MATH10242 Sequences and Series: Week 12 Tutorials
Revision: Extra Exercises

Please try these questions before your tutorial in the week beginning 4th May.
These questions are very much in the style of typical final exam questions, plus some additional, maybe harder, parts.

Question 1:
(a) Define what it means for a sequence $(a_n)_{n\in\mathbb{N}}$ to tend to $-\infty$.
(b) Define what it means for a sequence $(a_n)_{n\in\mathbb{N}}$ to be bounded below.
(c) The following statement is not correct; modify it to a correct statement.
"Every sequence contains a convergent subsequence."
(Adding "It is not true that" at the beginning is not what I have in mind.)
(d) Some students seem to believe that every sequence is convergent or tends to $+\infty$ or tends to $-\infty$ or switches between a finite number of values. Give an example of a sequence which has none of these properties.

Question 2:
(a) Fix $\epsilon > 0$. Find a natural number N such that $\left|\dfrac{2n^3 - \ln(n)}{n(n-1)^2 + 1} - 2\right| < \epsilon$ for all $n \geq N$.
What have you (if you managed to answer the question) just shown about the sequence $a_n = \dfrac{2n^3 - \ln n}{n(n-1)^2 + 1}$?
(b) Fix a real number K. Find a natural number N such that $\ln(n) - n < K$ for all $n \geq N$.
What does that prove?

Question 3: Find the limits of the following sequences.
(a) $\left(\dfrac{3^n - n^4}{2^n + n!}\right)_{n\in\mathbb{N}}$
(b) $\left((n + n^{\frac{1}{2}})^{\frac{1}{2}} - n^{\frac{1}{2}}\right)_{n\in\mathbb{N}}$

Question 4: Using L'Hôpital's Rule or otherwise, find
(a) $\lim_{n\to\infty} \dfrac{\ln(4n^{\frac{1}{3}} - 2)}{\ln(n+1)}$ and
(b) $\lim_{n\to\infty} \dfrac{\ln(e^{e^n} - n^6)}{n! - n^{10}}$.

Question 5: Determine whether the following series converge. In each case you should briefly justify your answer (for example by saying what test you are using).

(a) $\displaystyle\sum_{n=1}^{\infty} n^{10}e^{-n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{e^n}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{e^{n^2}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^3}{3^n}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{3}{2}}}$ (f) $\displaystyle\sum_{n=1}^{\infty} \tan(\frac{\pi}{2} - \frac{1}{n})$

(g) $\sum_{n=1}^{\infty} \frac{3n^4}{n(n+e)^2}$ (h) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ (i) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Question 6: Using partial fractions or otherwise, find $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$.

(b) Show that the following series converge and show (use partial fractions) that they have the same sum.

$$\sum_{n=1}^{\infty} \frac{1}{2n(2n+1)} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{n}$$

Question 7: (a) Define what it means for a sequence $(a_n)_{n\in\mathbb{N}}$ to (i) converge to a limit ℓ, (ii) tend to ∞ as $n \to \infty$ [the original wording said "converge to ∞" but it's not a good idea to use the word "converge" next to a divergent series].

(b) Given a sequence $(a_n)_{n\in\mathbb{N}}$, define a new sequence $(a_n^*)_{n\in\mathbb{N}}$ by $a_n^* = \frac{1}{2}(a_n + a_{n+1})$. Prove direct from your definitions above that (i) if $a_n \to \ell$ as $n \to \infty$ then $a_n^* \to \ell$ as $n \to \infty$, (ii) if $a_n \to \infty$ as $n \to \infty$ then $a_n^* \to \infty$ as $n \to \infty$.

(c) Show, by producing suitable examples, that the converse of each of (b)(i) and (b)(ii) is false.

Question 8: Let b be a positive real number and define the sequence $(a_n)_{n\in\mathbb{N}}$ inductively by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \frac{a_n}{a_n + b} \text{ for } n \geq 1.$$

(a) Prove by induction on n that $a_n > 0$ for all n.

(b) Prove that if $0 < b < 1$ then $a_n > 1 - b$ for all n.

(c) Deduce that, if $b > 0$, then the sequence $(a_n)_{n\in\mathbb{N}}$ is a decreasing sequence and, by quoting a suitable theorem, deduce that it converges.

(d) Prove that if $0 < b < 1$ then $a_n \to 1 - b$ as $n \to \infty$.

(e) Calculate $\lim_{n\to\infty} a_n$ in the case that $b \geq 1$.

Question 9:

(a) Find the radius of convergence for the series

(i) $\sum_{n=1}^{\infty} \frac{\sqrt{(2n)!}}{n!} x^n$ (ii) $\sum_{n=1}^{\infty} \frac{\sqrt{(2n)!}}{(n+1)!} x^n$

(b) Find the interval of convergence for the series

(i) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ (ii) $\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt{n}}$ (iii) $\sum_{n=1}^{\infty} \frac{(-x)^n}{7n-5}$