MATH10242 Sequences and Series: Exercises for Week 3 Tutorials

Attempt these questions before your tutorial in the week beginning 10th February. Questions 2 and 3 are important, so be sure to spend enough time on them. As usual, an asterisk indicates a significantly harder problem.

Question 1: Here you should justify a couple of formulas that we will often use. Prove:

(a) $\forall x, y, \quad |x - y| \geq |\, |x| - |y| \,|$;

(b) $\forall x, \ell$ and $\forall \epsilon > 0$, $|x - \ell| < \epsilon \iff \ell - \epsilon < x < \ell + \epsilon$.

Question 2: Let $\epsilon > 0$ be given. For each of the following sequences (a_n), find a natural number N such that $\forall n \geq N$, one has $|a_n| < \epsilon$ (thereby showing that $a_n \to 0$ as $n \to \infty$).

(a) $a_n = \frac{1}{n^2}$.

(b) $a_n = \frac{n+\sqrt{n}}{n^2+1}$.

(c) $a_n = \frac{\cos(n)}{n}$;

(d) $a_n = \sqrt{n+2} - \sqrt{n}$

(e)* $a_n = \frac{n}{2^n}$.

Hints: In parts (b) and c) find a nicer function $f(n)$ with $|a_n| < f(n)$. Most of part (d) has already been seen on the Week 2 sheet.

Question 3: Which of the following sequences converge and to what value? In each case you should properly justify your answers, making use of the formal definition of convergence to a limit, as we have been doing in class.

(a) $(1 + \frac{(-1)^n}{n})_{n\in\mathbb{N}}$;

(b) $(1 + \frac{3n^2+n}{2n^2})_{n\in\mathbb{N}}$;

(c) $(1 + (-1)^n)_{n\in\mathbb{N}}$;

(d) $(\dfrac{n + 4(-1)^n}{2n})$.

Question 4* (a) Let $x > 0$. Using the binomial theorem (or otherwise) prove that for all $n \in \mathbb{N}$, one has $(1 + x)^n \geq 1 + nx$.

(b) By taking $x = \frac{y}{n}$ in (a), deduce that for all $y > 0$ and $n \in \mathbb{N}$, $(1 + y)^{\frac{1}{n}} \leq 1 + \frac{y}{n}$.

(c) Hence show that for fixed $c > 1$, one has $c^{\frac{1}{n}} \to 1$ as $n \to \infty$.