MATH10242 Sequences and Series: Exercises for Week 4 Tutorials

Attempt these questions before your tutorial in the week beginning 17th February.

Question 0 is an easier question on which to get started. The solution to it is given on the next page — but don't look at the answers until you have worked seriously at the question! (a general point!) Question 1 is the most important on this sheet. Question 2 needs a proof but it's not a difficult one - certainly easy enough to be asked on the exam.

Question 0: Which of the following sequences converge (and to what number)? You should justify your answers. In some of the questions you may be able to use the theory we have been developing, but for others you may have to compute them explicitly as we did last week.

(a) $\left(1 - \dfrac{1}{2n^3}\right)_{n\in\mathbb{N}}$ (b) $(1 + n^3)_{n\in\mathbb{N}}$.

(c) $(\sqrt{n+1} - \sqrt{n})_{n\in\mathbb{N}}$ [See last week's solutions.]

(d) $\left(\dfrac{n^3}{5^n}\right)_{n\in\mathbb{N}}$ [See examples in the notes.]

Question 1: Which of the following sequences converge (and to what number)? Justify your answers.

(a) $(1 - \dfrac{3n^3 + n^2}{2n^3})_{n\in\mathbb{N}}$

(b) $(1 - \dfrac{3n^2 + n^3}{2n^2})_{n\in\mathbb{N}}$

(c) $(\sqrt{n^2+1} - n)_{n\in\mathbb{N}}$ [*Hint: Use the ideas we used for* $\sqrt{n+2} - \sqrt{n}$]

(d) $(\sqrt{2n} - \sqrt{n})_{n\in\mathbb{N}}$

(e) $(3^{-n})_{n\in\mathbb{N}}$

(f) $\left(\dfrac{n^3}{3^n + 4^n}\right)_{n\in\mathbb{N}}$

Question 2: Let $(a_n)_{n\in\mathbb{N}}$ be a bounded, decreasing sequence. Prove that $(a_n)_{n\in\mathbb{N}}$ is convergent.

Question 3: The definition that we gave in the notes for "the sequence $(a_n)_n$ converges to limit l" is (a). All of (b), (c) and (d) are equivalent definitions; prove that.

(a) $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $| a_n - l | < \epsilon$

(b) $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $| a_n - l | < \epsilon$

(c) $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $| a_n - l | \leq \epsilon$

(d) Given any positive integer k, there is $N \in \mathbb{N}$ such that $\forall n \geq N$, $| a_n - l | < \frac{1}{k}$

Solution to Question 0.

(a) Likely you guessed the limit to be $\ell = 1$. So, let's prove it. First,

$$|a_n - 1| = \left|1 - \frac{1}{2n^3} - 1\right| = \frac{1}{2n^3} < \frac{1}{n}.$$

Now, given $\epsilon > 0$ if we take $N = \left[\frac{1}{\epsilon}\right] + 1$, then for $n \geq N$ we get $\frac{1}{n} < \epsilon$ and hence $|a_n - 1| < \epsilon$ and we are done.

You could also use the notes more directly: Check that $(b_n) = (1 - \frac{1}{n})$ does indeed converge to 1 and then use the Sandwich Theorem.

(b) Since $1 + n^3 \geq n$ for all natural numbers n, the sequence is not bounded (see Example 2.4.8). Thus the sequence $(1 + n^3)_{n\in\mathbb{N}}$ does not converge, by Theorem 2.3.9.

(c)

$$\begin{aligned}
\sqrt{n+1} - \sqrt{n} &= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})} \\
&= \frac{(\sqrt{n+1}^2 - \sqrt{n}^2)}{\sqrt{n+1} + \sqrt{n}} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}
\end{aligned}$$

Thus $\sqrt{n+1} - \sqrt{n} = \dfrac{1}{\sqrt{n+1} + \sqrt{n}} \leq \dfrac{1}{2\sqrt{n}} \leq \dfrac{1}{\sqrt{n}}$. So, just as in the (very) similar example from last week, we need $\dfrac{1}{\sqrt{n}} < \epsilon$ or $n > \epsilon^{-2}$. So, take $N = 1 + [\epsilon^{-2}]$.

Then, again, tracing back through our computations, we see that if $n \geq N$ then $\dfrac{1}{\sqrt{n}} < \epsilon$ and so $\sqrt{n+1} - \sqrt{n} < \epsilon$. In other words, the limit is 0.

(d) Since $5^n > 4^n$, clearly $0 < \frac{n^3}{5^n} < \frac{n^3}{4^n} = \frac{n^3}{2^n \cdot 2^n}$.

Now, by the notes $2^n > n^2$, for $n \geq 5$ and hence

$$0 < \frac{n^3}{5^n} < \frac{n^3}{n^4} = \frac{1}{n} \qquad for\ n \geq 5.$$

By the notes, again, (see 3.1.5) we know that $(\frac{1}{n})$ is null and so by 3.1.4(ii) our given sequence $(a_n) = (\frac{n^3}{5^n})$ is null.