MATH10242 Sequences and Series: Revision: Extra Exercises, Solutions

I give, in square brackets, some references to results in the notes. These are to remind you where to find the relevant results; you are not expected to memorise these numbers for the exam! (But, of course, you should know the results.)

Question 1:
(a) Define what it means for a sequence $(a_n)_{n\in\mathbb{N}}$ to tend to $-\infty$.

(b) Define what it means for a sequence $(a_n)_{n\in\mathbb{N}}$ to be bounded below.

(c) The following statement is not correct; modify it to a correct statement.
"Every sequence contains a convergent subsequence."
(Adding "It is not true that" at the beginning is not what I have in mind.)

(d) Some students seem to believe that every sequence is convergent or tends to $+\infty$ or tends to $-\infty$ or switches between a finite number of values. Give an example of a sequence which has none of these properties.

Solutions:
(a) $(a_n)_{n\in\mathbb{N}} \to -\infty$ as $n \to \infty$ if, given any $K \in \mathbb{R}$ [or, given any $K < 0$], there is N such that, for all $n \geq N$, we have $a_n < K$ [$a_n \leq K$ is also ok (the "N" might be different but that's accommodated by the form of the definition)].

(b) $(a_n)_{n\in\mathbb{N}}$ is bounded below if there is $K \in \mathbb{R}$ such that $K < a_n$ [or $K \leq a_n$] for all $n \in \mathbb{N}$.

(c) "Every bounded [that is, bounded above and bounded below] sequence contains a convergent subsequence."

(d) There are lots of possibilities. We could start with the sequence $1, -1, 1, -1, \dots$ and modify it to, say, $1+1, -1-\frac{1}{2}, 1+\frac{1}{3}, -1-\frac{1}{4}, \dots$, that is, $a_n = (-1)^{n+1}(1+\frac{1}{n})$.

The sequence $\sin(n)$ probably is an example but this might not be so easy to prove.

Question 2:
(a) Fix $\epsilon > 0$. Find a natural number N such that $\left|\dfrac{2n^3 - \ln(n)}{n(n-1)^2+1} - 2\right| < \epsilon$ for all $n \geq N$.
What have you (if you managed to answer the question) just shown about the sequence $a_n = \dfrac{2n^3 - \ln n}{n(n-1)^2+1}$?

(b) Fix a real number K. Find a natural number N such that $\ln(n) - n < K$ for all $n \geq N$.
What does that prove?

Solutions: There will be more than one way of doing each of these. Remember that you're not looking for the least "N" that works, just some N that will do the job.

(a) This is the solution that I came up with: $\left|\dfrac{2n^3 - \ln(n)}{n(n-1)^2+1} - 2\right| = \left|\dfrac{2n^3 - \ln(n) - 2n(n-1)^2 - 2}{n(n-1)^2+1}\right| = \left|\dfrac{2n^3 - \ln(n) - 2n^3 + 4n^2 - 2n - 2}{n(n-1)^2+1}\right| = \left|\dfrac{4n^2 - 2n - \ln(n) - 2}{n(n-1)^2+1}\right| \leq \dfrac{4n^2}{(n-1)^3}$ for $n \geq 2$. So it will be enough to have $\dfrac{4n^2}{(n-1)^3} < \epsilon$, equivalently $n - 1 > \dfrac{4n^2}{\epsilon(n-1)^2}$. Note that if $n \geq 4$

then the fraction $\frac{n^2}{(n-1)^2}$ is ≤ 2, so it will be enough to have $N \geq 4$ and also $N - 1 \geq \frac{8}{\epsilon}$.
So take $N = \max\{4, [\frac{8}{\epsilon}] + 2\}$.
But here's a quicker solution, slightly modified from one found by a student in one of the tutorials: $\left|\frac{2n^3 - \ln(n)}{n(n-1)^2+1} - 2\right| = \left|\frac{2n^3 - \ln(n) - 2n(n-1)^2 - 2}{n(n-1)^2+1}\right| = \left|\frac{2n^3 - \ln(n) - 2n^3 + 4n^2 - 2n - 2}{n^3 - 3n^2 + n + 1}\right| = \left|\frac{4n^2 - 2n - \ln(n) - 2}{n^3 - 3n^2 + n + 1}\right| \leq \left|\frac{4n^2}{n^3 - 3n^2}\right| = \left|\frac{4n^2}{n^2(n-3)}\right| = \frac{4}{n-3}$ for $n \geq 4$. So, this will be $< \epsilon$ if $n - 3 > \frac{4}{\epsilon}$, so take $N = [\frac{4}{\epsilon}] + 3$.

[Another possibility goes as follows: We have $\left|\frac{2n^3 - \ln(n)}{n(n-1)^2+1}\right| = \left|\frac{2 - \frac{\ln(n)}{n^3}}{(1-\frac{1}{n})^2 + \frac{1}{n^3}}\right| = \frac{2 - \frac{\ln(n)}{n^3}}{(1-\frac{1}{n})^2 + \frac{1}{n^3}} < \frac{2}{(1-\frac{1}{n})^2}$. If we show that $\frac{2n^3 - \ln(n)}{n(n-1)^2+1} > 2$ (but that takes a bit more work, which I don't include here) then we will have $\left|\frac{2n^3 - \ln(n)}{n(n-1)^2+1} - 2\right| < \frac{2}{(1-\frac{1}{n})^2} - 2 = \frac{2 - 2 + \frac{4}{n} - \frac{2}{n^2}}{(1-\frac{1}{n})^2} = \frac{\frac{4}{n} - \frac{2}{n^2}}{(1-\frac{1}{n})^2} > \frac{\frac{4}{n} - \frac{4}{n^2}}{(1-\frac{1}{n})^2} = \frac{\frac{4}{n}(1-\frac{1}{n})}{(1-\frac{1}{n})^2} = \frac{4}{n(1-\frac{1}{n})} = \frac{4}{n-1}$. We want to make this $< \epsilon$, that is, $n - 1 > \frac{4}{\epsilon}$, that is $n > \frac{4}{\epsilon} + 1$, so choose $N = [\frac{4}{\epsilon}] + 2$. The computations can be reversed, so they show that if $n \geq N$ then $\left|\frac{2n^3 - \ln(n)}{n(n-1)^2+1} - 2\right| < \epsilon$.]

[Yet another approach is to write $\left|\frac{2n^3 - \ln(n)}{n(n-1)^2+1} - 2\right| = \left|\left(\frac{2n^3}{n(n-1)^2+1} - 2\right) - \frac{\ln(n)}{n(n-1)^2+1}\right| \leq \left|\left(\frac{2n^3}{n(n-1)^2+1} - 2\right)\right| + \left|\frac{\ln(n)}{n(n-1)^2+1}\right|$ (by the triangle inequality) and then, for each of the two terms, find an "N" after which it is $< \frac{\epsilon}{2}$ and take the larger of the two "N"s to make the sum $< \epsilon$. But I won't include the details.]

What all this shows is that the sequence $\frac{2n^3 - \ln n}{n(n-1)^2+1}$ converges to 2.

(b) [We use the fact that n grows faster than $\ln(n)$. We'll split the n in two: one piece to take care of the $\ln(n)$, the other to go off to $-\infty$.] Consider $n - \ln(n) = \frac{n}{2} + (\frac{n}{2} - \ln(n))$. Note that the function $\frac{x}{2} - \ln x$ is > 0 at $x = 2$ and has positive derivative from then on, so is strictly increasing. Therefore, $n - \ln(n) \geq \frac{n}{2}$ for $n \geq 2$. We want $n - \ln(n) > -K$, so take $N = -2K + 1$ (or $N = 1$ if $K \geq 0$). We check that works.

So suppose $n \geq -2K + 1$. Then $\ln(n) - n \leq -\frac{n}{2} \leq K - \frac{1}{2} < K$, as required.

Since K was arbitrary, that proves that the sequence $\ln(n) - n$ tends to $-\infty$ as $n \to \infty$.

Question 3: Find the limits of the following sequences.

(a) $\left(\frac{3^n - n^4}{2^n + n!}\right)_{n \in \mathbb{N}}$

(b) $\left((n + n^{\frac{1}{2}})^{\frac{1}{2}} - n^{\frac{1}{2}}\right)_{n \in \mathbb{N}}$

Solutions:

(a) Among the functions appearing, that with the highest order of growth is $n!$ [p.31, 4.1.4], so divide throughout by it, to get $\dfrac{3^n - n^4}{2^n + n!} = \dfrac{\frac{3^n}{n!} - \frac{n^4}{n!}}{\frac{2^n}{n!} + 1}$. As $n \to \infty$, all terms apart from the 1 go to 0 so, by the Algebra of Limits [3.2.1], the limit of the above as $n \to \infty$ is $\dfrac{0-0}{0+1} = 0$.

(b) $(n+n^{\frac{1}{2}})^{\frac{1}{2}} - n^{\frac{1}{2}} = \left((n+n^{\frac{1}{2}})^{\frac{1}{2}} - n^{\frac{1}{2}}\right) \times \dfrac{(n+n^{\frac{1}{2}})^{\frac{1}{2}} + n^{\frac{1}{2}}}{(n+n^{\frac{1}{2}})^{\frac{1}{2}} + n^{\frac{1}{2}}} = \dfrac{n + n^{\frac{1}{2}} - n}{(n+n^{\frac{1}{2}})^{\frac{1}{2}} + n^{\frac{1}{2}}} = \dfrac{n^{\frac{1}{2}}}{(n+n^{\frac{1}{2}})^{\frac{1}{2}} + n^{\frac{1}{2}}} = \dfrac{1}{(1+n^{\frac{-1}{2}})^{\frac{1}{2}} + 1}$ which, by the Algebra of Limits, goes to $\dfrac{1}{(1+0)^{\frac{1}{2}} + 1} = \dfrac{1}{2}$ as $n \to \infty$.

Question 4: Using L'Hôpital's Rule or otherwise, find

(a) $\lim_{n\to\infty} \dfrac{\ln(4n^{\frac{1}{3}} - 2)}{\ln(n+1)}$ and

(b) $\lim_{n\to\infty} \dfrac{\ln(e^{e^n} - n^6)}{n! - n^{10}}$.

Solutions:
(a) Both top and bottom lines go to ∞ as $n \to \infty$ (and both functions are, at least for $x \geq 1$, differentiable with nonzero derivative) so we can apply L'Hôpital to get $\lim_{n\to\infty} \dfrac{\ln(4n^{\frac{1}{3}} - 2)}{\ln(n+1)} = \lim_{n\to\infty} \dfrac{1}{4n^{\frac{1}{3}} - 2} \cdot \frac{4}{3} n^{-2/3} \cdot \dfrac{n+1}{1} = \lim_{n\to\infty} \dfrac{4}{3} \cdot \dfrac{n+1}{4n - 2n^{-2/3}} = \lim_{n\to\infty} \dfrac{4}{3} \cdot \dfrac{1 + \frac{1}{n}}{4 - 2n^{-5/3}} = \dfrac{4}{3} \cdot \dfrac{1}{4} = \dfrac{1}{3}$.
(b) If you try to use L'Hopital then you have the problem of trying to differentiate $n!$, which is not defined at non-integer values, so continuity does not even make sense [there are continuous interpolations but that's getting unnecessarily complicated]. So we should proceed "otherwise". Note that $\dfrac{\ln(e^{e^n} - n^6)}{n! - n^{10}} \leq \dfrac{\ln(e^{e^n})}{n! - n^{10}} = \dfrac{e^n}{n! - n^{10}}$ and $n!$ is the fastest-growing of these functions, so divide throughout by it to get $\dfrac{\frac{e^n}{n!}}{1 - \frac{n^{10}}{n!}} \to \dfrac{0}{1+0} = 0$ by the Algebra of Limits [3.2.1] so, by the Sandwich Theorem [3.1.1/3.1.4] (the original function is sandwiched between 0 and this function which has limit 0) the original limit is 0.

Question 5: Determine whether the following series converge. In each case you should briefly justify your answer (for example by saying what test you are using).

(a) $\displaystyle\sum_{n=1}^{\infty} n^{10} e^{-n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{e^n}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{e^{n^2}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^3}{3^n}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{3}{2}}}$ (f) $\displaystyle\sum_{n=1}^{\infty} \tan(\frac{\pi}{2} - \frac{1}{n})$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^4}{n(n+e)^2}$ (h) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ (i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Solutions: In some cases, other methods will also work.

(a) Let's try the ratio test [9.1.7], so $a_n = \frac{n^{10}}{e^n}$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{10}}{e^{n+1}} \cdot \frac{e^n}{n^{10}} = \left(1+\frac{1}{n}\right)^{10} \cdot \frac{1}{e} \to \frac{1}{e}$ as $n \to \infty$. Since $\frac{1}{e} < 1$ we conclude, by the ratio test, that the series is convergent.

[Note that saying that e^n grows faster than n^{10} is not enough - all that tells you is that the individual terms of the series tend to 0 as $n \to \infty$; the question here is what happens when you add them together. This is a question about a *series* rather than a *sequence*.]

(b) You could apply the ratio test but you might also notice from the outset that $\frac{n^n}{e^n} = \left(\frac{n}{e}\right)^n$ and, for $n \geq 3$, $\frac{n}{e} > 1$, so the sequence of terms $\left(\frac{n}{e}\right)^n$ does not go to 0 as $n \to \infty$ [this also follows from the table on p.31]. Hence [by 8.1.4] the series is divergent.

(c) Again the ratio test will work: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{e^{n+1}}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{e^n} = \cdots = e^{-2n}$ which tends to $0 < 1$ as $n \to \infty$, so the series is convergent.

(d) And, yet again, the ratio test does the job: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{2^n n^3} = \frac{2}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{2}{3} < 1$ as $n \to \infty$, so the series is convergent.

(e) You can probably see in advance that the ratio test will give the value 1, hence no conclusion can be drawn from it. But it is an alternating series with the nth term a_n going to 0 as $n \to \infty$ so, it is convergent (in fact, it's absolutely convergent by [9.2.3], that is, the version with all terms positive is also convergent since the power $\frac{3}{2}$ is < 1.)

(f) This is divergent because the nth terms don't converge to 0 - in fact they go off to ∞ (think of the graph of tan).

(g) The nth term is $\frac{3n}{(1+\frac{e}{n})^2}$ which does not tend to 0 as $n \to \infty$ (in fact it goes to ∞ as $n \to \infty$) so this diverges.

(h) The individual terms converge to 0 and the series is alternating, so this converges by the alternating series test [10.1.1]. [The original question had the series going from $n = 1$ but we should start at $n = 2$ since $\ln(1) = 0$.]

(i) This looks amenable to the integral test; let's check. The function is $f(x) = \frac{1}{x(\ln x)^2}$ and we're looking at in on the interval $[2, \infty)$ where it is continuous, positive and decreasing [the original version of the question had the sum going from $n = 1$ but that doesn't make sense since the term is not defined at $n = 1$, hence the change to the sum from $n = 2$]. So we look at $\int_2^\infty \frac{1}{x(\ln x)^2}\, dx$ to determine whether or not it exists; that is, whether or not $\lim_{K\to\infty} \int_2^K \frac{1}{x(\ln x)^2}\, dx$ exists.

We have (think of the substitution $u = \ln(n)$) $\int_2^K \frac{1}{x(\ln(x))^2}\, dx = \left[-\frac{1}{\ln(x)}\right]_2^K = -\frac{1}{\ln(K)} + \frac{1}{\ln(2)}$ which tends to $\frac{1}{\ln(2)}$ as $n \to \infty$. Thus, the integral converges and hence [9.2.1] so does the infinite series.

Question 6: (a) Using partial fractions or otherwise, find $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2-1}$.

(b) Show that the following series converge and show (use partial fractions) that they have the same sum.

$$\sum_{n=1}^{\infty}\frac{1}{2n(2n+1)} \quad \text{and} \quad \sum_{n=2}^{\infty}\frac{(-1)^n}{n}$$

Solutions:

(a) $\dfrac{1}{n^2-1}=\dfrac{1}{2}\left(\dfrac{1}{n-1}-\dfrac{1}{n+1}\right)$, so the Nth partial sum $s_N=\displaystyle\sum_{n=2}^{N}\frac{1}{n^2-1}=\frac{1}{2}\left(\sum_{n=2}^{N}\frac{1}{n-1}-\sum_{n=2}^{N}\frac{1}{n+1}\right)$: $\dfrac{1}{2}\left(\displaystyle\sum_{n=1}^{N-1}-\frac{1}{n}-\sum_{n=3}^{N+1}\frac{1}{n}\right)=\frac{1}{2}(1+\frac{1}{2}-\frac{1}{N}\frac{1}{N+1})$ and this tends to $\dfrac{3}{4}$ as $N\to\infty$, so $\displaystyle\sum_{n=2}^{N}\frac{1}{n^2-1}=\lim_{N\to\infty}s_N=\frac{3}{4}$.

[The following argument would get some of the marks but not all of them:

$$s_N=\sum_{n=2}^{\infty}\frac{1}{n^2-1}=\frac{1}{2}\left(\sum_{n=2}^{\infty}\frac{1}{n-1}-\sum_{n=2}^{\infty}\frac{1}{n+1}\right)=\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n}-\sum_{n=3}^{\infty}\frac{1}{n}\right)=\frac{1}{2}(1+\frac{1}{2})=\frac{3}{4}.$$

The problem with it is that, at the second "=" infinitely many terms of the series have been rearranged and we saw, in Section 12.2, that can lead to nonsense. If you had justified that step by saying that the series was absolutely convergent by comparison with $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2}$ and hence rearranging infinitely many terms is justified, then you would get the marks. But, if you go that route, rather than the first, recommended, route, make sure that your justification is valid and that you say enough.]

(b) The first series converges by comparison with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ [more precisely with $\displaystyle\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$] since $\dfrac{1}{2n(2n+1)}\le\dfrac{1}{4n^2}$. The second series converges since it is alternating and its terms have limit 0 [10.1.1].

Also, $\dfrac{1}{2n(2n+1)}=\dfrac{1}{2n}-\dfrac{1}{2n+1}$, so the Nth partial sum of the first series $s_N=\displaystyle\sum_{n=1}^{N}\frac{1}{2n(2n+1)}=\sum_{n=1}^{N}\frac{1}{2n}-\sum_{n=1}^{N}\frac{1}{2n+1}=\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\cdots-\frac{1}{2N+1}$ and that is the $(2N+1)$th partial sum, t_{2N+1} say, of the series $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n}$. So $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n(2n+1)}=\lim_{N\to\infty}s_N=\lim_{N\to\infty}t_{2N+1}=\lim_{2N+1\to\infty}t_{2N+1}=\sum_{n=2}^{\infty}\frac{(-1)^n}{n}$.

[In this case, just writing everything in terms of the infinite series $\displaystyle\sum_{n=2}^{\infty}$ and not going *via* partial sums would get comparatively fewer of the marks than doing the same in part (a) because it is definitely needed here - one of the series involved is not absolutely convergent, so rearranging infinitely many terms is not *a priori* valid. I've given full details of the argument; you could get full marks with a bit less, as long as you make it clear that you are considering the (finite) partial sums.]

Question 7: (a) Define what it means for a sequence $(a_n)_{n\in\mathbb{N}}$ to (i) converge to a limit ℓ, (ii) tends to ∞ as $n \to \infty$ [the original wording said "converge to ∞" but it's not a good idea to use the word "converge" next to a divergent series].

(b) Given a sequence $(a_n)_{n\in\mathbb{N}}$, define a new sequence $(a_n^*)_{n\in\mathbb{N}}$ by $a_n^* = \frac{1}{2}(a_n + a_{n+1})$. Prove direct from your definitions above that (i) if $a_n \to \ell$ as $n \to \infty$ then $a_n^* \to \ell$ as $n \to \infty$, (ii) if $a_n \to \infty$ as $n \to \infty$ then $a_n^* \to \infty$ as $n \to \infty$.

(c) Show, by producing suitable examples, that the converse of each of (b)(i) and (b)(ii) is false.

Solutions:

(a) (i) The sequence $(a_n)_{n\in\mathbb{N}}$ converges to ℓ if, given any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $\mid a_n - \ell \mid < \epsilon$ for all $n \geq N$. (ii) The sequence $(a_n)_{n\in\mathbb{N}}$ tends to ∞ if, given any $K \in \mathbb{R}$, there is $N \in \mathbb{N}$ such that $a_n > K$ for all $n \geq N$.

(b) (i) Given $\epsilon > 0$, choose N such that, for all $n \geq N$, we have $\mid a_n - \ell \mid < \epsilon$. Then, for $n \geq N$, we have $\mid a_n^* - \ell \mid = \mid \frac{1}{2}(a_n + a_{n+1}) - \ell \mid \leq \mid (\frac{1}{2}a_n - \frac{1}{2}\ell) + (\frac{1}{2}a_{n+1} - \frac{1}{2}\ell) \mid \leq \mid (\frac{1}{2}a_n - \frac{1}{2}\ell) \mid + \mid (\frac{1}{2}a_{n+1} - \frac{1}{2}\ell) \mid = \frac{1}{2} \mid (a_n - \ell) \mid + \frac{1}{2} \mid (a_{n+1} - \ell) \mid < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$ since both $n, n+1 \geq N$. As required.

(b) (ii) Given $K \in \mathbb{R}$, choose N such that, for all $n \geq N$, we have $a_n > K$. Let $n \geq N$; then $a_n^* = \frac{1}{2}a_n + \frac{1}{2}a_{n+1} > \frac{1}{2}K + \frac{1}{2}K = K$ since both $n, n+1 \geq N$. As required.

(c) For a counterexample to the converse of (i) you could take $a_n = (-1)^n$. This does not converge to a limit but the sequence a_n^* is the constant sequence 0, so converges. For a counterexample to the converse of (ii), we could take the sequence $a_n = n + (-1)^n n$ which does not tend to ∞ whereas $a_n^* = 2n$ does tend to ∞ as $n \to \infty$.

Question 8: Let b be a positive real number and define the sequence $(a_n)_{n\in\mathbb{N}}$ inductively by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \frac{a_n}{a_n + b} \text{ for } n \geq 1.$$

(a) Prove by induction on n that $a_n > 0$ for all n.

(b) Prove that if $0 < b < 1$ then $a_n > 1 - b$ for all n.

(c) Deduce that, if $b > 0$, then the sequence $(a_n)_{n\in\mathbb{N}}$ is a decreasing sequence and, by quoting a suitable theorem, deduce that it converges.

(d) Prove that if $0 < b < 1$ then $a_n \to 1 - b$ as $n \to \infty$.

(e) Calculate $\lim_{n\to\infty} a_n$ in the case that $b \geq 1$.

Solutions:

(a) Certainly the statement $a_n > 0$ is true for $n = 1$, so assume it is true for some value k. Then $a_{k+1} = \dfrac{a_k}{a_k + b}$ which is positive since both a_k and b are (a_k by the inductive hypothesis).

(b) Assume $0 < b < 1$. Then $a_1 = 1 > 1 - b$, so we have the case $n = 1$. Assume that $a_k > 1 - b$. Then $a_{k+1} > 1 - b$ iff $\dfrac{a_k}{a_k + b} > 1 - b$ iff $a_k > (1-b)(a_k + b) = a_k - a_k b + b - b^2 = (a_k + b) - (a_k + b)b = (1 - b)(a_k + b)$. But, by the inductive hypothesis $a_k + b > 1$, so $(1 - b)(a_k + b) > 1 - b$. Thus the condition has become $a_k > 1 - b$, which is true by the inductive hypothesis. So we do indeed have $a_{k+1} > 1 - b$ and we conclude by induction that $a_n > 1 - b$ for all n.

(c) First assume that $0 < b < 1$. Then we have $a_{n+1} = \dfrac{a_n}{a_n + b} < a_n$ since, by (b), $a_n + b > 1$. So the sequence $(a_n)_n$ is decreasing. But it is, by (a), bounded below (by

0), hence it converges by the Monotone Convergence Theorem [the decreasing sequence version of 2.5.3].

Otherwise $b \geq 1$, so we can't use part (b), but we do then have $a_n + b > 1$ since $a_n > 0$ by (a), so we still get $a_{n+1} < a_n$ and deduce convergence as above.

(d) By (c) the sequence converges to a limit, ℓ say, which must be ≥ 0 since $a_n > 0$ for all n. We have $\ell = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n}{a_n + b} = \frac{\ell}{\ell + b}$ by the Algebra of Limits. So we have $\ell^2 + b\ell = \ell$; rearrange to get $\ell(\ell - (1-b)) = 0$. This is where we have to use the assumption $0 < b < 1$ (so far, everything needed just $b > 0$); because then we have, by part (b), that $\ell > 1 - b > 0$, in particular $\ell \neq 0$ so we deduce $\ell = 1 - b$. That is, $a_n \to 1 - b$ as $n \to \infty$.

(e) Now assume $b \geq 1$. Then, at the point in (d) above where we had the equation $\ell(\ell - (1-b)) = 0$, we see that ℓ must equal 0, which is the limit of the a_n in this case.

Question 9:

(a) Find the radius of convergence for the series

(i) $\sum_{n=1}^{\infty} \frac{\sqrt{(2n)!}}{n!} x^n$ (ii) $\sum_{n=1}^{\infty} \frac{\sqrt{(2n)!}}{(n+1)!} x^n$

(b) Find the interval of convergence for the series

(i) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ (ii) $\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt{n}}$ (iii) $\sum_{n=1}^{\infty} \frac{(-x)^n}{7n-5}$

Solutions: Throughout, we use the Ratio Test to determine the radius of convergence and, in part (b), we look separately at each end-point of the interval of convergence.

(a)(i) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\sqrt{(2(n+1))!}}{(n+1)!} \cdot \frac{n!}{\sqrt{(2n)!}} \mid x \mid = \sqrt{\frac{(2n+2)!}{(2n)!}} \cdot \frac{n!}{(n+1)!} \mid x \mid = \sqrt{(2n+2)(2n+1)} \cdot \frac{1}{n+1} \mid x \mid = \sqrt{\frac{(2n+2)(2n+1)}{(n+1)^2}} \mid x \mid = \sqrt{\frac{(2+\frac{2}{n})(2+\frac{1}{n})}{(1+\frac{1}{n})^2}} \mid x \mid$. As $n \to \infty$, this converges to $\sqrt{4} \mid x \mid = 2 \mid x \mid$. This is < 1 when $\mid x \mid < \frac{1}{2}$, so $\frac{1}{2}$ is the radius of convergence.

(a)(ii) This is almost the same as (i); the only change in the calculation is that $\frac{(n+1)!}{(n+2)!} = \frac{1}{n+2}$ appears in place of $\frac{n!}{(n+1)!} = \frac{1}{n+1}$ and this does not affect the calculation of the limit, so we get the same radius of convergence, $\frac{1}{2}$.

(b)(i) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n} \mid x \mid$, which has limit $\mid x \mid$ as $n \to \infty$. Therefore the RoC is 1. We examine what happens at $x = \pm 1$.

At $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{1}{n}$ which we know is divergent [p.50, 9.2.2].

At $x = -1$ the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which convergent by the Alternating Series Test [10.1.1].

So the Interval of Convergence is $[-1, 1)$.

(b)(ii) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{2^n} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \mid x \mid = 2\sqrt{\frac{1}{1+\frac{1}{n}}} \mid x \mid \to 2 \mid x \mid$ as $n \to \infty$. So RoC$= \frac{1}{2}$.

At $x = \dfrac{1}{2}$ the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which, being an alternating series with terms going to 0, is convergent.

At $x = \dfrac{-1}{2}$ the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n(-1)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is divergent [9.1.6/9.2.4].

So the IoC is $(-1/2, 1/2]$.

(b)(iii) $\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \frac{7n-5}{7(n+1)-5} \mid x \mid$ which tends to $\mid x \mid$ as $n \to \infty$. So RoC$= 1$.

At $x = 1$ the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{7n-5}$ which is convergent by the alternating series test.

At $x = -1$ the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{1}{7n-5}$ which is divergent by comparison [9.1.2] with the divergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ (or, more precisely, by comparison with $\displaystyle\frac{1}{7}\sum_{n=1}^{\infty} \frac{1}{n}$).