MATH10242 Sequences and Series: Exercises for Week 9 Tutorials

Attempt these questions before your tutorial in the week beginning 23rd March.
It is essential that you be able to do computational questions like 1 and 2 but the proofs for Questions 3 and 4 are not hard and could reasonably be asked in exams.

Question 1: Use partial fractions to find: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$.

Question 2: Test the series below for convergence or divergence using the tests indicated. For the Comparison Test you need to decide whether you are expecting convergence (in which case you need to find a convergent series $\sum b_n$ with $b_n \geq a_n$ for all n) or divergence (in which case you need to find a divergent series $\sum b_n$ with $b_n \leq a_n$ for all n).

(a) *Comparison Test*

$$(i)\ \sum_{n\geq 1} \frac{n+1}{n^2+2}, \quad (ii)\ \sum_{n\geq 1} \frac{3n^2+2}{n^4+4}, \quad (iii)\ \sum_{n\geq 1} \frac{1}{2^n+n^2}, \quad (iv)\ \sum_{n\geq 2} \frac{1}{\ln n}.$$

(b) Ratio Test

$$(v)\ \sum_{n\geq 1} \frac{n^3}{3^n}, \quad (vi)\ \sum_{n\geq 1} \frac{3^n}{n!}, \quad (vii)\ \sum_{n\geq 1} \frac{n^n}{n!}, \quad (viii)\ \sum_{n=1}^{\infty} \frac{n!}{n^n}, \quad (ix)\ \sum_{n=1}^{\infty} \frac{n+1}{n^2+2}.$$

[Remark: you may use that $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$.]

Question 3: (a) Prove that if $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges.

(b) Suppose that $\sum_{n=1}^{\infty} (a_n + b_n)$ converges and that $a_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbb{N}$. Prove that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge.

(c) In part (b), why did we need $a_n \geq 0$ and $b_n \geq 0$?

Question 4: (a) Prove Theorem 8.1.5(ii): Suppose that $\sum_{n=1}^{\infty} a_n = s$ and that λ is any real number. Prove that the series $\sum_{n=1}^{\infty} \lambda a_n$ converges with sum λs.

(b) Prove 9.1.3: Given $N \geq 1$ and a series $\sum_{n\geq 1} a_n$, then $\sum_{n\geq 1} a_n$ converges $\iff$ $\sum_{n\geq N} a_n$ converges.

Question 5*: (a) Suppose that $\{a_n, b_n : n \geq 1\}$ are all positive and that $\lim_{n\to\infty} \frac{a_n}{b_n} = \ell$ exists. Prove that if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

(b) What happens in (b) if we allow negative terms? [You might find this easier after next week's lectures.]