$ or $n > \epsilon^{-2}$. So, take $N = 1 + [\epsilon^{-2}]$.

Then, again, tracing back through our computations, we see that if $n \geq N$ then $\dfrac{1}{\sqrt{n}} < \epsilon$ and so $\sqrt{n+2} - \sqrt{n} < \epsilon$.

(e) There will be a few ways of doing this; here's one. It's based on the observation that

$$\frac{n}{2^n} = 2\Big(\frac{n/2}{2^{n/2}.2^{n/2}}\Big) = \frac{2}{2^{n/2}} \cdot \frac{n/2}{2^{n/2}}.$$

The first term fairly obviously (but we'll prove it) has limit 0, whereas the second is bounded, so we should expect the product to have limit 0. (Later, when we have developed some theorems in the Algebra of Limits, we can apply them to give a 2-line proof making this argument precise. Since we haven't done that yet, we have to work a bit...)

First, we have (by an easy induction, or calculus) that $k < 2^k$ for all integers $k \geq 1$. Because we're going to need half-integer values in the computation below, use calculus (computing the derivative of $x/2^x$) to see that $x < 2^x$, so $x/2^x < 1$, for all $x \geq 1$, in fact for all $x \geq 1/2$, which is what we will need on the next line.

Then $\dfrac{n}{2^n} = \dfrac{2}{2^{n/2}} \cdot \dfrac{n/2}{2^{n/2}} \leq \dfrac{2}{2^{n/2}}$. So, given $\epsilon > 0$, we will have $| \dfrac{n}{2^n} | < \epsilon$ if $\dfrac{2}{2^{n/2}} < \epsilon$, hence if $2^{n/2} > 2/\epsilon$ hence if $n/2 > \log_2(2/\epsilon)$, that is $n > 2\log_2(2/\epsilon)$. So if we choose $N = [2\log_2(2/\epsilon)] + 1$ then we will have $n/2^n < \epsilon$ for all $n \geq N$. Therefore the sequence converges to 0.

And here's another method. Recall (from last semester) that $n^2 < 2^n$ for all $n \geq 5$. So, for $n \geq 5$, we have $\dfrac{n}{2^n} < \dfrac{n}{n^2} = \dfrac{1}{n}$, which will be $< \epsilon$ if $n > \dfrac{1}{\epsilon}$. So, if we take $N = \max\{5, [1/\epsilon] + 1\}$ then, for all $n \geq N$, we will have $\dfrac{n}{2^n} < \epsilon$, as required.

And here's yet another, even simpler, solution. Use the Binomial Theorem to expand $2^n = (1+1)^n = 1 + n + \dfrac{n(n-1)}{2} + ...$ so $2^n \geq \dfrac{n(n-1)}{2}$ (since all terms are non-negative) and hence $\dfrac{n}{2^n} \leq \dfrac{n}{n(n-1)/2} = \dfrac{2}{n-1}$ which will be $< \epsilon$ if $n > \dfrac{2}{\epsilon} + 1$.

Question 3: Which of the following sequences converge and to what value? In each case you should properly justify your answers, making use of the formal definition of convergence to a limit, as we have been doing in class.

(a) $(1 + \frac{(-1)^n}{n})_{n\in\mathbb{N}}$;

(b) $(1 + \frac{3n^2+n}{2n^2})_{n\in\mathbb{N}}$;

(c) $(1 + (-1)^n)_{n\in\mathbb{N}}$;

(d) $\left(\dfrac{n+4(-1)^n}{2n}\right)$.

Solution: (a) This sequence converges to 1. Likely this is clear intuitively, but we still need to give a proper argument. So rather like in 2(c), given $\epsilon > 0$ we try $N = N_\epsilon = [1/\epsilon] + 1$; thus $N > 1/\epsilon$ and so $1/N < \epsilon$.

Now for any $n \in \mathbb{N}$ with $n \geq N$ we get

$$\left|\left(1 + \frac{(-1)^n}{n}\right) - 1\right| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n} \leq \frac{1}{N} \leq \epsilon.$$

(b) As in Question 2 it is a good idea to manipulate the messy function of n into something nicer. The key idea is to divide top and bottom by n^2,

$$\frac{3n^2+n}{2n^2} = \frac{3+n^{-1}}{2} = \frac{3}{2} + \frac{1}{2n}.$$

Thus $1 + \dfrac{3n^2+n}{2n^2} = \frac{5}{2} + \dfrac{1}{2n}$.

Now it is easy—if we pick $N = \left[\frac{1}{2\epsilon}\right] + 1$ then (with the usual manipulation)) $N > \frac{1}{2\epsilon}$ and $\frac{1}{2N} < \epsilon$. For $n \geq N$ the earlier computations show that

$$\left|\left(1 + \frac{3n^2+n}{2n^2}\right) - \frac{5}{2}\right| = \left|\left(\frac{5}{2} - \frac{1}{2n}\right) - \frac{5}{2}\right| = \left|-\frac{1}{2n}\right| = \frac{1}{2n} \leq \frac{1}{2N} < \epsilon.$$

In other words $\lim\limits_{n\to\infty} a_n = 5/2$.

In this sort of question, it is probably best to do the computations in the displayed equation before getting the precise bound for N, as a kind of rough working. In other words, write down the displayed equation up to the $\leq 2N$. Then work out what is needed for this to be $< \epsilon$ and write out the solution, or go back and fill in if you've left space in what you've already written.

(c) The sequence is $0, 2, 0, 2, 0, 2, \ldots$ Just like $1, -1, 1, -1, \ldots$ this does not converge.

Here is the formal proof. Let $a_n = 1 + (-1)^n$ and suppose that (a_n) does converge to, say, x. Take $\epsilon = 1/2$. Then there exists N such that $|a_n - x| < 1/2$ for all $n \geq N$. Now for n even (and bigger than N) $a_n = 2$, so $|2 - x| < 1/2$ which certainly forces $x > 2 - 1/2 = 3/2$. On the other hand for n odd, $a_n = 0$ and now $x < 0 + 1/2 = 1/2$. A contradiction.

(d) First, we see that

$$a_n = \frac{n+4(-1)^n}{2n} = \frac{1+\frac{4}{n}(-1)^n}{2} = \frac{1}{2} + (-1)^n\frac{2}{n}.$$

So we might guess that the limit is $\frac{1}{2}$. So, let's prove it. Let $\epsilon > 0$. Notice that

$$|a_n - \frac{1}{2}| < \epsilon \iff |\frac{1}{2} + (-1)^n\frac{2}{n} - \frac{1}{2}| < \epsilon \iff \frac{2}{n} < \epsilon \iff n > \frac{2}{\epsilon}.$$

So, take $N = [\frac{2}{\epsilon}] + 1$. Then for $n \geq N$, and going back through these computations, we see that $|a_n - \frac{1}{2}| < \epsilon$.

Question 4* (a) Let $x > 0$. Using the binomial theorem (or otherwise) prove that for all $n \in \mathbb{N}$, one has $(1+x)^n \geq 1 + nx$.

(b) By taking $x = \frac{y}{n}$ in (a), deduce that for all $y > 0$ and $n \in \mathbb{N}$, $(1+y)^{\frac{1}{n}} \le 1 + \frac{y}{n}$.

(c) Hence show that for fixed $c > 1$, one has $c^{\frac{1}{n}} \to 1$ as $n \to \infty$.

Solution: (a) To prove that $(1+x)^n \ge 1 + nx$, just use the binomial theorem—which says that

$$(1+x)^n = 1 + nx + (\text{lots of positive terms}).$$

(b) It seems natural to take n^{th} roots, which gives[1] $(1+x) \ge (1+nx)^{1/n}$. This does not look quite right, but it also holds for $x = \frac{y}{n}$ giving $(1+\frac{y}{n}) \ge (1+y)^{1/n}$, which is what we wanted to prove.

What we needed above is the fact:

$$\text{if } 0 < \alpha < \beta \text{ then } 0 < \alpha^{1/n} < \beta^{1/n}.$$

You could reasonably assume that, the general rule being that "standard" results of arithmetic can be assumed. What is never entirely clear is what "standard" means. So, since we have been careful about the proofs of a number of such results, let's prove this one as well.

If it is false, then $\alpha^{1/n} \ge \beta^{1/n}$. Now we have earlier proved that if $0 < x \le y$ then $x^2 \le y^2$ and by an easy induction (can you write out the details?) it follows that $x^n \le y^n$ for all $n \ge 1$. So, apply this for $x = \alpha^{1/n}$ and $y = \beta^{1/n}$. Then you get $\alpha = x^n \le y^n = \beta$, giving a contradiction.

(c) For any $c > 1$ we can write $c = 1+y$ for $y > 0$. Now we choose $N > \dfrac{y}{\epsilon}$, so that $\dfrac{y}{N} < \epsilon$. Then

$$\begin{aligned}
\left|c^{1/n} - 1\right| &= c^{1/n} - 1 && \text{since clearly } c^{1/n} > 1 \text{ by the footnote} \\
&= (1+y)^{1/n} - 1 \\
&\le (1+\tfrac{y}{n}) - 1 && \text{by the first part of the question} \\
&= \frac{y}{n} \\
&\le \frac{y}{N} \\
&< \epsilon.
\end{aligned}$$

As before, a way to write out such an argument is to write out the first five lines of the display, and then go back and work out the relationship between N and ϵ.

[1] See the comment at the end of the solution

Extra Question (on continued fractions) for Week 3

What is $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{\cdots}}}}}$

Solutions to Extra Question (on continued fractions) for Week 3

What is $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}}}$

Solution: The question means, what is the limit of the sequence $\frac{1}{1}$, $\cfrac{1}{1+\cfrac{1}{1}}$, $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}$, $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}$, $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}}$, $\ldots$, that is, the sequence $\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \ldots$ (notice the Fibonacci Sequence there!).

If we assume that the limit exists (and there are ways to justify that), call it ℓ say, then we get, note, the equation $\ell = \dfrac{1}{1+\ell}$. Solving the resulting quadratic, we get two roots, one of which is negative so can't be the limit of a sequence of positive numbers. Therefore $\ell = \dfrac{-1+\sqrt{5}}{2}$ = the inverse of the Golden Ratio.