MATH10242 Sequences and Series: Exercises for Week 11 Tutorials, Solutions

Question 1: Find the radius of convergence R of the following power series.
In parts (i) and (ii), what is the *interval of convergence* of the given power series?

$$(i)\sum_{n\geq 1}\frac{x^n}{8^n}\qquad (ii)\sum_{n\geq 1}\frac{(-x)^n}{4n+1},\qquad (iii)\sum_{n\geq 1}\frac{(2n)!}{(n!)^2}x^n,\qquad (iv)\sum_{n\geq 1}\frac{n^n}{n!}x^n,$$

$$(v)\sum_{n\geq 1}n!\cdot x^n\qquad (vi)\sum_{n\geq 1}\frac{\sqrt{(2n)!}}{n!}x^n.$$

[You will need to use the formula for $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ from a previous Exercise Sheet.]

Solutions: (i) $\sum_{n\geq 1} c_n = \sum_{n\geq 1}\frac{x^n}{8^n}$. Here the (Modified) Ratio Test gives

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{x^{n+1}}{x^n}\frac{8^n}{8^{n+1}}\right| = \frac{|x|}{8}.$$

Thus, (the limit as $n\to\infty$ of) $\left|\frac{c_{n+1}}{c_n}\right| < 1 \iff |x| < 8$ (in which case it converges absolutely) and (the limit as $n\to\infty$ of) $\left|\frac{c_{n+1}}{c_n}\right| > 1 \iff |x| > 8$ (in which case it diverges).
Thus, the radius of convergence is $R = 8$.
When $x = 8$, respectively $x = -8$ the series becomes $\sum 1$, respectively $\sum(-1)^n$ and clearly neither of these converges (use the nth term test). Thus the interval of convergence is $(-8, 8)$.

(ii) $\sum_{n\geq 1} c_n = \sum_{n\geq 1}\frac{(-x)^n}{4n+1}$ Here the (Modified) Ratio Test gives

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{x^{n+1}}{x^n}\frac{4n+1}{4(n+1)+1}\right| = |x|\frac{4n+1}{4n+7} = |x|\frac{\left(1+\frac{1}{4n}\right)}{\left(1+\frac{7}{4n}\right)} \to |x| \quad as\ n\to\infty.$$

Thus, $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| < 1 \iff |x| < 1$ (in which case it converges absolutely) and $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| > 1 \iff |x| > 1$ (in which case it diverges).
Thus, the radius of convergence is $R = 1$.
When $x = 1$ the series $\sum_{n\geq 1} c_n = \sum_{n\geq 1}\frac{(-x)^n}{4n+1}$ becomes $\sum_{n\geq 1}\frac{(-1)^n}{4n+1}$, which converges by the Alternating Series Test.
When $x = -1$ the series $\sum_{n\geq 1} c_n = \sum_{n\geq 1}\frac{(-x)^n}{4n+1}$ becomes $\sum_{n\geq 1}\frac{(+1)}{4n+1}$, which diverges, for example by comparison with $\sum\frac{1}{n}$. Thus the interval of convergence is $(-1, 1]$.

(iii) Here $\sum c_n = \sum\frac{(2n)!}{(n!)^2}x^n$. Thus

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{x^{n+1}}{x^n}\frac{(2n+2)!}{(2n)!}\right|\cdot\left|\frac{(n!)^2}{((n+1)!)^2}\right| = |x|\frac{(2n+1)(2n+2)}{(n+1)^2} = |x|\frac{4+\frac{6}{n}+\frac{2}{n^2}}{1+\frac{2}{n}+\frac{1}{n^2}} \to 4|x|,$$

as $n \to \infty$.

Thus, $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| < 1 \iff |x| < \frac{1}{4}$ (in which case it converges absolutely) and $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| > 1 \iff |x| > \frac{1}{4}$ (in which case it diverges). Thus, the radius of convergence is $R = \frac{1}{4}$.

(iv) Here

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{x^{n+1}}{x^n}\frac{(n+1)^{n+1}n!}{n^n(n+1)!}\right| = |x|\frac{(n+1)(n+1)^n}{n^n(n+1)} = |x|\frac{(n+1)^n}{n^n} = |x|\Big(1+\frac{1}{n}\Big)^n \to |x|{\cdot}e,$$

as $n \to \infty$.

Thus, by the Ratio Test, again, the radius of convergence is $R = e^{-1}$.

(v) Here

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{x^{n+1}}{x^n}\right|\frac{(n+1)!}{n!} = (n+1)|x| \to \infty, \quad as\ n \to \infty.$$

So, by the Ratio Test there is no value of x for which the series converges (except of course for $x = 0$) and so the Radius of Convergence is $R = 0$.

(vi) $\displaystyle\sum_{n\geq 1} c_n = \sum_{n\geq 1}\frac{\sqrt{(2n)!}}{n!}x^n$. Thus

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{\sqrt{(2n+2)!}}{(n+1)!}x^{n+1}\cdot\frac{n!}{x^n\sqrt{(2n)!}}\right| = \left|\frac{\sqrt{(2n+2)(2n+1)(2n)!}}{(n+1)!}x^{n+1}\cdot\frac{n!}{x^n\sqrt{(2n)!}}\right|$$

$$= \frac{\sqrt{(2n+2)(2n+1)}}{(n+1)}\cdot|x| = \frac{\sqrt{(2+2/n)(2+1/n)}}{(1+1/n)}\cdot|x| \to 2|x| \quad as \quad n \to \infty.$$

Thus the radius of convergence is $R = \frac{1}{2}$.

Question 2: Let $r > 0$. Using Question 1(i) as a guide, find a series $\sum_{n=1}^{\infty} a_n x^n$ with radius of convergence r.

Solution: Replacing 8 by r in Question 1(i) we should guess that the series we want is $\displaystyle\sum_{n\geq 1}\frac{x^n}{r^n}$.

And, this does indeed work since using the Ratio Test again we see that:

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{x^{n+1}}{x^n}\frac{r^n}{r^{n+1}}\right| = \frac{|x|}{r}.$$

Thus, $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| < 1 \iff |x| < r$ (in which case it converges absolutely) and $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| > 1 \iff |x| > r$ (in which case it diverges).

Thus, the radius of convergence is $R = r$.

Question 3: Let $\sum_{n\geq 1} a_n$ be a series. We define two new series $\sum_{n\geq 1} a_n^+$, consisting of all the positive terms of the original series and and $\sum_{n\geq 1} a_n^-$, consisting of all the negative terms. To be specific, set

$$a_n^+ = \frac{a_n + |a_n|}{2} \qquad and \qquad a_n^- = \frac{a_n - |a_n|}{2},$$

and notice that if $a_n > 0$ then $a_n^+ = a_n$ and $a_n^- = 0$. Conversely, if $a_n < 0$ then $a_n^- = a_n$ and $a_n^+ = 0$.

(a) Prove that, if $\sum_{n\geq 1} a_n$ is absolutely convergent, then both $\sum_{n\geq 1} a_n^+$ and $\sum_{n\geq 1} a_n^-$ are convergent. Moreover, prove that

$$\sum_{n\geq 1} a_n = \sum_{n\geq 1} a_n^+ + \sum_{n\geq 1} a_n^-.$$

(b*) Prove that, if $\sum_{n\geq 1} a_n$ is only conditionally convergent, then both $\sum_{n\geq 1} a_n^+$ and $\sum_{n\geq 1} a_n^-$ are divergent.

Solution: We always use the description $a_n^+ = \frac{a_n+|a_n|}{2}$ and $a_n^- = \frac{a_n-|a_n|}{2}$,

(a) (This is really the same as the proof of Theorem 10.2.2, but let's prove it directly.) Suppose first that $\sum a_n$ is absolutely convergent; thus $\sum |a_n|$ converges. Hence by the Algebra of Infinite Sums (Theorem 8.1.5),

$$\frac{1}{2}\sum_{n\geq 1} a_n + |a_n|$$

also converges. By definition this is exactly $\sum_{n\geq 1} a_n^+$. The argument for $\sum_{n\geq 1} a_n^-$ is similar.

Finally, as $a_n = a_n^+ + a_n^-$ for all n, Theorem 8.1.5 then implies that

$$\sum_{n\geq 1} a_n = \sum_{n\geq 1} a_n^+ + \sum_{n\geq 1} a_n^-.$$

(b) Now suppose that $\sum_{n\geq 1} a_n$ is only conditionally convergent; thus $\sum |a_n|$ diverges. In this case, $\sum_{n\geq 1} \frac{1}{2}a_n$ is still convergent while $\sum \frac{1}{2}|a_n|$ diverges. Thus by Question 3(a) on the Week 9 Exercises Sheet,

$$\sum_{n\geq 1} \frac{1}{2}a_n + \frac{1}{2}|a_n|$$

also diverges. By definition this is exactly $\sum_{n\geq 1} a_n^+$. The argument for $\sum_{n\geq 1} a_n^-$ is similar.