MATH10242 Sequences and Series: Exercises for Week 7 Tutorials

For your tutorial in the week beginning 9th March. You should attempt (at the very least!) Question 1 from this sheet.

Question 1: Do the following sequences converge/diverge/tend to infinity or tend to minus infinity?

(*These also appear in the course notes, at the end of Chapter 5.*)

(a) $\big(\cos(n\pi)\sqrt{n}\big)_{n\in\mathbb{N}}$ (b) $\big(\sin(n\pi)\sqrt{n}\big)_{n\in\mathbb{N}}$

c) $\left(\dfrac{\sqrt{n^2+2}}{\sqrt{n}}\right)_{n\in\mathbb{N}}$ (d) $\left(\dfrac{n^3+3^n}{n^2+2^n}\right)_{n\in\mathbb{N}}$

e) $\left(\dfrac{n^2+2^n}{n^3+3^n}\right)_{n\in\mathbb{N}}$ (f) $\left(\dfrac{1}{\sqrt{n}-\sqrt{2n}}\right)_{n\in\mathbb{N}}$

Question 2: Complete the proof of Theorem 5.1.8, by proving the following result:

Theorem: Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ both are sequences that tend to infinity. Prove:

(i) $a_n + b_n \to \infty$ as $n \to \infty$;

(ii) $a_n \cdot b_n \to \infty$ as $n \to \infty$.

(iii) Let $M \in \mathbb{N}$. Assume that $(c_n)_{n\in\mathbb{N}}$ is a sequence such that $c_n \geq a_n$ for all $n \geq M$. Prove that $c_n \to \infty$ as $n \to \infty$.

Question 3: There are many variants on Question 2. Can you think of some? Here is one:

(i) Suppose that $(a_n)_{n\in\mathbb{N}} \to \infty$ as $n \to \infty$ and that $(b_n)_{n\in\mathbb{N}}$ is a sequence of non-zero numbers that converges to $\ell > 0$. Prove that $\dfrac{a_n}{b_n} \to \infty$ as $n \to \infty$.

(ii) What happens if $\ell = 0$ in part (i)?

Question 4: Use the subsequence test to show that:-

(i) the sequence $\left(\dfrac{n}{8} - \big[\dfrac{n}{8}\big]\right)_{n\in\mathbb{N}}$ does not converge;

(ii) the sequence $\left(\big[\sin(\frac{n\pi}{4})\big] - \sin(\frac{n\pi}{4})\right)_{n\in\mathbb{N}}$ does not converge.

Question 5: Assume that $n^{\frac{1}{\sqrt{n}}} \to \ell$ as $n \to \infty$.

Use the subsequence test to show that $\ell = 1$. [Hint: We do know $\lim\limits_{m\to\infty} m^{\frac{1}{m}}$.]

Question 6*: Prove that $(n!)^{-\frac{1}{n}} \to 0$ as $n \to \infty$. [Hint: Use 4.1.4 with $c = \frac{1}{\epsilon}$.]