MATH10242 Sequences and Series: Exercises for Week 11 Tutorials

Please do these questions before your tutorial in the week beginning 27th April.
Be sure that you can do examples like those in Question 1.

Question 1: Find the radius of convergence R of the following power series.
In parts (i) and (ii), what is the *interval of convergence* of the given power series?

$$(i) \sum_{n\geq 1} \frac{x^n}{8^n} \qquad (ii) \sum_{n\geq 1} \frac{(-x)^n}{4n+1}, \qquad (iii) \sum_{n\geq 1} \frac{(2n)!}{(n!)^2} x^n, \qquad (iv) \sum_{n\geq 1} \frac{n^n}{n!} x^n,$$

$$(v) \sum_{n\geq 1} n! \cdot x^n \qquad (vi) \sum_{n\geq 1} \frac{\sqrt{(2n)!}}{n!} x^n.$$

[You will need to use the formula for $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ from a previous Exercise Sheet.]

Question 2: Let $r > 0$. Using Question 1(i) as a guide, find a series $\sum_{n=1}^{\infty} a_n x^n$ with radius of convergence r.

Question 3: Let $\sum_{n\geq 1} a_n$ be a series. We define two new series $\sum_{n\geq 1} a_n^+$, consisting of all the positive terms of the original series and and $\sum_{n\geq 1} a_n^-$, consisting of all the negative terms. To be specific, set

$$a_n^+ = \frac{a_n + |a_n|}{2} \qquad \textit{and} \qquad a_n^- = \frac{a_n - |a_n|}{2},$$

and notice that if $a_n > 0$ then $a_n^+ = a_n$ and $a_n^- = 0$. Conversely, if $a_n < 0$ then $a_n^- = a_n$ and $a_n^+ = 0$.

(a) Prove that, if $\sum_{n\geq 1} a_n$ is absolutely convergent, then both $\sum_{n\geq 1} a_n^+$ and $\sum_{n\geq 1} a_n^-$ are convergent. Moreover, prove that

$$\sum_{n\geq 1} a_n = \sum_{n\geq 1} a_n^+ + \sum_{n\geq 1} a_n^-.$$

(b*) Prove that, if $\sum_{n\geq 1} a_n$ is only conditionally convergent, then both $\sum_{n\geq 1} a_n^+$ and $\sum_{n\geq 1} a_n^-$ are divergent.