MATH10242 Sequences and Series:
Solutions to Exercises for Week 2 Tutorials

Question 1: Let $x \in \mathbb{R}$. Using just the axioms for ordered fields (A0–9) and (Ord 1–4) from Chapter 1 of the Notes, and breaking into the cases when x is either positive, negative or zero, show that $x^2 \geq 0$.

Solution: We break into three cases. If $x = 0$ then $x^2 = 0 \cdot 0 = 0 \geq 0$ (we showed in the lectures that $x.0 = 0$ for every x.

If $x > 0$ then (Ord 4) gives $x^2 = x \cdot x > x \cdot 0 = 0$.

Finally, by (Ord 1), the only remaining possibility is that $x < 0$. But if $x < 0$ then, adding $-x$ to each side and using (Ord 3), we get $x + (-x) < -x$, that is, using (A4), $0 < -x$. The first paragraph then gives $0 < (-x)^2$. We need to show that we can "pull the minus signs out the the brackets". (Note that $(-x)^2$ means "the negative of x, squared", whereas, $-(-(x^2))$ means "the negative of the negative of x^2" and it's not obvious that these are equal, or that the latter is equal to x^2.)

So: first we prove the rule $(-a)b = -(ab)$: we have $(-a)b + ab = (-a + a)b = 0.b = 0$ (using (A6), (A9) and what's already been proved) so, by (A4), $(-a)b = -(ab)$. Therefore using this twice (and (A6)), we get $(-a)(-b) = ab$. In particular, $(-x)^2 = x^2$, which is what we needed to finish off this third case, and therefore the proof.

Question 2: Show, using just the axioms for ordered fields, that if $x, y > 0$ then $x > y \iff x^2 > y^2$.

Solution: Suppose that $x > y$. Then by (Ord 4) $x^2 = x{\cdot}x > x{\cdot}y$ and similarly $y{\cdot}x > y{\cdot}y$. Combining them (with (A6)) gives $x^2 > y^2$.

The same argument shows that, if $x < y$ then $x^2 < y^2$ (hence, by (Ord 1), $y^2 \not< x^2$) and if $x = y$ then (by what "=" means) $x^2 = y^2$ (so, again, $y^2 \not< x^2$). That is (again using (Ord 1)), if $x \not> y$ then $x^2 \not> y^2$. So we have shown $x^2 > y^2 \Leftrightarrow x > y$.

Question 3: Show, using just the axioms for ordered fields (including that $0 \neq 1$), that for all $x \in \mathbb{R}$ we have $x < x + 1$.

Solution: By (Ord 3) it will be enough to prove that $0 < 1$, so let's do that first.

I'll argue by contradiction, so suppose that $0 < 1$ is false. Then, by (Ord 1), either $0 = 1$ - which contradicts our assumption that $0 \neq 1$ - or $1 < 0$. So assume, aiming for a contradiction, that $1 < 0$. That is, $-(-1) < 0$ (we did the argument for that in the lectures). By (Ord 3) we deduce $-(-1) + (-1) < 0 + (-1)$ which, by (A4) and (A3), gives $0 < -1$. Then apply (Ord 4) to get $0.(-1) < (-1).(-1)$. The left-hand side is 0 (we showed this in the lecture) and, see the solution to Question 1, $(-1)(-1) = -(-1) = 1$. That is, $0 < 1$ - contradicting, by (Ord 1), the assumption that $1 < 0$.

Thus we deduce that $0 < 1$. Now add x to both sides and (Ord 3) gives us $x < x + 1$. [There will be many ways of proving this, maybe some more direct than the argument I've given.]

Another solution, found by a student: Take any $x \neq 0$; by Question 1, $x^2 \geq 0$, in fact, $x^2 > 0$ because, if $x^2 = 0$ then, multiplying both sides by $\frac{1}{x^2}$, we'd deduce $1 = 0$, contradiction. Since also $\frac{1}{x} \neq 0$ (otherwise, multiplying both sides by x, we'd again have the $0 = 1$ contradiction), we also have $\left(\frac{1}{x^2}\right)^2 > 0$. Next note that $\left(\frac{1}{x}\right)^2 = \frac{1}{x^2}$ (because both multiply x^2 to 1 so, by uniqueness of multiplicative inverse (A8), they're equal). Now apply (Ord 4) to the inequality $x^2 > 0$, multiplying both sides by $\frac{1}{x^2}$ (which we've just shown is positive), to deduce $1 > 0$. Then add x to both sides to finish.

Question 4: Show that for any $\delta > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$.
[Example 2.4.8 from the notes can be used here.]

Solution: Note that $\frac{1}{\delta}$ exists by (A8). We have $\frac{1}{\delta} > 0$ (if we had $\frac{1}{\delta} < 0$ then we'd deduce $1 < 0$, which contradicts the result of Question 1 since $1 = 1^2$) and so by the unboundedness of $\mathbb{N}$ (Example 2.4.8) there exists $n \in \mathbb{N}$ such that $n > \frac{1}{\delta}$. Now (Ord 4) shows that $dn > \frac{1}{\delta}d$ for any $d \in \mathbb{R}^+$. Take $d = \frac{1}{n}\delta$ - this is positive by (Ord 4) and (Ord 2) since δ and (by (Ord 2)) n are, hence (as in the first paragraph) so is $\frac{1}{n}$. With this value of d we get

$$\frac{1}{n} < \delta.$$

Question 5:* I said in the lecture that, from the construction of the reals $\mathbb{R}$ from the rationals $\mathbb{Q}$, it follows that $\mathbb{Q}$ is dense in $\mathbb{R}$ (meaning that, given any two real numbers $x < y$, there is a rational number, q, between them: $x < q < y$). Show that the set $\mathbb{R} \setminus \mathbb{Q}$ of **irrationals** is **dense** in $\mathbb{R}$ i.e. show that for all $x, y \in \mathbb{R}$ if $x < y$ then there exists $t \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < t < y$.

Solution: Here are a couple of, rather different, solutions. There may well be more.
[1]: Given $x < y$ in $\mathbb{R}$, first choose rational x', y' with $x \leq x' < y' \leq y$ - we can do that using density of $\mathbb{Q}$ in $\mathbb{R}$ twice (if both x, y are irrational). So it will be enough to show that there is an irrational strictly between x' and y'. We know from Foundations of Pure Maths that $\sqrt{2}$ is irrational; so (note), for every positive integer n, $\frac{\sqrt{2}}{n}$ is irrational. If we choose n large enough that $\frac{\sqrt{2}}{n} < y' - x'$ (the distance between x' and y') then we'll have $x' < x' + \frac{\sqrt{2}}{n} < y'$, as required since $x' + \frac{\sqrt{2}}{n}$ must be irrational (if it were rational, we'd quickly deduce, subtracting the rational x' and multiplying up by n, that $\sqrt{2}$ is rational - contradiction).
[2]: We could use the ideas around countable and uncountable sets that were discussed in Foundations of Pure Maths. Recall that the interval $(0, 1)$ is uncountable. Then scale it into the interval (x, y) by applying the bijection $z \in (0, 1) \mapsto x + \frac{z}{y-x}$. Since that is a bijection, it follows that the interval (x, y) is uncountable (as, indeed, is every non-empty open interval in $\mathbb{R}$). But recall also that the set $\mathbb{Q}$ is countable, so its subset $\mathbb{Q} \cap (x, y)$ is countable. Hence $(\mathbb{R} \setminus \mathbb{Q}) \cap (x, y)$ must be uncountable (if it were countable then (x, y) would be the union of two countable sets, hence countable, contradiction). In particular $(\mathbb{R} \setminus \mathbb{Q}) \cap (x, y)$ is non-empty, which is exactly what we wanted to prove.

Question 6: For each of the following sequences (a_n) and real numbers $\epsilon > 0$, find a natural number N such that $\forall n \geq N$ we have $|a_n| < \epsilon$.

(a) $a_n = \frac{1}{n}$, $\epsilon = 1/50$.

(b) $a_n = \frac{1}{n^2}$, $\epsilon = 1/100$.

(c) $a_n = \frac{1}{n^2}$, $\epsilon = 1/1000$.

(d) $a_n = \frac{1}{\sqrt{n}}$, $\epsilon = 1/1000$.

(e) $a_n = \frac{\cos(n)}{n}$, $\epsilon = 10^{-6}$.

(f) $a_n = \frac{\cos(n)}{n^2}$, $\epsilon = 10^{-6}$.

(g) $a_n = \sqrt{n+2} - \sqrt{n}$, $\epsilon = 10^{-6}$.

Solution: (a) We want $1/n < 1/50$, equivalently $n > 50$, so we can choose $N = 51$.
(b) We need $1/n^2 < 1/100$, equivalently $100 < n^2$. So we can choose $N = 11$.
(c) Similarly we need to have $1000 < n^2$. So we can choose, say, $N = 50$ (there's no requirement to choose the "best" value of N).
(d) We need $1/\sqrt{n} < 1/1000$, equivalently $1000 < \sqrt{n}$. So we can choose $N = 10^6 + 1$.
(e) We want to have $\mid \frac{\cos(n)}{n} \mid < 10^{-6}$, that is $\frac{|\cos(n)|}{n} < 10^{-6}$. Since the maximum value of $\mid \cos(n) \mid$ is 1, it will be enough to choose $N = 10^6 + 1$; let's just check that.

(Notice that, so far, we've kind of worked backwards to find the right value of N, and that's been enough since it's been obvious that the chosen value of N works. More commonly, you work backwards, or maybe even semi-guess, a value of N that will work, but then you do have to check that it really does work. So let's do that now, in this example.)

Suppose $n \geq 10^6 + 1$, so $\frac{1}{n} \leq \frac{1}{10^6+1} < \frac{1}{10^6}$, then $\mid \frac{\cos(n)}{n} \mid \leq \frac{1}{n} < 10^{-6} = \epsilon$. As required.
(f) We want $\mid \frac{\cos(n)}{n^2} \mid < 10^{-6}$ and, as in part (d), it will be enough to have $\frac{1}{n^2} < 10^{-6}$, that is $n > 10^3$, so take, say, $N = 10^4$ ($N = 10^3 + 1$ is the minimum choice). We check: if $n \geq 10^4$, then $n^2 > 10^6$, so $\frac{1}{n^2} < 10^{-6}$, hence $\mid \frac{\cos(n)}{n^2} \mid < \frac{1}{n^2} < 10^{-6}$, as required.
(g) We need to get a "nice" estimate of the difference $\sqrt{n+2} - \sqrt{n}$ between these square roots. Multiply by $\frac{\sqrt{n+2}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n}}$ and simplify to get $\sqrt{n+2} - \sqrt{n} = \frac{(\sqrt{n+2}-\sqrt{n})(\sqrt{n+2}+\sqrt{n})}{\sqrt{n+2}+\sqrt{n}} = \frac{2}{\sqrt{n+2}+\sqrt{n}}$. Now, since $\sqrt{n+2} > \sqrt{n}$, we have $\sqrt{n} + \sqrt{n+2} > \sqrt{n} + \sqrt{n} > 0$, so $\frac{1}{\sqrt{n}+\sqrt{n+2}} < \frac{1}{\sqrt{n}+\sqrt{n}}$, so $\frac{2}{\sqrt{n+2}+\sqrt{n}} < \frac{2}{2\sqrt{n}} = \frac{1}{\sqrt{n}}$. Therefore we should choose N such that $\frac{1}{\sqrt{N}} \leq 10^{-6}$, equivalently, $N \geq 10^{12}$. So take $N = 10^{12}$ (we don't need to check directly because the estimates we made had the form "it's enough to ..." and the final part of the estimation for N was "iff").

Question 7: For each of the following sequences (a_n) and real numbers $\epsilon > 0$, find a natural number N such that $\forall n \geq N$ we have $\mid a_n - 2 \mid < \epsilon$.

(a) $a_n = 2 - \frac{1}{2^n}$, $\epsilon = 1/1000$.

(b) $a_n = 2 + \frac{\sin(n)}{n}$, $\epsilon = 1/1000$.

Solution: (a) We want $\mid 2 - \frac{1}{2^n} - 2 \mid < \epsilon$, that is $\frac{1}{2^n} < \epsilon$, equivalently $2^n > \frac{1}{\epsilon}$ so, putting $\epsilon = 1/1000$, we need $2^n > 1000$, so take $N = 10$, say. (For a general ϵ, we'd take $N = \log_2 \frac{1}{\epsilon}$.)
(b) We want $\mid 2 - \frac{\sin(n)}{n} - 2 \mid < \epsilon$, that is $\mid \frac{\sin(n)}{n} \mid < \epsilon$. Since $\mid \sin(n) \mid \leq 1$, so $\mid \frac{\sin(n)}{n} \mid \leq \frac{1}{n}$, it will be sufficient to have $\frac{1}{n} < \epsilon$, equivalently $n > \frac{1}{\epsilon}$. In the case $\epsilon = 1/1000$, this becomes $n > 1000$, so take $N = 1001$, say. (For a general ϵ, we could take $N = \frac{1}{\epsilon} + 1$.)

Extra Question for Week 2 Prove, from the axioms, that $(-1).x = -x$.

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Solution: By (A4) we have $x + (-x) = 0$; in particular $1 + (-1) = 0$. So $0.x = (1+(-1))x = 1.x+(-1).x$ (by (A6) and (A9)) $= x+(-1).x$ (by (A7)). And $0.x = (0+0).x$ (by (A3)) $= 0.x + 0.x$ ((A6),(A9)) so, adding $-(0.x)$ to each side, we get, using (A4) and (A1), $0 = 0.x$.

Now we have both $x + (-x) = 0$ and $0 = x + (-1).x$ so, by the uniqueness part of (A4), we deduce $-x = (-1).x$.

There will be other proofs, maybe some shorter.