MATH10242 Sequences and Series: Exercises for Week 10 Tutorials, Solutions

Question 1: Use the Integral Test to test the series below for convergence or divergence. In (i) (ii) and (iii) is there some other test that would also work?

$$(i)\quad \sum_{n\geq 1}\frac{1}{n^2+1},\qquad (ii)\quad \sum_{n\geq 1}\frac{n}{n^2+1},\qquad (iii)\quad \sum_{n\geq 1} n^2e^{-n},\qquad (iv)\quad \sum_{n\geq 2}\frac{1}{n(\ln n)^p},\ \text{for } p>1.$$

Solutions: In each case we are clearly given $\sum a_n$ where $a_n = f(n)$ for a function $f(x)$ that is positive and continuous. In some cases it is not so clear whether it is decreasing, so one needs to say something about that. However, what is true in each case is that for some K the relevant function $f(x)$ is decreasing for $x \geq K$, and by comments in the notes, this suffices.

So modulo this preamble, the Integral Test does indeed apply.

(i) Here we consider $f(x) = \frac{1}{x^2+1}$ which is certainly decreasing (and positive and continuous). Now $\int_1^N \frac{1}{x^2+1}dx = \tan^{-1}(x)\big|_1^N$ which is bounded above by $\pi/2 - \tan^{-1}(1)$. So the series converges. (Of course, it would be easier to use the comparison test with $\sum n^{-2}$.)

(ii) Here use $\int_1^N \frac{x}{x^2+1}dx = \frac{1}{2}\ln(x^2+1)\big|_1^N$. Now as $\ln(N^2+1)\to\infty$ as $N\to\infty$ the series diverges. (Here one could also use the comparison test with $\sum n^{-1}$.)

(iii) Here we use $f(x) = x^2e^{-x}$ for which we do have to think a little about whether it is eventually decreasing. The best way to do this is to find the max and min using $f'(x) = (2x - x^2)e^{-x}$ to see that it has a local maximum only at $x = 2$. Thus it decreases for $x \geq 2$.

Integrating by parts twice gives $\int_1^N x^2e^{-x}dx = -(x^2+2x+2)e^{-x}\big|_1^N \to 0 + (1+2+2)e^{-1} = 5e^{-1}$ as $N\to\infty$ (by using results from Chapter 3). So the series converges. (Here the Ratio Test would also work and is surely easier.)

(iv) Here $\int_2^N \frac{1}{x(\ln x)^p}dx = \frac{1}{-p+1}\ln(x)^{-p+1}\Big|_2^N$. Since $p > 1$, we have $-p+1 < 0$ and so $\ln(N)^{-p+1} \to 0$ as $N\to\infty$. So the improper integral is indeed finite and the series converges.

Question 2: Test the series below for convergence or divergence

(i) $\displaystyle\sum_{n\geq 1}\sin\left(\frac{1}{n^2}\right)$ [Hint: first show that $\sin(x) < x$ for all $x > 0$.]

(ii) $\displaystyle\sum_{n\geq 1}\tan\left(\frac{1}{n^2}\right)$ (iii) $\displaystyle\sum_{n\geq 1}\cos\left(\frac{1}{n^2}\right)$ (iv) $\displaystyle\sum_{n\geq 1}\frac{n!}{(n+1)!}$ (v) $\displaystyle\sum_{n\geq 1}\frac{n!}{(n+2)!}$

(vi) $\displaystyle\sum_{n\geq 1} n^{-2}\cos(1/n)e^{\sin(1/n)}$ (vii) $\displaystyle\sum_{n\geq 2} n^3e^{-n^4}$ (viii) $\sum_{n\geq 2}\frac{1}{n(\ln n)}$ (ix) $\displaystyle\sum_{n\geq 2}\frac{1+\ln(n)}{n(\ln n)^2}$

Solutions: (i) To check that $\sin(x) < x$ set $y = \sin(x) - x$. Then $y' = \cos(x) - 1 \leq 0$ for all x and so y is a decreasing function. Since $y(0) = 0$ this implies that $y(x) \leq 0$ for

$x > 0$. (With a little more work one can see that y is strictly decreasing for $0 < x < \pi$ and hence $y(x) < 0$ for all $x > 0$, but this is not necessary.)

So, we can compare $0 \le \sin(\frac{1}{n^2}) \le \frac{1}{n^2}$. (Note that the terms really are all positive since $\frac{1}{n^2} < \pi$). Since $\sum \frac{1}{n^2}$ converges, so does $\sum_{n\ge1} \sin(\frac{1}{n^2})$.

(ii) $\tan(\frac{1}{n^2}) = \dfrac{\sin(\frac{1}{n^2})}{\cos(\frac{1}{n^2})}$. But since $\frac{1}{n^2} < \pi/3$, we have $\cos(\frac{\pi}{3}) < \cos(\frac{1}{n^2}) < 1$ for all $n \ge 1$. So

$$0 < \tan(\frac{1}{n^2}) \le \frac{\sin(\frac{1}{n^2})}{\cos(\frac{\pi}{3})} \le \frac{1}{\cos(\frac{\pi}{3})}\,\frac{1}{n^2}.$$

Since $\sum_{n\ge1} \frac{1}{\cos(\frac{\pi}{3})}\,\frac{1}{n^2}$ converges by the AoL and a standard example (specifically 8.1.5 and 9.1.5), $\sum_{n\ge1} \tan(\frac{1}{n^2})$ converges by the Comparison Test.

(iii) Since $\lim_{n\to\infty} \cos(\frac{1}{n^2}) = 1$, $\sum_{n\ge1} \cos(\frac{1}{n^2})$ diverges (by the nth term test 8.1.4).

(iv) $\sum_{n\ge1} \frac{n!}{(n+1)!} = \sum_{n\ge1} \frac{1}{(n+1)}$ which diverges.

(v) $\displaystyle\sum_{n\ge1} \frac{n!}{(n+2)!} = \sum_{n\ge1} \frac{1}{(n+1)(n+2)} = \sum_{n\ge1} \frac{1}{(n^2+3n+2)} \le \sum_{n\ge1} \frac{1}{n^2}$. Since $\displaystyle\sum_{n\ge1} \frac{1}{n^2}$ converges, so does $\displaystyle\sum_{n\ge1} \frac{n!}{(n+2)!}$ by the comparison test.

(vi) Here one should note that $\int_1^N x^{-2}\cos(1/x)e^{\sin(1/x)}dx = -e^{\sin(1/n)}\Big|_1^N$. As $n \to \infty$, clearly $e^{\sin(1/n)} \to e^{\sin(0)} = 1 < \infty$. So the sum converges by the integral test.

As usual, one should also check that the terms $a_n = n^{-2}\cos(1/n)e^{\sin(1/n)}$ are positive—which is clear—and that it is decreasing for large n—which is a little messier, but easy enough. As often happens you could also use the Comparison test with $\sum n^{-2}$.

(vii) Here we can use the integral test (and the 3 conditions are obviously satisfied for large x. Thus:

$$\int_{n=2}^{N} x^3 e^{-x^4} = \frac{-1}{4}e^{-x^4}\Big|_2^N = \frac{1}{4}(e^{-2^4} - e^{-N^4}) \to \frac{1}{4}e^{-2^4} < \infty,$$

as $n \to \infty$. So, our series also converges. (The ratio test would also work,)

(viii) In this case it is easy to check that the function $\frac{1}{x\ln(x)}$ is positive and decreasing. Moreover, $\int_2^N \frac{1}{x\ln(x)}dx = \ln(\ln(x))\Big|_2^N \to \infty$ as $N \to \infty$. So, it diverges by the integral test.

Remark This result should be compared with Question 1(iv); the "dividing line" between convergent and divergent series really is quite subtle!

(ix) This is slightly less obvious. Here

$$\sum_{n\ge2} \frac{1+\ln(n)}{n(\ln n)^2} = \sum_{n\ge2} \frac{1}{n(\ln n)^2} + \sum_{n\ge2} \frac{1}{n(\ln n).}$$

Thus we get the sum of a convergent and a divergent series and so by Question 3 from the previous exercise sheet, it is divergent. (Alternatively, you could just compare it to $\displaystyle\sum_{n\ge2} \frac{1}{n(\ln n).}$)

Question 3: Test the series below for convergence and for absolute convergence. Which are conditionally convergent?

For this question, try first just writing down the answer with only a brief reason why it is true—for example if one had the series $\sum \frac{n^2+1}{n^4+3n^2}$ you might write "absolutely convergent and hence convergent by comparison with $\sum \frac{1}{n^2}$". Then you can check some or all of them by doing the details.

$$(i)\ \sum_{n\geq 1}(-1)^n\left(\frac{n+1}{n+2}\right), \qquad (ii)\ \sum_{n\geq 1}(-1)^n\left(\frac{n+1}{n^2+2}\right), \qquad (iii)\ \sum_{n\geq 1}(-1)^n\left(\frac{n+1}{n^3+1}\right)$$

$$(iv) \sum_{n\geq 1}(-1)^n\frac{\cos(n)}{n^2} \qquad (iv)\ \sum_{n\geq 1}\frac{1}{(-2)^n}.$$

Solutions: (i) $\displaystyle\sum_{n\geq 1}(-1)^n a_n = \sum_{n\geq 1}(-1)^n\left(\frac{n+1}{n+2}\right)$. Here $a_n \to 1$ as $n \to \infty$ so the series diverges, by Theorem 8.1.4.

(ii) $\displaystyle\sum_{n\geq 1}(-1)^n a_n = \sum_{n\geq 1}(-1)^n\left(\frac{n+1}{n^2+2}\right)$. Here $a_n \to 0$ as $n \to \infty$ and the a_n are positive, so the Alternating Series Test says it converges. However,

$$\sum_{n\geq 1}\left|(-1)^n\left(\frac{n+1}{n^2+2}\right)\right| = \sum_{n\geq 1}\frac{n+1}{n^2+2} = \sum_{n\geq 1}\frac{1+\frac{1}{n}}{n+\frac{2}{n}}$$

and it is easy to see that this diverges by comparison with $\sum \frac{1}{n}$. In more detail,

$$\frac{1+\frac{1}{n}}{n+\frac{2}{n}} \ \geq\ \frac{1}{n+\frac{2}{n}} \ \geq \frac{1}{n+2} \qquad \text{for } n \geq 1$$

Since $\sum_{n\geq 1}\dfrac{1}{n+2} = \sum_{n\geq 3}\dfrac{1}{n}$, it diverges (for example by 9.1.3). Thus our series $\displaystyle\sum_{n\geq 1}\left|(-1)^n\left(\frac{n+1}{n^2+2}\right)\right|$ diverges by the comparison test, and $\displaystyle\sum_{n\geq 1}(-1)^n\left(\frac{n+1}{n^2+2}\right)$ is conditionally convergent.

(iii) $\displaystyle\sum_{n\geq 1}|(-1)^n a_n| = \sum_{n\geq 1}\left|(-1)^n\frac{n+1}{n^3+1}\right| = \sum_{n\geq 1}\frac{1+\frac{1}{n}}{n^2+\frac{1}{n}}$, which converges by comparison with $\sum \frac{1}{n^2}$. Hence the series converges absolutely. (It is left to you to fill in the details of how one explicitly does the comparison.)

(iv) $\displaystyle\sum_{n\geq 1}(-1)^n a_n = \sum_{n\geq 1}(-1)^n\frac{\cos(n)}{n^2}$. Here the cosine terms will be a mess, but at least they have absolute value ≤ 1. Hence

$$|(-1)^n a_n| \ = \ \left|(-1)^n\frac{\cos(n)}{n^2}\right| \ \leq \ \frac{1}{n^2}.$$

Hence the series converges absolutely by comparison with $\sum \frac{1}{n^2}$.

(iv) $\displaystyle\sum_{n\geq 1}(-1)^n a_n = \sum_{n\geq 1}\frac{1}{(-2)^n} = \sum_{n\geq 1}(-1)^n\frac{1}{2^n}$. We already know that the geometric series $\sum_{n\geq 1}\dfrac{1}{2^n}$ converges. Hence our given series converges absolutely.