MATH10242 Sequences and Series: Exercises for Week 2 Tutorials (Exercises on Real Numbers and Convergence)

Have a go at these questions before your examples classes (= tutorials) in the week starting February 3rd. (You can find your tutorial group on Blackboard.) An asterisk indicates a problem that could be hard in some sense - it might involve an intricate argument, or an idea that doesn't arise directly from the notes or even a bit of inspiration. Don't spend too long on a question with an asterisk at the expense of spending enough time on the more straightforward problems.

Questions 1-4 are about working from axioms. Questions 6 and 7 are the most important on the sheet.

Question 1: Let $x \in \mathbb{R}$. Using just the axioms for ordered fields (A0–9) and (Ord 1–4) from Chapter 1 of the Notes, and breaking into the cases when x is either positive, negative or zero, show that $x^2 \geq 0$.

Question 2: Show, using just the axioms for ordered fields, that if $x, y > 0$ then $x > y \iff x^2 > y^2$.

Question 3: Show, using just the axioms for ordered fields (including that $0 \neq 1$), that for all $x \in \mathbb{R}$ we have $x < x + 1$.

Question 4: Show that for any $\delta > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$.
[Example 2.4.8 from the notes can be used here.]

Question 5:* I said in the lecture that, from the construction of the reals $\mathbb{R}$ from the rationals $\mathbb{Q}$, it follows that $\mathbb{Q}$ is dense in $\mathbb{R}$ (meaning that, given any two real numbers $x < y$, there is a rational number, q, between them: $x < q < y$). Show that the set $\mathbb{R} \setminus \mathbb{Q}$ of **irrationals** is **dense** in $\mathbb{R}$ i.e. show that for all $x, y \in \mathbb{R}$ if $x < y$ then there exists $t \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < t < y$.

Question 6: For each of the following sequences (a_n) and real numbers $\epsilon > 0$, find a natural number N such that $\forall n \geq N$ we have $|a_n| < \epsilon$.

(a) $a_n = \frac{1}{n}$, $\epsilon = 1/50$.

(b) $a_n = \frac{1}{n^2}$, $\epsilon = 1/100$.

(c) $a_n = \frac{1}{n^2}$, $\epsilon = 1/1000$.

(d) $a_n = \frac{1}{\sqrt{n}}$, $\epsilon = 1/1000$.

(e) $a_n = \frac{\cos(n)}{n}$, $\epsilon = 10^{-6}$.

(f) $a_n = \frac{\cos(n)}{n^2}$, $\epsilon = 10^{-6}$.

(g) $a_n = \sqrt{n+2} - \sqrt{n}$, $\epsilon = 10^{-6}$.

Question 7: For each of the following sequences (a_n) and real numbers $\epsilon > 0$, find a natural number N such that $\forall n \geq N$ we have $\mid a_n - 2 \mid < \epsilon$.

(a) $a_n = 2 - \frac{1}{2^n}$, $\epsilon = 1/1000$.

(b) $a_n = 2 + \frac{\sin(n)}{n}$, $\epsilon = 1/1000$.