MATH10242 Sequences and Series

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[1]This set of notes is a slightly modified version of notes developed by Prof. J. T. Stafford and, before him, Prof. A. J. Wilkie.

Contents

0.1 Introduction

Maybe you can see that

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 2$$

and even that

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n} + \cdots = \frac{3}{2}.$$

But what exactly do these formulas mean? What does it mean to add infinitely many numbers together? Is that even meaningful?

You might recognize the numbers above as being terms of geometric progressions and know the relevant general formula, for $|x| < 1$,

$$1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x}.$$

But how do we prove this? And what if $|x| \geq 1$?

In this course we shall answer these, and related, questions. In particular, we shall give a rigorous definition of what it means to add up infinitely many numbers and then we shall find rules and procedures for finding the sum in a wide range of particular cases.

Here are some more, rather remarkable, such formulas:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6},$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots = \log 2,$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \infty.$$

We shall prove the second and third of these formulas in this course unit, but the first one is too difficult and will be done in your lectures on real and complex analysis in the second year. Here, "real analysis" means the study of functions from real numbers to real numbers, from the point of view of developing a rigorous foundation for calculus (differentiation and integration) and for other infinite processes.[1] The study of sequences and series is the first step in this programme.

This also means there are two contrasting aspects to this course. On the one hand we will develop the machinery to produce formulas like the ones above. On the other hand it is also crucial to understand the theory that lies behind that machinery. This rigorous approach forms the second aspect of the course—and is in turn the first step in providing a solid foundation for real analysis.

[1] and "complex analysis" refers to the complex numbers, not (necessarily) complexity in the sense of "complicated"!

Chapter 1

Before We Begin

1.1 Some Reminders about Mathematical Notation

1.1.1 Special sets

We use the following notation throughout the course.

$\mathbb{R}$ - the set of real numbers;

$\mathbb{R}^+$ - the set of strictly positive real numbers, i.e. $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$;

$\mathbb{Q}$ - the set of rational numbers;

$\mathbb{Z}$ - the set of integers (positive, negative and 0);

$\mathbb{N}$ (or $\mathbb{Z}^+$) - the set of natural numbers, or **positive** integers $\{x \in \mathbb{Z} : x > 0\}$. (In this course, we do not count 0 as a natural number. We can use some other notation like $\mathbb{Z}^{\geq 0}$ for the set of integers greater than or equal to 0.)

$\emptyset$ - the empty set.

1.1.2 Set theory notation

The expression "$x \in X$" means x is an **element** (or **member**) of the set X. For sets A, B, we write $A \subseteq B$ to mean that A is a **subset** of B (i.e. every element of A is also an element of B). Thus $\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

Standard **intervals** in $\mathbb{R}$: if $a, b \in \mathbb{R}$ with $a \leq b$, then

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$;
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$;
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$;
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$;
- $(a, \infty) = \{x \in \mathbb{R} : a < x\}$;
- $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$;
- $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$;
- $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$;
- $(-\infty, \infty) = \mathbb{R}$.

1.1.3 Logical notation

The expression "$\forall x \ldots$" means "for all $x \ldots$" and "$\exists x \ldots$" means "there exists at least one x such that ...". These are usually used in the context "$\forall x \in A \ldots$" meaning "for all elements x of the set $A \ldots$", and "$\exists x \in A \ldots$" meaning "there exists at least one element x in the set A such that ...".

Thus, for example, "$\forall x \in \mathbb{R}\ x > 1$" means "for all real numbers x, x is greater than 1" (which happens to be false) and "$\exists x\ x > 1$" means "there exists a real number x such that x is greater than 1" (which happens to be true).

1.1.4 Greek letters

The two most commonly used Greek letters in this course are δ (delta) and ϵ (epsilon). They are reserved exclusively for (usually small) positive real numbers.

Others are α (alpha), β (beta), γ (gamma), λ (lambda), θ (theta-usually an angle), η (eta) and Σ (capital sigma - the summation sign which will be used when we come to study series in Part II).

1.1.5 Structure of the course

This course unit has two lectures each week; each student has one tutorial per week - you will be assigned to one of the four tutorial groups (this should be shown in your space on the student system).

Lectures:

Tuesday 10:00–11:00 in University Place, Theatre B;

Thursday 9:00–10:00 in Crawford House, Theatre 1.

Tutorials: These start on 5th February and are at the following times/places; you will be assigned to one of the five tutorial groups.

Wednesday 11:00–12:00 in AT G.207;

Wednesday 11:00–12:00 in University Place 1.218;

Wednesday 11:00-12:00 in University Place 1.219;

Wednesday 12:00–1:00 in AT G.207;

Friday 11:00–12:00 in AT G.107.

The tutorials start in Week 2 and, typically, Week n tutorials concentrate on the subject matter introduced in Week $n-1$.

I will put the weekly exercise sheets on my teaching webpage website. (Complete course notes and some links are already there.) It is very important that you work at these sheets before the weekly tutorials; try to have a serious attempt at all the problems. But don't waste time going round in circles: try to recognise when you're doing that, have a break/do something else and, maybe, when you come back to the question you'll see your way around what seemed to be a problem. In general, for each lecture or tutorial hour, you should expect to spend two to three hours working at understanding the material and testing/strengthening your understanding by doing problems. It is not enough to read and summarise the notes; it is only by testing your understanding by doing examples that you will really understand the material. Moreover, the exam questions will be similar to these problems!

When you get really stuck on something, discuss it with someone! As well as me and the people who help in the tutorials, there are your fellow-students - they are an excellent resource and bear in mind that one of the most useful exercises you can do is to try to explain something to someone else.

Office Hour: see **my teaching webpage**
https://personalpages.manchester.ac.uk/staff/mike.prest/teaching.html

Assessment:

1. **Final exam** carrying 80% of the course weight.
2. **Coursework** carries 20% of the course weight and will consist of:

an in-class test on Wednesday 13th March.

1.1.6 Where we're headed and some things we'll see on the way

In Part I we aim to understand the behaviour of infinite sequences of real numbers, meaning what happens to the terms as we go further and further on in the sequence. Do the terms all gradually get as close as we like to a limiting value (then the sequence is said to converge to that value) or not? The "conceptual" aim here is to really understand what this means. To do that, we have to be precise and avoid some plausible but misleading ideas. It's worthwhile trying to develop, and refine, your own "pictures" of what's going on. We also have to understand the precise definition well enough to be able to use it when we calculate examples, though we will gradually build up a stock of general results (the "Algebra of Limits"), general techniques and particular cases, so that we don't have to think so hard when faced with the next example.

Part II is about "infinite sums" of real numbers: how we can make a sensible definition of that vague idea and then how we can calculate the value of an infinite sum - if it exists. We also need to be able to tell whether a particular "infinite sum" does or doesn't make sense/exist. Sequences appear here in two ways: first as the *sequence* of numbers to be "added up" (and the order of adding up does matter, as we shall see); second as a crucial ingredient in the actual definition of an "infinite sum" ("infinite series" is the official term). What we actually do is add up just the first n terms of such an infinite series - call this value the n-th partial sum - and then see what happens to this sequence (note) of partial sums as n gets bigger and bigger. If that sequence of partial sums converges to a limit then that limit is what we define to be the sum of the infinite series. Hopefully that makes sense to you and seems like it should be the right definition to use. Anyway, it works and, again, we have the conceptual aspect to get to grips with as well as various techniques that we can (try to) use in computations of examples.

Here are some of the things we prove about our concept of limit: a sequence can have *at most* one limit; if a sequence is increasing but never gets beyond a certain value, then it has a limit; if a sequence is squeezed between two other sequences which have the same limit l, then it has limit l. These properties help clarify the concept and are frequently used in arguments and calculations. We also show arithmetic properties like: if we have two sequences, each with a limit, and produce a new sequence by adding corresponding terms, then this new sequence has a limit which is, as you might expect, the sum of the limits of the two sequences we started with.

Then we turn to methods of calculating limits. We compare standard functions (polynomials, logs, exponentials, factorials, ...) according to how quickly they grow, but according to a very coarse measure - their "order" of growth, rather than rate of growth (i.e. derivative). That lets us see which functions in a complicated expression for the n-th term of a sequence are most important in calculating the limit of the sequence. There will be lots of examples, so that you can gain some facility in computing limits, and there are various helpful results, L'Hôpital's Rule being particularly useful.

While the properties of sequences are, at least once you've absorbed the concept, quite natural, infinite series hold quite a few surprises and really illustrate the need to be careful about definitions in mathematics (many mathematicians made errors by mishandling infinite series, especially before the concepts were properly worked out in the 1800s). Given an infinite

series, there are two questions: does it have a sum? (then we say that it "converges", meaning that the sequence of partial sums has a limit - the value of that infinite sum) and, if so, what is the value of the sum? There are a few series (e.g. a geometric series with ratio < 1) where we can quite easily compute the value but, in general this is hard. It is considerably easier to determine *whether* a series has a sum or not by comparing it with a series we already know about. Indeed, the main test for convergence that we will use, the Ratio Test, is basically comparison with a geometric series.

Many infinite series that turn up naturally are "alternating", meaning that the terms are alternately positive and negative. So, in contrast with the corresponding series where all the terms are made positive, there's more chance of an alternating sequence converging, because the terms partly cancel each other out. Indeed, remarkably, it's certain provided the absolute values of the individual terms shrink monotonically to 0!

We'll finish with power series: infinite series where each term involves some variable x - you could think of these as "infinite polynomials in x". Whether or not a power series converges (i.e. has a sum), depends very much on the value of x, convergence being more likely for smaller values of x. In fact, the picture is as good as it could be: there's a certain "radius of convergence" R (which could be 0 or ∞ or anything in between, depending on the series) such that, within that radius we have convergence, outside we have divergence, and on the boundary ($x = \pm R$) it could go either way for each boundary point (so we have to do some more work there).

1.1.7 Reading outside the Course Notes

You should do some reading outside the course notes.

If you read only the course notes, do only the exercises set and look just at past exams for this course unit, then you should pass and you might even get a very good mark. But your understanding and enjoyment of mathematics will be enhanced if you become more of an "independent learner" by reading around a bit, asking your own questions and looking beyond the course notes for answers. You might see connections with other courses and you're likely to develop a stronger sense of "ownership" of the mathematics you're learning. It takes more effort, but more engagement is likely to result in more enjoyment and, in the long run, better marks. In particular, you're less likely to feel overwhelmed by all the new material in front of you because you will have had some practice in extracting what's most useful from a variety of sources.

The time to start reading outside the course notes is now! Not when exams are close - at that time you will probably want to focus your efforts on what is most directly relevant to exams but that is the time when the perspective and understanding that you have developed *during* the semester will pay off and make revision easier.

Notation in this course is largely consistent with the Foundations of Pure Mathematics course unit text:

[PJE] Peter J Eccles, '*An Introduction to Mathematical Reasoning.*'

1.1.8 Basic properties of the real numbers

It will be assumed that you are familiar with the elementary properties of $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ that were covered last semester in MATH10101/10111. These include, in particular, the basic facts about the arithmetic of the integers and a familiarity with the Principle of Mathematical Induction. One may then proceed to construct the set $\mathbb{R}$ of real numbers. There are many ways of doing this which, remarkably, all turn out to be equivalent in a sense that can be made mathematically

precise. One method with which you should be familiar is to use infinite decimal expansions as described in Section 13.3 of [PJE].

Here we extract some of the basic arithmetic and order properties of the reals.

First, just as for the set of rational numbers, $\mathbb{R}$ is a **field**. That means that it satisfies the following conditions.

(A0) $\forall a, b \in \mathbb{R}$ one can form the sum $a + b$ and the product $a \cdot b$ (also written as just ab). We have that $a + b \in \mathbb{R}$ and $a \cdot b \in \mathbb{R}$;

(A1) $\forall a, b, c \in \mathbb{R}$, $a + (b + c) = (a + b) + c$ (*associativity* of $+$);

(A2) $\forall a, b \in \mathbb{R}$, $a + b = b + a$ (*commutativity* of $+$);

(A3) $\forall a \in \mathbb{R}$, $a + 0 = 0 + a = a$;

(A4) $\forall a \in \mathbb{R}$, there is a unique element in $\mathbb{R}$ (denoted $-a$) such that $a + (-a) = 0 = (-a) + a$;

(A5) $\forall a, b, c \in \mathbb{R}$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (*associativity* of $\cdot$);

(A6) $\forall a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$ (*commutativity* of $\cdot$);

(A7) $\forall a \in \mathbb{R}$, $a \cdot 1 = 1 \cdot a = a$;

(A8) $\forall a \in \mathbb{R}$, if $a \neq 0$ then there is a unique element in $\mathbb{R}$ (denoted a^{-1} or $\frac{1}{a}$) such that $a \cdot (a^{-1}) = 1 = a^{-1} \cdot a$;

(A9) $\forall a, b, c \in \mathbb{R}$, $a \cdot (b + c) = a \cdot b + a \cdot c$ (the *distributive* law).

These **axioms** (A0)-(A9) list the basic arithmetic/algebraic properties that hold in $\mathbb{R}$ and from which all the other such properties may be deduced. Further identities, such as $x \cdot 0 = 0$, $x \cdot (-y) = -(x \cdot y)$, $-(-x) = x$, $(x + y)(x - y) = x^2 - y^2$, $(x + y)^2 = x^2 + 2xy + y^2$, ... follow from these axioms. Here we are using the usual notation: for $a \in \mathbb{R}$, a^2 is an abbreviation for $a \cdot a$ (similarly a^3 is an abbreviation for $a \cdot (a \cdot a)$, etc.).

Example 1.1.1. For example, let us prove the last identity $(x + y)^2 = x^2 + 2xy + y^2$ above.

So, we have

$$\begin{aligned}
(x+y)^2 &= (x+y)(x+y) \quad \text{(by definition)}, \\
&= (x+y) \cdot x + (x+y) \cdot y \quad \text{(by A9)}, \\
&= x \cdot (x+y) + y \cdot (x+y) \quad \text{(by A6)}, \\
&= x \cdot x + x \cdot y + y \cdot x + y \cdot y \quad \text{(by A9)}, \\
&= x \cdot x + x \cdot y + x \cdot y + y \cdot y \quad \text{(by A6)}, \\
&= x \cdot x + 1 \cdot (x \cdot y) + 1 \cdot (x \cdot y) + y \cdot y \quad \text{(by A7)}, \\
&= x \cdot x + (1+1) \cdot (x \cdot y) + y \cdot y \quad \text{(by A9 and A6)}, \\
&= x \cdot x + 2 \cdot (x \cdot y) + y \cdot y \quad \text{(by definition)}, \\
&= x^2 + 2xy + y^2 \quad \text{(by definition)}.
\end{aligned}$$

Actually we have also used A1 many times here. This allowed us to ignore brackets in expressions involving many $+$ symbols.

Note that if we replace $\mathbb{R}$ in these axioms by $\mathbb{Q}$ then they do hold; that is, $\mathbb{Q}$ is also a field (but $\mathbb{Z}$ is not since it fails (A8)). The point of giving a name ("field") to an "arithmetic system" where these hold is that many more examples appear in mathematics, so it proved to be worth isolating these properties and investigating their general implications. Other fields that you will have encountered by now are the complex numbers and the integers modulo 5 (more generally, modulo any prime). But the reals and rationals have some further properties not shared by these latter examples.

Namely, the reals form an **ordered field**. This means that we have a **total order relation** $<$ on $\mathbb{R}$. All the properties of this relation and how it interacts with the arithmetic operations follow from the following axioms:

(Ord 1) $\forall a, b \in \mathbb{R}$, exactly one of the following is true: $a < b$ or $b < a$ or $a = b$;

(Ord 2) $\forall a, b \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$;

(Ord 3) $\forall a, b \in \mathbb{R}$, if $a < b$ then $\forall c \in \mathbb{R}$, $a + c < b + c$;

(Ord 4) $\forall a, b \in \mathbb{R}$, if $a < b$ then $\forall d \in \mathbb{R}^+$, $a \cdot d < b \cdot d$.

As usual, we write $a \leq b$ to mean either $a = b$ or $a < b$.

You do not need to remember exactly which rules have been written down here. The important point is just that this short list of properties is all that one needs in order to be able to prove all the other standard facts about $<$. For example, we have:

Example 1.1.2. Let us prove that if x is **positive** (i.e. $0 < x$) then $-x$ is **negative** (i.e. $-x < 0$). So suppose that $0 < x$. Then by (Ord 3), $0 + (-x) < x + (-x)$. Simplifying we get $-x = 0 + (-x) < x + (-x) = 0$, as required.

Other facts that follow from these properties include:
$\forall x \in \mathbb{R} \quad x^2 \geq 0$,
$\forall x, y \in \mathbb{R} \quad x \leq y \Leftrightarrow -x \geq -y$, *et cetera.*
It left as an exercise to prove these facts using just the axioms (Ord 1–4). Also see the first exercise sheet.

However, there are more subtle facts about the real numbers that cannot be deduced from the axioms discussed so far. For example, consider the theorem that $\sqrt{2}$ is irrational. This really contains two statements: first that $\sqrt{2} \notin \mathbb{Q}$ (that is, no rational number squares to 2; this was proved in MATH10101/10111); second that there really is such a number in $\mathbb{R}$ - there is a (positive) solution to the equation $X^2 - 2 = 0$ in $\mathbb{R}$. And this definitely is an extra property of the reals—simply because it not true for the rationals (which satisfies all the algebraic and order axioms that we listed above).

So, we need to formulate a property of $\mathbb{R}$ that expresses that there are no "missing numbers" (like the "missing number" in $\mathbb{Q}$ where $\sqrt{2}$ should be). Of course, we have to say what "numbers" should be there, in order to make sense of saying that some of there are "missing". The example of $\sqrt{2}$ might suggest that we should have "enough" numbers so as to be able to take n-th roots of positive numbers and perhaps to solve other polynomial equations and following that idea does lead to another field - the field of real algebraic numbers - but we have a much stronger condition ("completeness") in mind here. We will introduce it in Chapter 2, just before we need it.

1.1.9 The Integer Part (or 'Floor') Function

From any of the standard constructions of the real numbers one has the fact that any real number is sandwiched between two successive integers in the following precise sense:

$\forall x \in \mathbb{R}$, $\exists n \in \mathbb{Z}$ *such that* $n \leq x < n + 1$.

The integer n that appears here is unique and is denoted $[x]$; this is called the **integer part** of x. The function $x \mapsto [x]$ is called the integer part, or floor, function. Note that $0 \leq x - [x] < 1$.

Example 1.1.3. $[1.47] = 1$, $[\pi] = 3$, $[-1.47] = [-2]$.

Part I

Sequences

Chapter 2

Convergence

2.1 What is a Sequence?

Definition 2.1.1. *A* **sequence** *is a list* $a_1, a_2, a_3, \ldots, a_n, \ldots$ *of real numbers labelled (or* **indexed***) by natural numbers. We usually write such a sequence as* $(a_n)_{n\in\mathbb{N}}$*, or as* $(a_n)_n$ *or just as* (a_n)*. We say that* a_n *is the* n*th* **term of the sequence**.

The word "sequence" suggests time, with the numbers occurring in temporal sequence: first a_1, then a_2, *et cetera*. Indeed, some sequences arise this way, for instance as successive approximations to some quantity we want to compute. Formally, a sequence is simply a function $f{:}\mathbb{N} \to \mathbb{R}$, where we write a_n for $f(n)$.

We shall be interested in the *long term behaviour* of sequences, i.e. the behaviour of the numbers a_n when n is very large; in particular, do the approximations converge on some value?

Example 2.1.2. Consider the sequence $1, 4, 9, 16, \ldots n^2, \ldots$. Here, the nth term is n^2. So we write the sequence as $(n^2)_{n\in\mathbb{N}}$ or $(n^2)_n$ or just (n^2). What is the nth term of the sequence $4, 9, 16, 25, \ldots$? What are the first few terms of the sequence $\big((n-1)^2\big)_n$?

Example 2.1.3. Consider the sequence $2,\ \frac{3}{2},\ \frac{4}{3},\ \frac{5}{4},\ \ldots$. Here, $a_n = \dfrac{n+1}{n}$. The sequence is $\left(\dfrac{n+1}{n}\right)_{n\in\mathbb{N}}$.

Example 2.1.4. Consider the sequence $-1, 1, -1, 1, -1, \ldots$. A precise and succinct way of writing it is $\big((-1)^n\big)_{n\in\mathbb{N}}$. The nth term is 1 if n is even and -1 if n is odd.

Example 2.1.5. Consider the sequence $\left(\dfrac{(-1)^n}{3^n}\right)_{n\in\mathbb{N}}$. The 5th term, for example, is $\dfrac{-1}{243}$.

Example 2.1.6. Sometimes we might not have a precise formula for the nth term but rather a rule for generating the sequence, E.g. consider the sequence $1, 1, 2, 3, 5, 8, 13, \ldots$, which is specified by the rule $a_1 = a_2 = 1$ and, for $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$. (This is the **Fibonacci sequence**.)

Long term behaviour: In Examples 2.1.2 and 2.1.6 the terms get huge, with no bound on their size (we shall say that they **tend to** ∞).
However, for 2.1.3, the 100th term is $\dfrac{101}{100} = 1 + \dfrac{1}{100}$, the 1000th term is $\dfrac{1001}{1000} = 1 + \dfrac{1}{1000}$. It looks as though the terms are getting closer and closer to 1. (Later we shall express this by saying that $\left(\dfrac{n+1}{n}\right)_{n\in\mathbb{N}}$ **converges** to 1.)
In Example 2.1.4 , the terms alternate between -1 and 1 so don't converge to a single value.

In Example 2.1.5, the terms alternate between being positive and negative, but they are also getting very small in *absolute* value (i.e. in their size when we ignore the minus sign): so $\left(\frac{(-1)^n}{3^n}\right)_{n\in\mathbb{N}}$ converges to 0.

Before giving the precise definition of "convergence" of a sequence, we require some technical properties of the modulus (i.e. absolute value) function.

We now make the convention that unless otherwise stated, all variables (x, y, l, ϵ, δ ...) range over the set $\mathbb{R}$ of real numbers.

2.2 The Triangle Inequality

We define the **modulus**, $|x|$ of x, also called the **absolute value** of x.

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Note that $|x| = \max\{x, -x\}$, so $x \leq |x|$ and $-x \leq |x|$.

Theorem 2.2.1 (**The Triangle Inequality**). *For all x, y, we have*

$$|x + y| \leq |x| + |y|.$$

Proof. We have $x \leq |x|$ and $y \leq |y|$. Adding, we get $x + y \leq |x| + |y|$.
Also, $-x \leq |x|$ and $-y \leq |y|$, so $-(x + y) = (-x) + (-y) \leq |x| + |y|$.
It follows that $\max\{x + y, -(x + y)\} \leq |x| + |y|$, i.e. $|x + y| \leq |x| + |y|$, as required. □

Remark: For $x, y \in \mathbb{R}$, $|x - y|$ is the distance from x to y along the "line" $\mathbb{R}$. The triangle inequality is saying that the sum of the lengths of any two sides of a triangle is at least as big as the length of the third side, as is made explicit in the following corollary. (Of course we are dealing here with rather degenerate triangles: the name really comes from the fact that in this form, the triangle inequality is also true for points in the plane.)

Corollary 2.2.2 (Also called the **Triangle Inequality**). *For all a, b, c, we have*

$$|a - c| \leq |a - b| + |b - c|$$

.

Proof. $|a - c| = |(a - b) + (b - c)| \leq |a - b| + |b - c|$ (by Theorem 2.2.1 with $x = (a - b)$ and $y = (b - c)$)). □

Lemma 2.2.3. *Some further properties of the modulus function:*

(a) $\forall x, y \quad |x \cdot y| = |x| \cdot |y|$ *and, if* $y \neq 0$, $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$;

(b) $\forall x, y \quad |x - y| \geq \big||x| - |y|\big|$;

(c) $\forall x, l$ *and* $\forall \epsilon > 0, \quad |x - l| < \epsilon \iff l - \epsilon < x < l + \epsilon$.

Proof. Exercises: (a) - consider the cases; (b), (c) - see the Exercise Sheet for Week 3. □

Now we come to the definition of what it means for a sequence $(a_n)_{n\in\mathbb{N}}$ to converge to a real number l. We want a precise mathematical way of saying that "as n gets bigger and bigger, a_n gets closer and closer to l" (in the sense the distance between a_n and l tends towards 0).

2.3 The Definition of Convergence

Definition 2.3.1. *We say that a sequence* $(a_n)_{n\in\mathbb{N}}$ **converges to the real number** l *if the following holds:*
$\forall\epsilon > 0 \quad \exists N \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$ *with* $n \geq N$ *we have* $|a_n - l| < \epsilon$.

We use various expressions and notations for when this holds:

- a_n **tends to** l as n tends to infinity, written $a_n \to l$ as $n \to \infty$.
- the **limit** of $(a_n)_{n\in\mathbb{N}}$ as n tends to infinity equals l, written $\lim_{n\to\infty} a_n = l$.

Example 2.3.2. Consider the sequence $\left(\dfrac{n+1}{n}\right)_{n\in\mathbb{N}}$ of 2.1.3.
Claim: $\dfrac{n+1}{n} \to 1$ as $n \to \infty$.
Proof Let $\epsilon > 0$ be given. The definition requires us to show that there is a natural number N such that for all $n \geq N$, $\left|\dfrac{n+1}{n} - 1\right| < \epsilon$. So we look for N so that $\forall n \geq N$, $\left|\left(1 + \dfrac{1}{n}\right) - 1\right| < \epsilon$, which is equivalent to requiring that $\forall n \geq N$, $\left|\dfrac{1}{n}\right| < \epsilon$.

The N here will depend on ϵ (in fact in this example, as in most, no choice of N will work for all ϵ, so any choice of value for N will be in terms of ϵ). Since the last inequality is equivalent to $\forall n \geq N$, $\dfrac{1}{\epsilon} < n$ we take N to be any natural number greater than $\dfrac{1}{\epsilon}$, say for definiteness $[\epsilon^{-1}] + 1$.

Example 2.3.3. Now consider the sequence $\left(\dfrac{(-1)^n}{3^n}\right)_{n\in\mathbb{N}}$ of 2.1.5. The *claim* is that $\dfrac{(-1)^n}{3^n} \to 0$ as $n \to \infty$.
Proof: Let $\epsilon > 0$ be given. To prove the claim we must find $N \in \mathbb{N}$ so that $\forall n \geq N$, $\left|\dfrac{(-1)^n}{3^n} - 0\right| < \epsilon$. In fact $N = [\epsilon^{-1}] + 1$ (which implies $\frac{1}{N} < \epsilon$) works here. For suppose that $n \geq N$. Then $\left|\dfrac{(-1)^n}{3^n} - 0\right| = \left|\dfrac{(-1)^n}{3^n}\right| = \dfrac{|(-1)^n|}{|3^n|}$ (by Lemma 2.2.3(a)) $= \dfrac{1}{3^n} \leq \dfrac{1}{n}$ (since it is easy to show (by induction) that $\forall n \in \mathbb{N}$, $3^n \geq n$, and hence that $\dfrac{1}{3^n} \leq \dfrac{1}{n}$). But $\dfrac{1}{n} \leq \dfrac{1}{N} < \epsilon$, and we are done.

The definition of convergence is notoriously difficult for students to take in, and rather few grasp it straight away, especially to the extent of being able to apply it. So here's a different, but equivalent, way of saying the same thing. Maybe one of the definitions might make more sense to you than the other - it can be useful to look at something from different angles to understand it. And, of course, the more examples you do, the more quickly you will get the picture.

By Lemma 2.2.3(c) the condition $|a_n - l| < \epsilon$ is equivalent to saying that $l - \epsilon < a_n < l + \epsilon$. This in turn is equivalent to saying that $a_n \in (l - \epsilon, l + \epsilon)$. So, to say that $a_n \to l$ as $n \to \infty$ is saying that no matter how small an interval we take around l, the terms of the sequence $(a_n)_{n\in\mathbb{N}}$ will eventually lie in it, meaning that, from some point on, *every* term lies in that interval. You might like to look at the formula for N (in terms of ϵ) in Example 2.3.2 and check that (taking $\epsilon = \frac{1}{10}$), $\dfrac{n+1}{n} \in (1 - \frac{1}{10}, 1 + \frac{1}{10})$ for all $n \geq 11$, and that (taking $\epsilon = \frac{3}{500}$)

$\dfrac{n+1}{n} \in (1 - \frac{3}{500}, 1 + \frac{3}{500})$ for all $n \geq 167$ (so, for $\epsilon = \frac{3}{500}$ we can take $N = 167$ or any integer ≥ 167 - the definition of convergence doesn't require us to choose the least N that will work).

Example 2.3.4. This is more of a nonexample really. Consider the sequence $((-1)^n)_{n\in\mathbb{N}}$ (of 2.1.4). Then there is no l such that the terms will eventually *all* lie in the interval $(l - \frac{1}{2}, l + \frac{1}{2})$. This is because if $l \leq \frac{1}{2}$ then the interval does not contain the number 1, yet infinitely many of the terms of the sequence are equal to 1, and if $l \geq \frac{1}{2}$ then the same argument applies to the number -1. Hence there is *no* number l such that $(-1)^n \to l$ as $n \to \infty$. In general, if there is no l such that $a_n \to l$ as $n \to \infty$ then we say that the sequence $(a_n)_{n\in\mathbb{N}}$ **does not converge** or **is divergent**.

Example 2.3.5. Another nonexample. Consider the sequence $(n^2)_{n\in\mathbb{N}}$ of Example 2.1.2. This does not converge either. Here is a rigorous proof of this fact directly using the definition of convergence.

Suppose, for a contradiction, that there is some l such that $a_n \to l$ as $n \to \infty$. Choose $\epsilon = 1$ in Definition 2.3.1. So there must exist some $N \in \mathbb{N}$ such that $|n^2 - l| < 1$ for all $n \geq N$. In particular $|N^2 - l| < 1$ and $|(N+1)^2 - l| < 1$. Therefore $|(N+1)^2 - N^2| = |(N+1)^2 - l + l - N^2| \leq |(N+1)^2 - l| + |l - N^2|$ (by the Triangle Inequality). But each of the terms in the last expression here is less than 1, so we get that $|(N+1)^2 - N^2| < 1 + 1 = 2$. However, $|(N+1)^2 - N^2| = 2N + 1$, so $2N + 1 < 2$, which is absurd since $N \geq 1$.

This last example is a particular case of a general theorem. Namely, if the set of terms of a sequence is not bounded (as is certainly the case for the sequence $(n^2)_{n\in\mathbb{N}}$) then it cannot converge. We develop this remark precisely now.

Definition 2.3.6. *Let S be any non-empty subset of $\mathbb{R}$.*

- *We say that S is* **bounded above** *if there is a real number M such that $\forall x \in S$, $x \leq M$. Such an M is an* **upper bound** *for S.*
- *Similarly, S is* **bounded below** *if there is a real number m such that $\forall x \in S$, $x \geq m$. Such an m is a* **lower bound** *for S.*
- *If S is both bounded above and bounded below, then S is* **bounded**.

Example 2.3.7. (1) Let $S = \{17, -6, -25, 25, 0\}$. Then S is bounded: an upper bound is 25 (or any larger number) and a lower bound is -25 (or any smaller number). In fact one can show easily by induction on the size of X that if X is a non-empty, finite subset of $\mathbb{R}$ then X is bounded.
(2) Intervals like $(a, b]$ or $[a, b]$, etc, for $a < b$ are bounded above, and below, by a, respectively b. However an interval like (a, ∞) is only bounded below.
(3) Let $S = \{x \in \mathbb{R} : x^2 < 2\}$. Since $1.5^2 > 2$ it follows that if $x^2 < 2$ then $-1.5 < x < 1.5$. Of course there are better bounds, but this is certainly sufficient to show that S is bounded.

Applying this to the set $\{a_n : n \in \mathbb{N}\}$ of terms of a sequence we make the following definition.

Definition 2.3.8. *A sequence $(a_n)_{n\in\mathbb{N}}$ is* **bounded** *if there exists $M \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, $|a_n| \leq M$.*

Theorem 2.3.9 (Convergent implies Bounded)**.** *Suppose that $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence (i.e. for some l, $a_n \to l$ as $n \to \infty$). Then $(a_n)_{n\in\mathbb{N}}$ is a bounded sequence.*

Proof. Choose l so that $a_n \to l$ as $n \to \infty$. Now take $\epsilon = 1$ in the definition of convergence. Then there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - l| < 1$.

But $|a_n| = |(a_n - l) + l| \leq |a_n - l| + |l|$ (by the triangle inequality). Thus for all $n \geq N$, $|a_n| \leq 1 + |l|$. So if we take $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |l|\}$ we have that $|a_n| \leq M$ for *all* $n \in \mathbb{N}$, as required. $\square$

We give one last example before we prove some more general theorems about convergence.

Example 2.3.10. Let $a_n = \dfrac{8n^{\frac{1}{3}}}{n^3 + \sqrt{n}}$. Then $a_n \to 0$ as $n \to \infty$.

Proof: Let $\epsilon > 0$ be given. We must find N (which will depend on the given ϵ) such that for all $n \geq N$, $\left| \dfrac{8n^{\frac{1}{3}}}{n^3 + \sqrt{n}} - 0 \right| < \epsilon$, i.e. so that $\dfrac{8n^{\frac{1}{3}}}{n^3 + \sqrt{n}} < \epsilon$ (since the terms are positive, we may remove the modulus signs).

So the game is to find a decent looking upper bound for $\dfrac{8n^{\frac{1}{3}}}{n^3 + \sqrt{n}}$ so that one can easily read off what N must be. Well, we certainly have $n^3 \geq \sqrt{n}$ (for all n), so $\dfrac{8n^{\frac{1}{3}}}{n^3 + \sqrt{n}} \leq \dfrac{8n^{\frac{1}{3}}}{\sqrt{n} + \sqrt{n}} = \dfrac{4n^{\frac{1}{3}}}{\sqrt{n}}$.

(You must get used to tricks like this: replacing an expression in the denominator by a smaller expression and/or replacing the numerator by a larger term always increases the size of the fraction, *provided* everything is positive.)

Now $\dfrac{4 \cdot n^{\frac{1}{3}}}{\sqrt{n}} = \dfrac{4 \cdot n^{\frac{1}{3}}}{n^{\frac{1}{2}}} = \dfrac{4}{n^{\frac{1}{2} - \frac{1}{3}}} = \dfrac{4}{n^{\frac{1}{6}}}$.

So we have that $\dfrac{8n^{\frac{1}{3}}}{n^3 + \sqrt{n}} \leq \dfrac{4}{n^{\frac{1}{6}}}$.

So we will have the desired inequality, namely $\left| \dfrac{8n^{\frac{1}{3}}}{n^3 + \sqrt{n}} - 0 \right| < \epsilon$, provided that $\dfrac{4}{n^{\frac{1}{6}}} < \epsilon$ which, after rearrangement, is provided that $\left(\dfrac{4}{\epsilon}\right)^6 < n$. So to complete the argument we just take N to be any natural number greater than $\left(\dfrac{4}{\epsilon}\right)^6$, say $N = \left[\left(\dfrac{4}{\epsilon}\right)^6\right] + 1$.

2.4 The Completeness Property for $\mathbb{R}$

If we consider the set $S = \{x \in \mathbb{Q} : x^2 < 2\}$ of rational numbers with square less than 2, then we can find upper bounds for S: 10 or, better, 1.5 or, better still, 1.4142135623731, ... but there is no optimal (in the sense of least) upper bound in $\mathbb{Q}$.

Definition 2.4.1. *Let S be a non-empty subset of $\mathbb{R}$ and assume that S is bounded above. Then a real number M is a* **supremum** *or* **least upper bound** *of S if the following two conditions hold:*

(i) M is an upper bound for S, and

(ii) if $M' \in \mathbb{R}$ satisfies $M' < M$ then M' is not an upper bound for S.

It is easy to see that a set S cannot have more than one supremum. For if M_1 and M_2 were both suprema and $M_1 \neq M_2$, then either $M_1 < M_2$ or $M_2 < M_1$ (this is the sort of situation where one automatically uses rule (Ord 1)). But, in either case, this contradicts condition 2.4.1(ii) above.

Thus we have proved:

Lemma 2.4.2. *The supremum of a set S, if it exists, is unique and we then denote it by* $\sup(S)$. □

Lemma 2.4.3. *Let S be a non-empty subset of $\mathbb{R}$ and assume that $M = \sup(S)$ exists. Then $\forall \epsilon \in \mathbb{R}^+$, $\exists x \in S$ such that $M - \epsilon < x \leq M$.*

Proof. Let $\epsilon > 0$ be given. Let $M' = M - \epsilon$. Then $M' < M$, so by 2.4.1(ii), M' is *not* an upper bound for S. So there must be some $x \in S$ such that $x > M'$. Since M is an upper bound for S (by 2.4.1(i)), we also have $x \leq M$. So $M' < x \leq M$, i.e. $M - \epsilon < x \leq M$, as required. □

Example 2.4.4. For $S = \{17, -6, -25, 25, 0\}$ we clearly have that $\sup(S) = 25$. Indeed, if S is any non-empty, finite subset of $\mathbb{R}$ then it contains a greatest element and this element is necessarily its supremum.

Example 2.4.5. However, it need not be the case that a set contains its supremum as a member. Indeed, if $a < b$ then we claim that $\sup(a, b) = b$ despite the fact that $b \notin (a, b)$.

Proof. Let us prove this. Certainly b is an upper bound for (a, b). To see that 2.4.1(ii) is also satisfied, suppose that $M' < b$. Let $c = max\{a, M'\}$. Then certainly $c < b$ and so the *average*, $d = \dfrac{c+b}{2}$ of c and b satisfies $M' \leq c < d < b$. Similarly, $a < d < b$. So d is an element of the set (a, b) which is strictly greater than M'. Hence M' is not an upper bound for (a, b), as required. □

It is now time to introduce the final property that characterises the reals. It is a succinct, but sweeping property of $\mathbb{R}$. It is strong enough to imply facts as diverse as "the number 2 has a square root" on the one hand and apparently easy things like "$\mathbb{N}$ has no upper bound in $\mathbb{R}$" on the other.

Property 2.4.6. *(The* **Completeness** *property of $\mathbb{R}$) (A14) Any non-empty subset of $\mathbb{R}$ which is bounded above has a supremum.*

Remark. We will not prove the completeness property. After all, we've not actually defined what real numbers are, so we can't begin to prove that they have such-and-such a property. The usual route is to carefully define the real numbers in terms of so-called *Dedekind cuts* (an alternative approach is in terms of *Cauchy sequences*). The completeness axiom is then almost part of the definition. We don't have the time to give the details in this course but you can find the construction in most analysis text books.

Let us list some consequences of the completeness property.

Example 2.4.7. Consider now the set $S = \{x \in \mathbb{R} : x^2 < 2\}$ of Example 2.3.7. We have already noted that it is bounded above, for example by 1.5. Thus by Property 2.4.6 it has a supremum, say s. We will see below that $s^2 = 2$ and so s does indeed equal $\sqrt{2}$. (Clearly s is the positive square root of 2 since $1 \in S$, so $1 \leq s$).

So suppose, for a contradiction, that $s^2 \neq 2$. Then either $s^2 > 2$ or $s^2 < 2$. Assume first that $s^2 > 2$. What we want to do is find some (positive) number t that is small enough so that

$(s-t)$ is still an upper bound for S. It might be tempting to take t to be the average $t = \frac{s^2-2}{2}$, but this does not quite work. However the idea is good:

Let $t = \dfrac{s^2-2}{4s}$. Then $t > 0$. Also $t < \dfrac{s^2}{4s} = \dfrac{s}{4}$, so $t < s$ and $s - t > 0$. Then we check that $s - t$ is also an upper bound for S, which contradicts the fact that s is the *least* upper bound for S. The key point is to compute

$$\begin{aligned}(s-t)^2 = \quad & s^2 - 2st + t^2 \geq s^2 - 2st = s^2 - 2s\left(\frac{s^2-2}{4s}\right) \\ & = s^2 - \left(\frac{s^2-2}{2}\right) = \frac{s^2}{2} + 1 > \frac{2}{2} + 1 = 2.\end{aligned}$$

This does it, since if $x \in S$ then $x^2 < 2 < (s-t)^2$. As $(s-t) > 0$, it follows (see the first example sheet) that $x < (s-t)$ contradicting the fact that s is a supremum for S.

We should also dispose of the case $s^2 < 2$; that is left for the exercise sheet.

In fact, by using a similar method (though the details are rather more complicated), one can also show that

- for any $x \in \mathbb{R}$ with $x > 0$, and any natural number n, there exists $y \in \mathbb{R}$, with $y > 0$, such that $y^n = x$. Such a y is unique and is denoted $x^{\frac{1}{n}}$ or $\sqrt[n]{x}$ (the positive real nth root of x). Furthermore,
- for any $x \in \mathbb{R}$, and any *odd* natural number n, there exists a unique $y \in \mathbb{R}$ (of the same sign as x) such that $y^n = x$. Again, we write this y as $x^{\frac{1}{n}}$.

Finally, here is a property of the reals that you probably took for granted. However let's *prove* it using the completeness property.

Example 2.4.8. *$\mathbb{N}$ is not bounded above.*

Proof. To see this suppose, for a contradiction, that $\mathbb{N}$ is bounded above. Then the completeness property says that $\mathbb{N}$ has a supremum, say $s = \sup(S)$. Apply Lemma 2.4.3 with $\epsilon = \frac{1}{2}$. Then the lemma guarantees that there is some $x \in \mathbb{N}$ such that $s - \frac{1}{2} < x$. Adding 1 to both sides we get $s + \frac{1}{2} < x + 1$. But then we have $x + 1 \in \mathbb{N}$ and $s < x + 1$ which contradicts our assumption that s was the supremum of, and hence an upper bound for, S. $\square$

Using similar ideas one can prove:

Example 2.4.9. (i) (See Problems for Week 2) *$\forall x, y \in \mathbb{R}$, if $x < y$ then $\exists q \in \mathbb{Q}$ such that $x < q < y$.*

We should also make the obvious corresponding definition for *lower* bounds for sets of real numbers.

Definition 2.4.10. *Suppose that S is a non-empty subset of $\mathbb{R}$ and that S is bounded below. Then a real number m is an* **infimum** *(or* **greatest lower bound***) of S if*

(i) m is a lower bound for S, and

(ii) if $m' \in \mathbb{R}$ and $m < m'$, then m' is not a lower bound for S.

Again, one can show that the infimum of a set S is unique if it exists, and we write $\inf(S)$ for it when it does exist. One might think that we need a result like Property 2.4.6 to postulate the existence of infima. But we can directly prove from that proposition.

Theorem 2.4.11 (Existence of infima)**.** *Every non-empty subset T of $\mathbb{R}$ which is bounded below has an infimum.*

Proof. See the Problem Sheet for Week 1. As a hint, define $T^- = \{-x : x \in T\}$. Then you should prove that T^- has a supremum, M say. Now show that $-M$ is the infimum of the original set T. $\square$

So, to summarise, we are not going to define the real numbers from first principles - you can find their construction in various sources and, in any case, you are used to dealing with them - but it is the case that everything we need about the reals follows from the axioms that we listed above for an ordered field, together with the completeness property. Indeed, these axioms completely characterise the reals. That is, and this is a rather remarkable fact: *although there are many different fields, and even many different ordered fields, if we add the completeness axiom then there is just one mathematical structure which satisfies all these conditions - namely the reals $\mathbb{R}$.*

2.5 Some General Theorems about Convergence

Theorem 2.5.1 (Uniqueness of Limits)**.** *A sequence can converge to at most one limit.*

Proof. What we have to show is that if $a_n \to l_1$ as $n \to \infty$ and $a_n \to l_2$ as $n \to \infty$, then $l_1 = l_2$.

So suppose that $l_1 \neq l_2$. Say, wlog ("without loss of generality"), that $l_1 < l_2$. Let $\epsilon = \dfrac{l_2 - l_1}{2}$, so $\epsilon > 0$.

Since $a_n \to l_1$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - l_1| < \epsilon$.

Also, since $a_n \to l_2$ as $n \to \infty$, there exists $N' \in \mathbb{N}$ such that for all $n \geq N'$, $|a_n - l_2| < \epsilon$.

Now, let $m = \max\{N, N'\}$.

Then $|a_m - l_1| < \epsilon$ and $|a_m - l_2| < \epsilon$.

But $l_2 - l_1 = |l_2 - l_1| = |(l_2 - a_m) + (a_m - l_1)| \leq |l_2 - a_m| + |a_m - l_1|$ (by the triangle inequality). But the right hand side here is strictly less than $\epsilon + \epsilon$. That is, substituting in our value $\epsilon = \dfrac{l_2 - l_1}{2}$, we obtain $l_2 - l_1 < l_2 - l_1$, a contradiction! $\square$

So a sequence can have at most one limit. But when does it actually have a limit? The next piece of theory provides us with a criterion that covers many particular cases.

Definition 2.5.2. *A sequence $(a_n)_{n\in\mathbb{N}}$ is said to be*

(i) **increasing** *if for all $n \in \mathbb{N}$, $a_n \leq a_{n+1}$;*

(ii) **decreasing** *if for all $n \in \mathbb{N}$, $a_{n+1} \leq a_n$;*

(iii) **strictly increasing** *if for all $n \in \mathbb{N}$, $a_n < a_{n+1}$;*

(iv) **strictly decreasing** *if for all $n \in \mathbb{N}$, $a_{n+1} < a_n$.*

A sequence satisfying any of these four conditions is called **monotone** *or* **monotonic**. *If it satisfies (iii) or (iv) it is called* **strictly monotonic**.

Theorem 2.5.3 (Monotone Convergence Theorem)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence that is both increasing* and *bounded. Then $(a_n)_{n\in\mathbb{N}}$ converges.*

Proof. Since the set $\{a_n : n \in \mathbb{N}\}$ is bounded, it has a supremum, l say. We show that $a_n \to l$ as $n \to \infty$.

So let $\epsilon > 0$ be given. We now use Lemma 2.4.3 which tells us that there is some $x \in \{a_n : n \in \mathbb{N}\}$ such that $l - \epsilon < x \leq l$.

Obviously $x = a_N$ for some $N \in \mathbb{N}$. So $l - \epsilon < a_N \leq l$.

Now for any $n \geq N$ we have that $a_n \geq a_N$ (since the sequence $(a_n)_{n\in\mathbb{N}}$ is increasing) and of course we also have that $a_n \leq l$ (since l is certainly an upper bound for the set $\{a_n : n \in \mathbb{N}\}$, being its supremum).

So, for all $n \geq N$, we have $l - \epsilon < a_n \leq l$, and hence $l - \epsilon < a_n \leq l + \epsilon$. But (by 2.2.3(c)) this is equivalent to saying that for all $n \geq N$, we have $|l - a_n| < \epsilon$. Thus $a_n \to l$ as $n \to \infty$ as required. $\square$

Example 2.5.4. Let $a_n = \dfrac{n^2-1}{n^2}$. Then $(a_n)_{n\in\mathbb{N}}$ is an increasing sequence since $a_n = 1 - \dfrac{1}{n^2} \leq 1 - \dfrac{1}{(n+1)^2} = a_{n+1}$. Further, $0 \leq a_n < 1$ for all n, so $(a_n)_{n\in\mathbb{N}}$ is also a bounded sequence. Hence $(a_n)_{n\in\mathbb{N}}$ converges.

In fact it is fairly easy to show directly from the definition of convergence that, with a_n as in the example above, $a_n \to 1$ as $n \to \infty$. However, Theorem 2.5.3 comes into its own in situations where it is far from easy to guess what the limit is:

Example 2.5.5. Let $a_n = \left(\dfrac{n+1}{n}\right)^n = \left(1 + \dfrac{1}{n}\right)^n$. Then one can show (though it's not particularly easy) that $(a_n)_{n\in\mathbb{N}}$ is an increasing, bounded sequence. So it converges. But what is its limit? It turns out to be e (the base for natural logarithms).

2.6 Exponentiation - *a digression*

This subsection is really a digression, so you do not need to remember the details but it uses the ideas we have developed in an interesting way.

There are also some theoretical applications of Theorem 2.5.3 which allow one to give rigorous definitions of some classical functions of analysis. For example, if x and y are positive real numbers what does x^y, or "x to the power y", exactly mean? The answer "x multiplied by itself y times" doesn't really make sense if y is not a natural number! Here we see the idea of how to define this rigorously using Monotone Convergence although, since this is a bit of a digression, some steps will be skipped.

Let x, y be positive real numbers. We aim to give a rigorous definition of the real number x^y. If $y \in \mathbb{Q}$, say $y = \dfrac{n}{m}$ with $n, m \in \mathbb{N}$, then we may use the existence of roots (see Section 2.4) to define x^y to be $\sqrt[m]{x^n}$. But if $y \notin \mathbb{Q}$ how should we proceed? We shall use Question 5 of the Exercise sheet for Week 2, which tells us that we can approximate y arbitrarily closely by rational numbers q. We then show that the x^q converge and we call the limit x^y. The precise details are as follows.

Lemma 2.6.1. *(a) Let $y \in \mathbb{R}$. Then there exists a sequence $(q_n)_{n\in\mathbb{N}}$ which (i) is strictly increasing, (ii) has rational terms (i.e. $q_n \in \mathbb{Q}$ for each $n \in \mathbb{N}$) and (iii) converges to y.*

(b) If q and s are positive rational numbers such that $q < s$, and if $x \geq 1$, then $x^q < x^s$.

Proof. (a) Firstly, let q_1 be any rational number strictly less than y (e.g. $q_1 = [y] - 1$.)

Let q_2 be any rational number satisfying $\max\{q_1, y - \frac{1}{2}\} < q_2 < y$ (using the result from the problem sheet). Now let q_3 be any rational number satisfying $\max\{q_2, y - \frac{1}{3}\} < q_3 < y$ (as above).

We continue: once $q_1 < q_2 < \cdots < q_n < y$ have been constructed, we choose a rational number q_{n+1} satisfying $\max\{q_n, y - \frac{1}{n+1}\} < q_{n+1} < y$.

Clearly our construction has ensured that the sequence $(q_n)_{n\in\mathbb{N}}$ is (strictly) increasing and is bounded above (by y). It therefore converges by the Monotone Convergence Theorem. However, we have also guaranteed that for all $n \in \mathbb{N}$, $y - \frac{1}{n} < q_n < y$, from which it follows that the limit is, in fact, y.[1]

(b) This is left as an exercise; see the solution to Question 4 from the Exercise sheet for Week 3. $\square$

Construction of x^y (outline).

We first use the lemma to construct an increasing sequence $(q_n)_{n\in\mathbb{N}}$ of rational numbers that converges to y. Part (b) of the lemma implies that $(x^{q_n})_{n\in\mathbb{N}}$ is a strictly increasing sequence which is bounded above by x^N where N is any natural number greater than y (e.g. $[y]+1$). Hence, by the Monotone Convergence Theorem, there is some l such that $x^{q_n} \to l \quad \text{as} \quad n \to \infty$. We define x^y to be this l.

We can extend this definition to negative y by the formula $x^{-y} = \dfrac{1}{x^y}$, and we also set $x^0 = 1$. Finally, if $0 < x < 1$, so $x^{-1} \geq 1$, then we define x^y to be $(x^{-1})^{-y}$. We do not give any value to x^y for negative x. $\square$

With these definitions all the usual laws for exponentiation can now be established. One first proves them for rational exponents and then shows that the laws carry over to arbitrary real exponents upon taking the limit described above. This latter process is made easier by developing an "algebra of limits" which we shall do in the next chapter.

The laws of exponentiation being referred to above are as follows (where x is assumed positive throughout).

(E1) $x^{y+z} = x^y \cdot x^z$;

(E2) $(x \cdot y)^z = x^z \cdot y^z$ when $x, y > 0$;

(E3) $(x^y)^z = x^{y \cdot z}$;

(E4) if $0 < x < y$ and $0 < z < t$, then $x^z < y^z$.
Similarly, if $x \geq 1$ and $0 < z < t$, then $x^z < x^t$.

Since E1-E4 encapsulate all we will need to know about exponention, we don't need to refer back to our particular construction. In particular, we will use (E4) a number of times, without particular comment.

[1] Can you see this final step? As a hint, note that for any $\epsilon > 0$ there exists $n \in \mathbb{N}$ with $0 < \dfrac{1}{n+1} < \epsilon$. Now use Lemma 2.2.3(c) as usual.

Chapter 3

The Calculation of Limits

We now develop a variety of methods and general results that will allow us to calculate limits of particular sequences without always having to go back to the original definition.

3.1 The Sandwich Rule

Roughly stated, this rule states that if a sequence is sandwiched between two sequences each of which converges to the same limit, then the sandwiched sequence converges to that limit as well. More precisely:

Theorem 3.1.1 (The Sandwich Rule)**.** *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences and suppose that they converge to the same real number l. Let $(c_n)_{n\in\mathbb{N}}$ be another sequence such that for all sufficiently large n, $a_n \leq c_n \leq b_n$. Then $c_n \to l$ as $n \to \infty$.*

Proof. The hypotheses state that for some $N \in \mathbb{N}$ we have

$$a_n \leq c_n \leq b_n \qquad \text{for all } n \geq N.$$

Now we write down what it means for $\lim\limits_{n\to\infty} a_n = \ell = \lim\limits_{n\to\infty} b_n$. So, fix $\epsilon > 0$. Then there exists N_1 such that if $n \geq N_1$ then $|\ell - a_n| < \epsilon$. Equivalently

$$\ell - \epsilon < a_n < \ell + \epsilon \qquad \text{for all } n \geq N_1.$$

Similarly, there exists N_2 such that if $n > N_2$ then

$$\ell - \epsilon < b_n < \ell + \epsilon \qquad \text{for all } n \geq N_2.$$

Now, let $M = \max\{N, N_1, N_2\}$. Then the displayed inequalities combine to show that:

$$\text{for all } n \geq M \text{ we have } \ell - \epsilon < a_n \leq c_n \leq b_n < \ell + \epsilon.$$

In particular, $\ell - \epsilon < a_n \leq c_n \leq b_n < \ell + \epsilon$, which is what we needed to prove to show that $\lim\limits_{n\to\infty} c_n = \ell$. $\square$

Remark 3.1.2. The Sandwich Rule is often applied when either $(a_n)_{n\in\mathbb{N}}$ or $(b_n)_{n\in\mathbb{N}}$ is a *constant* sequence: if $a \leq c_n \leq b_n$ for sufficiently large n and $b_n \to a$ as $n \to \infty$, then $c_n \to a$ as $n \to \infty$. (Just take $a_n = a$ for all $n \in \mathbb{N}$ in 3.1.1.) Similarly, if $a_n \leq c_n \leq b$ for sufficiently large n and $a_n \to b$ as $n \to \infty$, then $c_n \to b$ as $n \to \infty$.

Definition 3.1.3. *A sequence $(a_n)_{n\in\mathbb{N}}$ is a* **null** *sequence if $a_n \to 0$ as $n \to \infty$.*

Theorem 3.1.4. *(i) If $(a_n)_{n\in\mathbb{N}}$ is a null sequence then so are the sequences $(|a_n|)_{n\in\mathbb{N}}$ and $(-|a_n|)_{n\in\mathbb{N}}$.*
(ii) (Sandwich rule for null sequences.) *Suppose that $(a_n)_{n\in\mathbb{N}}$ is a null sequence. Let $(b_n)_{n\in\mathbb{N}}$ be any sequence such that for all sufficiently large n, $|b_n| \leq |a_n|$. Then $(b_n)_{n\in\mathbb{N}}$ is also a null sequence.*

Proof. (i) This follows from the definitions. For any $\epsilon > 0$, by definition there exists N such that $|a_n| < \epsilon$ for all $n \geq N$. But then $|\,|a_n|\,| = |a_n| < \epsilon$ and so, by definition $\lim\limits_{n\to 0} |a_n| = 0$. The same works for $-|a_n|$ since $|-|a_n|\,| = |a_n|$.

(ii) We are given that for some $N \in \mathbb{N}$ we have $|b_n| \leq |a_n|$ for all $n \geq N$. It follows that for all $n \geq N$, $\;-|a_n| \leq b_n \leq |a_n|$. But by (i) both $(|a_n|)_{n\in\mathbb{N}}$ and $(-|a_n|)_{n\in\mathbb{N}}$ are null sequences. Hence by the Sandwich Rule 3.1.1 (with $l = 0$) it follows that $(b_n)_{n\in\mathbb{N}}$ is null. $\square$

Of course, this is really what we were doing in explicit cases like Question 2 of the Week 3 Exercise Sheet.

Example 3.1.5. (i) $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is null.
(ii) $\left(\dfrac{1}{n^2+n^{\frac{3}{2}}}\right)_{n\in\mathbb{N}}$ is null.
Proof (i) We did this as part of Example 2.3.2.
(ii) For all $n \in \mathbb{N}$, $\left|\dfrac{1}{n^2+n^{\frac{3}{2}}}\right| = \dfrac{1}{n^2+n^{\frac{3}{2}}} \leq \dfrac{1}{n^2} = \left|\dfrac{1}{n^2}\right| \leq \left|\dfrac{1}{n}\right|$. So the result follows from Theorem 3.1.4 and Part (i) (Or you could use Question 2(a) of the Week 3 Exercise Sheet.)

Lemma 3.1.6. *(a) For all $m \geq 5$ we have $2^m > m^2$.*
(b) In particular, if $a_n = 2^{-n}$, then $\lim_{n\to\infty} a_n = 0$.

Proof. (a) You may have seen this in the *Foundations of Pure Mathematics* course.
Base case: For $m = 5$ we have $2^5 = 32 > 25 = 5^2$.

So, suppose for some $k \geq 5$ we have $2^k > k^2$. Then in trying to relate 2^{k+1} and $(k+1)^2$ one gets

$$2^{k+1} = 2 \cdot 2^k \;>\; 2 \cdot k^2 = (k+1)^2 + k^2 - 2k - 1.$$

Now the key point is that, as $k \geq 5$ we have $(k^2 - 2k - 1) = (k-1)^2 - 2 > 2$. So from the displayed equation we get

$$2^{k+1} > (k+1)^2 + \big(k^2 - 2k - 1\big) > (k+1)^2 + 2.$$

Thus, by induction $2^m > m^2$ for all $m \geq 5$.
(b) Of course, you can prove this more directly, but part (a) implies that $0 < a_n = 2^{-n} \leq n^{-2}$. Thus, by Theorem 3.1.4 and Question 2(a) (of Problem Sheet 2) $\lim_{n\to\infty}(a_n) = 0$. $\square$

Example 3.1.7. $\left(\dfrac{1}{2^n+3^n}\right)_{n\in\mathbb{N}}$ is null.
Proof: For all $n \in \mathbb{N}$, we have $\left|\dfrac{1}{2^n+3^n}\right| = \dfrac{1}{2^n+3^n} \leq \dfrac{1}{2^n}$. So the result follows from Theorem 3.1.4 and Lemma 3.1.6.

Example 3.1.8. $\left(\dfrac{n^3}{4^n}\right)_{n\in\mathbb{N}}$ is null.
Proof: By the lemma, $\forall n \geq 5$,

$$\left|\frac{n^3}{4^n}\right| = \frac{n^3}{4^n} = \frac{n^3}{2^n \cdot 2^n} \leq \frac{n^3}{n^2 \cdot n^2} = \frac{1}{n}.$$

So the result follows from 3.1.4 (and the fact that $(\frac{1}{n})$ is null).

3.2 The Algebra of Limits.

Hopefully you now have a feel for testing whether a given sequence is convergent or not. What should also be apparent is that we tend to repeat the same sorts of tricks. When that happens, one should suspect that there are general rules in play. This is the case here and we can convert the sorts of manipulations we have been doing into theorems telling us when certain types of functions converge (or not). The next theorem can be paraphrased as saying that: *Limits of convergent series satisfy the same rules of addition and multiplication as numbers.*

Theorem 3.2.1. *[The Algebra of Limits Theorem] Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be sequences and let a, b be real numbers. Suppose that $\lim_{n\to\infty} a_n = a$ and that $\lim_{n\to\infty} b_n = b$. Then:*

(i) $\lim_{n\to\infty} |a_n| = |a|$;

(ii) for any $k \in \mathbb{R}$, $\lim_{n\to\infty} ka_n = ka$;

(iii) $\lim_{n\to\infty}(a_n + b_n) = a + b$;

(iv) $\lim_{n\to\infty}(a_n \cdot b_n) = a \cdot b$;

(v) if $b_n \neq 0$ (for all n), and $b \neq 0$, then $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$.

(vi) in particular, if $b_n \neq 0$ (for all n), and $b \neq 0$, then $\lim_{n\to\infty} \dfrac{1}{b_n} = \dfrac{1}{b}$.

Remarks: (i) In many of the proofs we are going to want to change ϵ in the definition of convergence, so let's rephrase the definition of convergence as: a sequence (c_n) converges to ℓ if, for all $\eta > 0$ there exists M such that,

$$\text{for all } m \geq M \text{ we have } \ell - \eta < a_m < \ell + \eta.$$

(ii) Secondly, we will frequently use the various versions of the triangle inequality from 2.2.1 and 2.2.2, generally without explicit mention.

Proof. (i) Let $\epsilon > 0$ be given. Choose N so that for all $n \geq N$, $|a_n - a| < \epsilon$. Now by Lemma 2.2.3(b), $\big||a_n| - |a|\big| \leq |a_n - a|$. Hence, for all $n \geq N$, $\big||a_n| - |a|\big| < \epsilon$, as required.

(ii) Firstly, if $k = 0$ the result is obvious since $\lim_{n\to\infty} 0 = 0$. So suppose that $k \neq 0$.

Let $\epsilon > 0$ be given. Here we take $\eta = \dfrac{\epsilon}{|k|}$. Then $\eta > 0$ so we may apply the Remark to obtain $M \in \mathbb{N}$ such that for all $m \geq M$, we have $|a_m - a| < \eta$. Multiplying by $|k|$ (which is > 0) we get that $|k| \cdot |a_n - a| < |k| \cdot \eta = \epsilon$ for all $n \geq M$. Therefore, for $n \geq M$, we have $|ka_n - ka| = |k(a_n - a)| < \epsilon$. This shows that $ka_n \to ka$ as $n \to \infty$, as required.

(iii) Let $\epsilon > 0$ be given. We take (for reasons that will soon become apparent) $\eta = \dfrac{\epsilon}{2}$ in the Remark applied to the two sequences. Thus we can find M_1 such that

$$a - \eta \leq a_n \leq a + \eta \quad \text{for all } n \geq M_1$$

and M_2 such that

$$b - \eta \leq b_n \leq b + \eta \quad \text{for all } n \geq M_2.$$

Adding these equations (and using the triangle inequality as always) we find that, for $N = \max\{M_1, M_2\}$

$$(a + b) - 2\eta \leq (a_n + b_n) \leq (a + b) + 2\eta \quad \text{for all } n \geq N.$$

Since $\epsilon = 2\eta$ this means that, for $N = \max\{M_1, M_2\}$

$$(a+b) - \epsilon \leq (a_n + b_n) \leq (a+b) + \epsilon \quad \text{for all } n \geq N.$$

So $a_n + b_n \to a + b$ as $n \to \infty$, as required.

(iv) This is harder. We first use the fact (Theorem 2.3.9) that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are bounded sequences. So we may choose $B \in \mathbb{R}^+$ such that:

$$\text{for all } n \in \mathbb{N},\ |a_n| \leq B \qquad (1)$$

Now let $\epsilon > 0$ be given. Let $M = \max\{B, |b|\}$ and take $\eta = \dfrac{\epsilon}{2M}$; so $\eta > 0$. Now use the Remark to choose $M_1 \in \mathbb{N}$ such that

$$\forall n \geq M_1, \quad |a_n - a| < \eta \qquad (2).$$

Similarly, choose $M_2 \in \mathbb{N}$ such that

$$\forall n \geq M_2, \quad |b_n - b| < \eta \qquad (3).$$

Let $N = \max\{M_1, M_2\}$. Then for all $n \geq N$ we have

$$\begin{aligned}
|a_nb_n - ab| &= |a_nb_n - a_nb + a_nb - ab| \\
&\leq |a_nb_n - a_nb| + |a_nb - ab| \quad \text{(by the triangle inequality)} \\
&= |a_n||b_n - b| + |a_n - a||b| \quad \text{(since } |xy| = |x|\cdot|y|\text{, see Lemma 2.2.3(a))} \\
&\leq |B||b_n - b| + |a_n - a||b| \quad \text{(by (1))} \\
&\leq |M||b_n - b| + |a_n - a||M| \quad \text{(by the definition of } M\text{)} \\
&< M\cdot\eta + \eta\cdot M \quad \text{(by (3), (2))} \\
&= 2\cdot M\cdot\eta = \epsilon.
\end{aligned}$$

Thus $a_nb_n \to ab$ as $n \to \infty$, as required.

(v) This is immediate from parts (iv) and (vi). Indeed, if $\dfrac{1}{b_n} \to \dfrac{1}{b}$ as $n \to \infty$, then we can then apply part (iv) to conclude that $a_n \cdot \dfrac{1}{b_n} \to a \cdot \dfrac{1}{b}$ as $n \to \infty$.

(vi) We start with:

Claim: There exists $N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, one has $|b_n| > \frac{1}{2}|b|$.

Proof of Claim: Use the Remark with $\eta = \frac{1}{2}|b|$ (which is strictly positive as $|b| \neq 0$). This gives $N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, we have $|b| - \frac{1}{2}|b| < |b_n| < |b| + \frac{1}{2}|b|$, which is what we wanted.

We now complete the proof. So let $\epsilon > 0$ be given.

Let $\eta = \dfrac{|b|^2}{2}\cdot\epsilon$. Then $\eta > 0$ and hence by the Remark there exists $N_2 \in \mathbb{N}$ such that, for all $n \geq N_2$, we have $|b_n - b| < \eta$. Now let $N = \max\{N_1, N_2\}$. Then, for $n \geq N$ we have

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b_nb}\right| = \frac{|b - b_n|}{|b_n||b|}$$

However, $|b_n| > \dfrac{|b|}{2}$ (because $n \geq N_1$) and hence $|b_n|\cdot|b| > \dfrac{|b|^2}{2}$. Therefore

$$\frac{|b - b_n|}{|b_n||b|} < \frac{2|b - b_n|}{|b|^2} < \frac{2\cdot\eta}{|b|^2} \quad \text{(because } n \geq N_2\text{)}.$$

Since $\eta = \dfrac{|b|^2}{2} \cdot \epsilon$ we obtain that $\left|\dfrac{1}{b_n} - \dfrac{1}{b}\right| < \epsilon$ for all $n \geq N$, as required. $\square$

Theorem 3.2.2. *Let $(a_n)_{n\in\mathbb{N}}$ be a null sequence and let $(b_n)_{n\in\mathbb{N}}$ be a bounded sequence (not necessarily convergent). Then $(a_n \cdot b_n)_{n\in\mathbb{N}}$ is a null sequence.*

Proof. Exercise. $\square$

Example 3.2.3. For any fixed positive real number p, $\dfrac{1}{n^p} \to 0$ as $n \to \infty$.

Proof: Let $\epsilon > 0$ be given. We want to show that $\dfrac{1}{n^p} < \epsilon$ for all large n. But

$$\frac{1}{n^p} < \epsilon \iff n^p > \frac{1}{\epsilon} \iff n > \left(\frac{1}{\epsilon}\right)^{1/p},$$

(where the final equivalence uses E4 of Section 2.6). So, we take N to be $[\epsilon^{-\frac{1}{p}}] + 1$. Thus, if $n \geq N$ then $n > \epsilon^{-\frac{1}{p}}$ and so the above computation shows that $\dfrac{1}{n^p} < \epsilon$. Therefore $\left|\dfrac{1}{n^p} - 0\right| < \epsilon$ and hence $\lim_{n\to 0} \dfrac{1}{n^p} = 0$.

Aside: By the way, if you do not want to use E4 here, you could also use the argument from the solutions to the Exercise sheet for Week 3. So, (using that remark) if we take $q = [p]$ then $q \in \mathbb{Q}^+$ and $n^{-p} \leq n^{-q}$. So (by the Sandwich Theorem again) it is enough to prove Example 3.2.3 for q or, as is the same statement, for p rational. This is the case of E4 proved explicitly on that solution sheet.

Example 3.2.4. $\dfrac{n^2+n+1}{n^2-n+1} \to 1$ as $n \to \infty$

Proof: Divide top and bottom of the nth term by n^2. (This trick, and variations of it, is the main idea in all examples like this.) Thus

$$\frac{n^2+n+1}{n^2-n+1} = \frac{1+\frac{1}{n}+\frac{1}{n^2}}{1-\frac{1}{n}+\frac{1}{n^2}}.$$

We now apply the above example with $p = 1$ and $p = 2$ to deduce that $\frac{1}{n} \to 0$ and that $\frac{1}{n^2} \to 0$ as $n \to \infty$. (Of course, these are also examples we have done several times before!)

So by the Algebra of Limits Theorem 3.2.1(ii, iii)

$$1 + \frac{1}{n} + \frac{1}{n^2} \longrightarrow 1 + 0 + 0 = 1 \quad \text{as } n \to \infty$$

and

$$1 - \frac{1}{n} + \frac{1}{n^2} \longrightarrow 1 + (-1)\cdot 0 + 0 = 1 \quad \text{as } n \to \infty.$$

So by using Algebra of Limits Theorem 3.2.1(vi)) again we obtain that

$$\frac{1+\frac{1}{n}+\frac{1}{n^2}}{1-\frac{1}{n}+\frac{1}{n^2}} \to \frac{1}{1} = 1,$$

that is, $\dfrac{n^2+n+1}{n^2-n+1} \to 1$ as $n \to \infty$.

For many functions given as polynomials or fractions of polynomials we can apply methods as in the above example. It is however very important to use this result (and earlier results) only for *convergent* sequences.

For example, we have seen that (a_n) is not convergent when $a_n = (-1)^n$. This is an example of a bounded sequence that is not convergent (so the converse of Theorem 2.5.3 fails).

There are however cases where we can deduce negative statements.

Example 3.2.5. If $p > 0$ then (n^p) is an unbounded sequence (and hence is not convergent by Theorem 2.3.9).

Proof. Given any ℓ we can certainly find a natural number $n > (\ell)^{1/p}$ since the natural numbers are unbounded. However, using (E4) of Section 2.6 we have $n^p > \ell \iff n > (\ell)^{1/p}$. Thus this also proves that (n^p) is unbounded.

It is worth emphasising that in proving unboundedness (or any counterexample) it's not necessary that every value of n is "bad", just that we can always find a larger "bad" value of n.

Example 3.2.6. (i) If (a_n) is unbounded and (b_n) is convergent, then $(a_n + b_n)$ is unbounded. Similarly, (ka_n) is unbounded whenever $k \neq 0$.

Proof. Since (b_n) is convergent, it is bounded, say $|b_n| < B$ for all n. But now given any ℓ there exists some a_n with $|a_n| > B + \ell$. Hence by the triangle inequality, $|a_n + b_n| > \ell$, as required. The proof for products is similar.

Chapter 4

Some Special Sequences

In the previous chapter we saw how to establish the convergence of certain sequences that were built out of ones that we already knew about. In this chapter we build up a stock of basic convergent sequences that will recur throughout your study of analysis.

4.1 Basic Sequences

Lemma 4.1.1. *For any $c > 0$, $c^{1/n} \to 1$ as $n \to \infty$.*

Proof. This proof has several steps, as follows:

Step I: Prove that $(1+x)^n \geq 1 + nx$ for all $x \geq 1$ and $n \in \mathbb{N}$.

Step II: By taking $x = \dfrac{y}{n}$ in (a), deduce that for all $y > 0$ and $n \in \mathbb{N}$, $(1+y)^{\frac{1}{n}} \leq 1 + \dfrac{y}{n}$.

Step III: Hence show that for fixed $c > 1$, one has $c^{\frac{1}{n}} \to 1$ as $n \to \infty$.

Step IV: Complete the proof.

Proofs of Steps:

Step I: Just use the binomial theorem—which says that

$$(1+x)^n = 1 + nx + (\text{lots of positive terms}).$$

Step II: Take n^{th} roots in Part I and use (E4) of Section 2.6 to get $(1+x) \geq (1+nx)^{1/n}$. Substituting $x = \dfrac{y}{n}$ gives $\left(1 + \dfrac{y}{n}\right) \geq (1+y)^{1/n}$, which is what we want.

Step III: Fix $\epsilon > 0$. For any $c > 1$ we can write $c = 1 + y$ for $y > 0$. Now we choose $N = \left[\dfrac{y}{\epsilon}\right] + 1$, so that $\frac{y}{N} < \epsilon$. Then for any $n \geq N$ we have:

$$\begin{aligned}
\left|c^{1/n} - 1\right| &= c^{1/n} - 1 && \text{since clearly } c^{1/n} > 1 \text{ (or use E4 of Section 2.6)} \\
&= (1+y)^{1/n} - 1 \\
&\leq \left(1 + \frac{y}{n}\right) - 1 && \text{by part II} \\
&= \tfrac{y}{n} \leq \tfrac{y}{N} < \epsilon.
\end{aligned}$$

Step IV: So it remains to consider the case when $0 < c < 1$. But then $d = c^{-1} > 1$ so by Part III, $d^{1/n} \to 1$ as $n \to \infty$. By the Algebra of Limits Theorem 3.2.1(vi), $\dfrac{1}{d^{1/n}} \to \frac{1}{1} = 1$ as $n \to \infty$. Finally, since $\dfrac{1}{d^{1/n}} = c^{1/n}$, we have shown that $c^{1/n} \to 1$ as $n \to \infty$. $\square$

For completeness, recall Question 4 from the Week 4 Example Sheet:

Lemma 4.1.2. *For $0 < c < 1$ we have $c^n \to 0$ as $n \to \infty$.* $\square$

Proof. Here is a different (and much simpler) proof. Let $d = 1/c > 1$ and write $d = 1 + x$, for $x > 0$. Then $d^n = (1+x)^n = 1 + nx + \cdots x^n$ by the binomial theorem. As all the terms are positive, $d^n > nx$. Thus for any number E, we have $d^n > E$ whenever $n \geq E/x$.

Now suppose that ϵ is given. Then $c^n < \epsilon \iff d^n = \dfrac{1}{c^n} > \dfrac{1}{\epsilon}$. So, using the computations of the last paragraph, take $N = 1 + \left[\dfrac{1}{x\epsilon}\right]$, where $x = 1/c - 1$. Then, for $n \geq N$ we have $n > \dfrac{1}{x\epsilon}$ and so $d^n > \dfrac{1}{\epsilon}$ by the last paragraph. In other words, $c^n < \epsilon$, as required. $\square$

We now consider several sequences where it is harder to see what is going on; typically they are of the form $a_n = \dfrac{b_n}{c_n}$ where both b_n and c_n get large (or small) as n gets large. What matters, for the limit, is how quickly b_n grows (or shrinks to 0) in comparison with c_n. Roughly, if $(b_n)_n$ has a higher "order of growth"[1] than c_n then the modulus of the ratio will tend to infinity and there will be no limit; if $(b_n)_n$ and $(c_n)_n$ have roughly the same order of growth, then we might get a limiting value for $(a_n)_n$; if $(c_n)_n$ has the higher order of growth, then $(a_n)_n \to 0$ as $n \to \infty$.

Lemma 4.1.3. *Suppose that (a_n) is a convergent sequence with limit ℓ. For any integer M, let $b_n = a_{n+M}$ (if M happens to be negative, we just take $b_n = 0$ or any other number for $1 \leq n \leq -M$). Then $\lim\limits_{n\to\infty} b_n = \ell$.*

Proof. The point is that (apart from starting at different places) the two sequences are really the same, so ought to have the same limit. More formally, let $\epsilon > 0$. Then we can find $N \in \mathbb{N}$ such that $|\ell - a_n| < \epsilon$ for all $n \geq N$. Hence $|\ell - b_n| = |\ell - a_{n+M}| < \epsilon$ for all for all $n \geq \max\{1, N - M\}$. So the sequence (b_n) converges. $\square$

Lemma 4.1.4. *For any c, $\dfrac{c^n}{n!} \to 0$ as $n \to \infty$.*

Proof. Set $a_n = \dfrac{c^n}{n!}$. Since it suffices to prove that $|a_n| \to 0$ (see Theorem 3.1.4), we can replace c by $|c|$ and assume that $c > 0$. In particular, $a_n > 0$ for all n.

Now, $a_{n+1} = a_n \cdot \dfrac{c}{n+1}$. For all $n \geq 2c$, we have $\dfrac{c}{n+1} < \frac{1}{2}$ and hence $a_{n+1} = a_n \cdot \dfrac{c}{n+1} < \dfrac{a_n}{2}$. So, fix an integer $N \geq c$. Then for all $m > 0$ a simple induction says that $0 < a_{m+N} < \dfrac{a_N}{2^m} = a_N 2^{-m}$.

But we know that $\lim_{m\to 0}(2^{-m}) = 0$ by Lemma 4.1.2. Thus by the Algebra of Limits Theorem 3.2.1 $\lim\limits_{m\to 0} a_N(2^{-m}) = 0$ and by the Sandwich Theorem $\lim\limits_{m\to 0}(a_{m+N}) = 0$. By Lemma 4.1.3, $\lim\limits_{n\to 0}(a_n) = 0$. $\square$

Next, two special sequences where the proofs are harder.

Lemma 4.1.5. *The sequence $(n^{\frac{1}{n}})_{n\in\mathbb{N}}$ converges to 1 as $n \to \infty$.*

Proof. This is not so obvious.

Throughout the argument we consider only $n \geq 2$.

[1] By "order of growth" we mean something coarser than "rate of growth", that is, derivative. (The rate of growth = derivative is, however, relevant to computing limits, see L'Hôpital's Rule which we discuss later - it says that we can compute limits by replacing functions by their derivatives.

Let $k_n = n^{1/n} - 1$. Then E4 of Section 2.6 says that $k_n > 0$ and clearly $n = (1 + k_n)^n$. By the binomial theorem

$$n \;=\; (1 + k_n)^n \;=\; 1 + nk_n + \frac{n(n-1)}{2}k_n^2 + \cdots + k_n^n.$$

Since all the terms are positive, $n > \dfrac{n(n-1)}{2}k_n^2$, and hence $k_n^2 < \dfrac{2n}{n(n-1)} = \dfrac{2}{n-1}$. So, for all $n \geq 2$ we have that $0 < k_n < \dfrac{\sqrt{2}}{\sqrt{n-1}}$.

Now $\dfrac{\sqrt{2}}{\sqrt{n-1}} \to 0$ as $n \to \infty$ (exercise), so by the Sandwich Rule, $k_n \to 0$ as $n \to \infty$. But $n^{\frac{1}{n}} = 1 + k_n$, so by the Algebra of Limits Theorem, $n^{\frac{1}{n}} \to 1$ as $n \to \infty$. $\square$

Lemma 4.1.6. *Fix c with $0 < c < 1$ and fix k. Then* $\lim_{n\to\infty} n^k \cdot c^n = 0$.

Remark. This proof is quite subtle, and it is included for completeness. Once we have L'Hôpital's Theorem, we can give a very quick proof.

Proof. If $k = 0$ then the result is Lemma 4.1.5. Hence the result is also true for $k \leq 0$ by the Sandwich Rule (as $0 < n^k \cdot c^n \leq c^n$ if $k \leq 0$).

So we may assume that $k > 0$.

Let us first assume that $k \in \mathbb{N}$. Now $n^{\frac{1}{n}} \to 1$ as $n \to \infty$ (by 4.1.2). Hence, by the AoL (Algebra of Limits Theorem), $n^{\frac{1}{n}} \cdot n^{\frac{1}{n}} \to 1 \cdot 1 = 1$ as $n \to \infty$. By repeatedly multiplying by $n^{\frac{1}{n}}$ we see that $(n^{\frac{1}{n}})^m \to 1$ as $n \to \infty$ for any positive integer m, and in particular for $m = k$. Another use of AoL gives that $(n^{\frac{1}{n}})^k \cdot c \to c$ as $n \to \infty$ or, to put this another way, $(n^k)^{\frac{1}{n}} \cdot c \to c$ as $n \to \infty$.

Now let $\epsilon = \dfrac{1-c}{2}$ so that $\epsilon > 0$ by our assumption on c.

Choose $N \in \mathbb{N}$ so that for all $n \geq N$ we have that

$$|(n^k)^{\frac{1}{n}} \cdot c - c| < \epsilon.$$

We may rewrite this as

$$c - \epsilon < (n^k)^{\frac{1}{n}} \cdot c < c + \epsilon \qquad (\forall n \geq N).$$

Raising to the power n gives

$$0 < n^k \cdot c^n < (c + \epsilon)^n \qquad (\forall n \geq N).$$

However, $c + \epsilon = c + \dfrac{1-c}{2} = \dfrac{1+c}{2}$ and $\left|\dfrac{1+c}{2}\right| = \dfrac{1+c}{2} < 1$ (as $0 < c < 1$) and so by by Lemma 4.1.2 we have that $(c + \epsilon)^n \to 0$ as $n \to \infty$. Thus by the Sandwich Rule $n^k \cdot c^n \to 0$ as $n \to \infty$ as required.

Now in the case that k is (positive and) not an integer we simply observe that $0 < n^k \cdot c^n \leq n^{[k]+1} \cdot c^n$ and, since $[k] + 1$ *is* an integer we have that $n^{[k]+1} \cdot c^n \to 0$ as $n \to \infty$ by what we've just shown.

Hence $n^k \cdot c^n \to 0$ as $n \to \infty$ by the Sandwich Rule. $\square$

Example 4.1.7. Find: (1) $\lim\limits_{n\to\infty} \sqrt[n]{3n}$.

(2) $\lim\limits_{n\to\infty} (-\frac{1}{2})^n$.

(3) $\lim\limits_{n\to\infty} \dfrac{n!}{n^n}$.

Proofs: (1) Here we can break it up to get $\sqrt[n]{3n} = 3^{1/n}n^{1/n}$. Now Lemmas 4.1.1 and 4.1.5 show that $\lim_{n\to\infty} 3^{1/n} = 1 = \lim_{n\to\infty} n^{1/n}$. Thus by the Algebra of Limits Theorem, $\lim_{n\to\infty} \sqrt[n]{3n} = 1$.
(2) We know from Lemma 4.1.2 that $\lim_{n\to\infty} (\frac{1}{2})^n = 0$, and so Theorem 3.1.4 (or 3.2.2 implies that $\lim_{n\to\infty} \left(\frac{-1}{2}\right)^n = 0$.
(3) This does not follow directly from our results, but think about individual terms:

$$\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} = \frac{1}{n}\frac{2}{n} \cdots \frac{n}{n} \leq \frac{1}{n}.$$

Since all the terms are positive and $\lim_{n\to\infty} \frac{1}{n} = 0$, we can use the Sandwich Theorem to get $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

4.2 New Sequences from Old

Example 4.2.1.

$$\frac{n^{100} + 2^n}{2^n + n^2} \to 1 \text{ as } n \to \infty.$$

The general method with sequences like this is to find the fastest growing term, and then divide top and bottom by it. The fastest-growing term is usually found using the following observation.

If we have a fraction $\frac{a_n}{b_n}$ where we know that $\frac{a_n}{b_n} \to 0$ as $n \to \infty$, then (b_n) has higher order of growth than (a_n).

For example, if we just have powers of n, then the highest positive power grows fastest and we can divide all terms by it. If n occurs in the exponent, as in c^n, then provided that $c > 1$, such a term will always have higher order of growth than any fixed power of n (by 4.1.6). If there are several terms of the form c^n, then the one with the largest c will grow fastest. If, however, there are $n!$'s around (or n^n terms) then they will grow faster than any c^n term, no matter how large the constant c is. Also Example 4.1.7 shows that n^n has higher order of growth than $n!$.

So in this example 2^n is the fastest-growing term. Therefore we divide top and bottom by it:

$$\frac{n^{100} + 2^n}{2^n + n^2} = \frac{\frac{n^{100}}{2^n} + 1}{1 + \frac{n^2}{2^n}}$$

Now $\frac{n^{100}}{2^n} = n^{100} \cdot (\frac{1}{2})^n \to 0$ by Lemma 4.1.6 (with $k = 100$ and $c = \frac{1}{2}$), and $\frac{n^2}{2^n} = n^2 \cdot (\frac{1}{2})^n \to 0$ by 4.1.6 (with $k = 2$ and $c = \frac{1}{2}$).

Hence, by AoL,

$$\frac{n^{100} + 2^n}{2^n + n^2} \to \frac{0 + 1}{1 + 0} = 1 \text{ as } n \to \infty.$$

Example 4.2.2.

$$\frac{10^{6n} + n!}{3 \cdot n! - 2^n} \to \frac{1}{3} \text{ as } n \to \infty.$$

The fastest-growing term is $n!$ so we divide top and bottom by it:

$$\frac{10^{6n} + n!}{3 \cdot n! - 2^n} = \frac{\frac{10^{6n}}{n!} + 1}{3 - \frac{2^n}{n!}} \to \frac{0 + 1}{3 - 0} = \frac{1}{3} \text{ as } n \to \infty.$$

Here we applied Lemma 4.1.4 with $c = 10^6$ to deduce that $\dfrac{10^{6n}}{n!} \to 0$, and again with $c = 2$ to deduce that $\dfrac{2^n}{n!} \to 0$, and finally we applied AoL.

Alternative viewpoint. Another way to think about "relative orders of growth" is to draw up a table.

We will say that a_n goes to zero faster than b_n if $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$. The first table shows common functions ordered by how quickly they slip or plunge to zero.

Goes to zero fastest (top to bottom)	comments
$\dfrac{1}{n^n}$	See 4.1.7
$\dfrac{1}{n!}$	See 4.1.4
c^n for $0 < c < 1$	See 4.1.6 and think of $e^{-n} = \dfrac{1}{e^n}$
$\dfrac{1}{n^p} = n^{-p}$ for $p > 0$	you can put any polynomial in the denominator
$\dfrac{1}{\ln(n)}$	see below
$c^{\frac{1}{n}}$ and $n^{\frac{1}{n}}$ for $c > 0$	These come last as they tend to 1; see 4.1.1 and 4.1.5.

Taking reciprocals we can get a table with fastest-growing terms first.

Fastest-growing (top to bottom)	comments
n^n	See 4.1.7
$n!$	See 4.1.4
d^n for $1 < d$	See 4.1.6 and think of e^n
n^p for $p > 0$	See 3.2.3; you can also take any polynomial here
$\ln(n)$	see below
$c^{\frac{1}{n}}$ and $n^{\frac{1}{n}}$ for $c > 0$	These come last as they tend to 1; see 4.1.1 and 4.1.5.

These tables include $\ln(n)$, which we have yet to discuss but you should have the intuition that "Since exponentials grow faster than any polynomial, so logs grow slower." We show this in the next example.

Example 4.2.3. For any $0 < c$, we have $\lim_{n\to\infty} \dfrac{\ln(n)}{n^c} = 0$.

Proof: Just like Lemma 4.1.6, this is best done using L'Hôpital's rule. You can prove it using (a variant of) Lemma 4.1.6, but it is sufficiently messy I will leave it to L'Hôpital's rule.

Examples 4.2.4. Find the limits of the sequences (a_n) where:

$$(a)\ a_n = \frac{n^3}{3^n} \qquad (b)\ a_n = \frac{3^n}{n^3} \qquad (c)\ a_n = \frac{3^n + n!}{n!} \qquad (d)\ a_n = \frac{3!}{n^3} \qquad (e)\ \frac{n^{28} + 5n^7 + 1}{2^n}.$$

Answers (a) By the table, exponentials grow faster than polynomials (or use 4.1.6), so this tends to zero.

(b) This is really something we deal with in the next chapter, but notice it is the reciprocal of (a). So we would expect that it tends to $\frac{1}{0} = \infty$. This is indeed a true statement, but can you see how to prove it?

(c) $\dfrac{3^n + n!}{n!} = \dfrac{3^n}{n!} + 1$. By the table, $\lim_{n\to\infty} \dfrac{3^n}{n!} = 0$ and so by the AoL Theorem $\lim_{n\to\infty} \left(\dfrac{3^n}{n!} + 1\right) = 1$.

(d) Careful! This is just a constant times $\frac{1}{n^3}$ and so (by the AoL or directly) it tends to zero.

(e) Here the table says that the limit is 0. Note that to prove it using Lemma 4.1.6,, you should write it as $\dfrac{n^{28}}{2^n} + 5\dfrac{n^7}{2^n} + \dfrac{1}{2^n}$. Then 4.1.6, says each fraction has limit zero and so the AoL says that

$$\lim_{n\to\infty} \frac{n^{28} + 5n^7 + 1}{2^n} = 0 + 5 \cdot 0 + 0 = 0.$$

In the next example we will use the following general result:

Lemma 4.2.5. *Suppose that $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence, say with $\lim_{n\to\infty} a_n = \ell$. Let r and s be real numbers such that $r \le a_n \le s$ for all sufficiently large $n \in \mathbb{N}$, say for all $n \ge M$. Then $r \le \ell \le s$.*

Proof. Suppose (for a contradiction) that, $\ell < r$. Let $\epsilon = r - \ell$ and choose N_1 so that for all $n \ge N_1$, $|a_n - \ell| < \epsilon$. Set $N = \max\{N_1, M\}$. Then for any $n \ge N$ we have

$$\ell - \epsilon < a_n < \ell + \epsilon = \ell + (r - \ell) = r.$$

This contradicts the hypothesis that $a_n \ge r$. The same argument shows that $\ell \le s$. $\square$

Here is a more complicated type of example which we will see quite a lot.

Example 4.2.6. Let the sequence $(a_n)_{n\in\mathbb{N}}$ be defined inductively by

$$a_1 = 2,$$

and for $n \ge 1$

$$a_{n+1} = \frac{a_n^2 + 2}{2a_n + 1}.$$

We show that $(a_n)_{n\in\mathbb{N}}$ converges and then find the limit.
To do this we first show that

(A) $\forall n \in \mathbb{N}$, $a_n \ge 1$, and

(B) $\forall n \in \mathbb{N}$, $a_{n+1} \leq a_n$.

Finally,

(C) This forces (a_n) to converge. Now use the AoL to solve an equation for the limit ℓ.

Proof of A: Now obviously $a_1 \geq 1$. So, suppose, for some natural number $n \geq 1$ we have $a_n \geq 1$. Then $a_{n+1} = \dfrac{a_n^2+2}{2a_n+1}$, so $a_{n+1} \geq 1 \iff \dfrac{a_n^2+2}{2a_n+1} \geq 1 \iff a_n^2+2 \geq 2a_n+1$ (here we are using the fact that by induction $2a_n+1$ is positive because a_n is). But

$$a_n^2+2 \geq 2a_n+1 \iff a_n^2-2a_n+1 \geq 0 \iff (a_n-1)^2 \geq 0,$$

which is certainly true. So working back through these equivalences we see that $a_{n+1} \geq 1$ as required.

Remark. It is very important in this sort of argument that one has $\iff$ or at least $\Leftarrow$ since one is trying to prove the first statement using the validity of the final statement. If you just have $\Rightarrow$ you would not be justified in drawing these conclusions.

Proof of B: We have

$$a_{n+1} \leq a_n \iff \frac{a_n^2+2}{2a_n+1} \leq a_n \iff a_n^2+2 \leq 2a_n^2+a_n \iff 0 \leq a_n^2+a_n-2.$$

But $a_n \geq 1$ (by A), so $a_n^2 \geq 1$, and hence $a_n^2+a_n \geq 2$, so we are done.

Step C. So we have now shown that $(a_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence which is bounded (above by 2 and below by 1). Hence, by the Monotone Convergence Theorem (or, rather, by its variant in the Exercises for Week 3) it converges. Let its limit be ℓ. We calculate ℓ explicitly as follows.

We first note that, by Lemma 4.2.5, as $a_n \geq 0$ for all n, then $\ell \geq 0$ also holds.

By repeated use of the Algebra of Limits theorem, we have $a_n^2 \to \ell^2$, $a_n^2+2 \to \ell^2+2$, $2a_n+1 \to 2\ell+1$ and, finally,

$$\frac{a_n^2+2}{2a_n+1} \to \frac{\ell^2+2}{2\ell+1} \text{ as } n \to \infty$$

(note that the denominator is not zero since $\ell \geq 0$.) Thus $\lim_{n\to\infty} a_{n+1} = \dfrac{\ell^2+2}{2\ell+1}$.

However, Lemma 4.1.3 shows that $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n$. So putting these equations together we see that $\dfrac{\ell^2+2}{2\ell+1} = \ell$.

We can now solve this for ℓ: rearranging the equation $\dfrac{\ell^2+2}{2\ell+1} = \ell$ gives $\ell^2+2 = 2\ell^2+\ell$ and hence gives the quadratic equation $\ell^2+\ell-2=0$, or $(\ell+2)(\ell-1)=0$. Thus, either $\ell=1$ or $\ell=-2$. But we know that $\ell \geq 0$, so $\ell \neq -2$ and hence $\ell = 1$. Thus

$$\lim_{n\to\infty} a_n = 1.$$

Remarks 4.2.7. It is important to realise that *the calculation of the limit in Example 4.2.3 was valid only because we already knew that the limit* existed.

To bring this point home, reflect on the following (incorrect!) proof that $(-1)^n \to 0$ as $n \to \infty$ (which we know to be false from Example 2.1.4): Let $a_n = (-1)^n$. Then the sequence $(a_n)_{n\in\mathbb{N}}$ is defined inductively by $a_1 = -1$ and $a_{n+1} = -a_n$. Let $l = \lim_{n\to\infty} a_n$. Then by the Algebra of Limits Theorem we have $\lim_{n\to\infty}(-a_n) = -l$, i.e. $\lim_{n\to\infty} a_{n+1} = -l$. But, as above, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$, so $l = -l$, whence $2l = 0$ and so $l = 0$ as required!

More subtle examples will appear on future Exercise sheets.

4.3 Newton's Method for Finding Roots of Equations - *optional*

This sub-section is an aside that will not be covered in the class; as such, it is not examinable, but you might find it interesting and/or useful.

Suppose that we wish to find a solution to an equation of the form

$$f(x) = 0$$

where $f : \mathbb{R} \to \mathbb{R}$ is some function (e.g. we shall take $f(x)$ to be $x^2 - 2$ below and thereby look for a square root of 2).

Newton's method is as follows.

Let x_1 be a reasonable approximation to a solution of the equation. For $n \geq 1$, let

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Then, under suitable assumptions on x_1 and f, it can be shown that the sequence $(x_n)_{n\in\mathbb{N}}$ will converge to a solution and, further, is a very good way of finding closer and closer approximations to a solution.

Example 4.3.1. Let $x_1 = 1$ and $f(x) = x^2 - 2$. Then Newton's method suggests looking at the sequence defined inductively by

$$x_{n+1} = x_n - \left(\frac{x_n^2 - 2}{2x_n}\right)$$

Simplifying this gives:

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}.$$

Now one can show that this sequences converges but let us just *assume* that here and let λ denote the limit. We check that $\lambda = \sqrt{2}$.

Now one can easily show that $\lambda \neq 0$, so we may apply the Algebra of Limits Theorem to obtain

$$\lim_{n\to\infty}\left(\frac{x_n}{2} + \frac{1}{x_n}\right) = \frac{\lambda}{2} + \frac{1}{\lambda}.$$

But also $x_{n+1} \to \lambda$ as $n \to \infty$, so $\lambda = \frac{\lambda}{2} + \frac{1}{\lambda}$ and it is a simple matter to solve this equation to obtain $\lambda = \sqrt{2}$ as required.

Let us now evaluate a few terms to see how well they approximate $\sqrt{2}$.

We have

$$\begin{aligned}
x_1 &= 1 \\
x_2 &= \frac{1}{2} + \frac{1}{1} = \frac{3}{2} \simeq 1 \cdot 5000000 \\
x_3 &= \frac{3}{4} + \frac{2}{3} = \frac{17}{12} \simeq 1 \cdot 4166667 \\
x_4 &= \frac{17}{24} + \frac{12}{17} = \frac{577}{408} \simeq 1 \cdot 4142156 \\
x_5 &= \frac{577}{816} + \frac{408}{577} \simeq 1 \cdot 4142136.
\end{aligned}$$

In fact, to seven decimal places we have that $\sqrt{2}$ is indeed equal to $1 \cdot 4142136$.

Example 4.3.2. Let us calculate $\sqrt{10}$ by Newton's method. We take $f(x) = x^2 - 10$ and $x_1 = 3$. The Newton sequence for this function is defined inductively by

$$x_{n+1} = x_n - \left(\frac{x_n^2 - 10}{2x_n}\right)$$

which, upon simplification, becomes

$$x_{n+1} = \frac{x_n}{2} + \frac{5}{x_n}.$$

[Check: If $x_n \to \lambda$ as $n \to \infty$, then by the usual argument, $\lambda = \frac{\lambda}{2} + \frac{5}{\lambda}$, which solves to $\lambda = \sqrt{10}$. So the limit, if it exists (which we assume here), is $\sqrt{10}$.]

We have

$$\begin{aligned} x_1 &= 3 \\ x_2 &= \frac{3}{2} + \frac{5}{3} = \frac{19}{6} \simeq 3 \cdot 1666667 \\ x_3 &= \frac{19}{12} + \frac{30}{19} = \frac{721}{228} \simeq 3 \cdot 1622807 \end{aligned}$$

In fact, to seven decimal places, $\sqrt{10} = 3 \cdot 1622776$.

Chapter 5

Divergence

Example: Recall from Theorem 2.3.9 that if a sequence is convergent then it is bounded. We can turn this around and see that certain sequences are not convergent simply because they are not bounded; for example $\left(\frac{e^n}{n^2}\right)_{n\in\mathbb{N}}$ (or some of the examples on the Exercise sheet for Week 6). To see this: we know from Lemma 4.1.6 that $\left(\frac{n^2}{e^n}\right) \to 0$. So for any ϵ; in particular $\epsilon = \frac{1}{K}$ there exists N such that $\frac{n^2}{e^n} < \frac{1}{K}$ for $n \geq N$. Taking reciprocals we get

$$\frac{e^n}{n^2} > K \qquad \text{for all } n \geq N.$$

So the sequence $\left(\frac{e^n}{n^2}\right)_{n\in\mathbb{N}}$ is unbounded and hence not convergent. Of course, this argument is completely general, so we will make a definition and a theorem out of it.

5.1 Sequences that Tend to Infinity

Definition 5.1.1. *A sequence $(a_n)_{n\in\mathbb{N}}$ is* **divergent** *if it is not convergent, i.e. if there is no $l \in \mathbb{R}$ such that $a_n \to l$ as $n \to \infty$.*

Definition 5.1.2. *We say that a sequence $(a_n)_{n\in\mathbb{N}}$* **tends to infinity as n tends to infinity** *if for each positive real number K, there exists an integer N (depending on K) such that for all $n \geq N$, $\;a_n > K$. In this case we write $a_n \to \infty$ as $n \to \infty$, or $\lim_{n\to\infty} a_n = \infty$.*

Example 5.1.3. The sequence $((-1)^n)_{n\in\mathbb{N}}$ is divergent since there is no $l \in \mathbb{R}$ such that $(-1)^n \to l$ as $n \to \infty$. (see Example 2.3.4). But it does not tend to infinity as $n \to \infty$ because one may simply take $K = 1$: there is clearly no N such that for all $n \geq N$, $\;(-1)^n > 1$.

Example 5.1.4. The sequence $(\sqrt{n})_{n\in\mathbb{N}}$ is not bounded and so is divergent (by Theorem 2.3.9). It does tend to infinity as $n \to \infty$. For let $K > 0$ be given. Choose $N = [K^2] + 1$. Then if $n \geq N$ we have that $n \geq [K^2] + 1 > K^2$. Hence $\sqrt{n} > \sqrt{K^2} = K$, as required.

Example 5.1.5. Here is an example one needs to be careful about.

Consider the sequence $((-1)^n \cdot n)_{n\in\mathbb{N}}$. This is clearly not a bounded sequence, so is not convergent. But it does not tend to infinity either. For if it did, taking $K = 1$ we would have an N such that for all $n \geq \mathbb{N}$, $\;(-1)^n \cdot n > 1$. Which is absurd since half the time it is negative.

Theorem 5.1.6 (The Reciprocal Rule). *(i) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-zero real numbers such that $a_n \to \infty$ as $n \to \infty$. Then $\dfrac{1}{a_n} \to 0$ as $n \to \infty$.*
(ii) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-zero real numbers such that for all sufficiently large n, $a_n > 0$. Assume that the sequence $(\dfrac{1}{a_n})_{n\in\mathbb{N}}$ is null. Then $a_n \to \infty$ as $n \to \infty$.

Proof. (i) Suppose that $a_n \to \infty$ as $n \to \infty$ and let $\epsilon > 0$ be given.

Then $\dfrac{1}{\epsilon} > 0$, so taking $K = \dfrac{1}{\epsilon}$ in Definition 5.1.2, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n > K$, i.e. for all $n \geq N$, $a_n > \dfrac{1}{\epsilon}$. Hence for all $n \geq N$, $0 < \dfrac{1}{a_n} < \epsilon$. Thus for all $n \geq N$ we have that $\left|\dfrac{1}{a_n} - 0\right| < \epsilon$, which proves that $\dfrac{1}{a_n} \to 0$ as $n \to \infty$.

(ii) Now suppose that $\dfrac{1}{a_n} \to 0$ as $n \to \infty$. Let $K > 0$ be given.

Then $\dfrac{1}{K} > 0$, so taking $\epsilon = \dfrac{1}{K}$ we see that there exists an N such that for all $n \geq N$, $\left|\dfrac{1}{a_n} - 0\right| < \epsilon$. We may also take N large enough so that we have $a_n > 0$ for all $n \geq N$. Thus for all $n \geq N$, $0 < \dfrac{1}{a_n} < \epsilon = \dfrac{1}{K}$. Therefore for all $n \geq N$, $a_n > K$, so $\lim_{n\to\infty} a_n = \infty$ as required. □

Examples 5.1.7. One may invert the special null sequences of Section 4.1. For example, we have that for any $c > 0$, $n! \cdot c^n \to \infty$ as $n \to \infty$. Similarly, if $c > 1$, we have $\dfrac{c^n}{n^k} \to \infty$ as $n \to \infty$. (In both cases the proof is left as an exercise.)

Consider now the sequence $(n! - 8^n)_{n\in\mathbb{N}}$. We have $\dfrac{1}{n! - 8^n} = \dfrac{1}{n!(1 - \frac{8^n}{n!})} = \dfrac{1}{n!} \cdot \dfrac{1}{(1 - \frac{8^n}{n!})} \to 0 \cdot \dfrac{1}{1-0} = 0$ (by Lemma 4.1.4 with $c = 8$, and AoL). Also, the fact that $1 - \dfrac{8^n}{n!} \to 1$ (as $n \to \infty$), ensures that for sufficiently large n, $n! - 8^n > 0$. Thus, by The Reciprocal Rule 5.1.6(ii), $n! - 8^n \to \infty$ as $n \to \infty$.

We now prove an Algebra of Limits Theorem for sequences tending to infinity.

Theorem 5.1.8. *(i) Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ both tend to infinity. Then*
(a) $a_n + b_n \to \infty$ as $n \to \infty$;
(b) if $c > 0$, then $c \cdot a_n \to \infty$ as $n \to \infty$;
(c) $a_n \cdot b_n \to \infty$ as $n \to \infty$.
(ii) (The Infinite Sandwich Rule.) If $(b_n)_{n\in\mathbb{N}} \to \infty$ as $n \to \infty$, and $(a_n)_{n\in\mathbb{N}}$ is any sequence such that $a_n \geq b_n$ for all sufficiently large n, then $a_n \to \infty$ as $n \to \infty$.

Proof. For (i)(b), let $K > 0$ be given. Then $\dfrac{K}{c} > 0$, so there exists $N \in \mathbb{N}$ such that $a_n > \dfrac{K}{c}$ for all $n \geq N$. Hence $c \cdot a_n > K$ for all $n \geq N$, and we are done.

The rest of the proofs are left as exercises. □

Definition 5.1.9. *We say that a sequence $(a_n)_{n\in\mathbb{N}}$* **tends to $-\infty$ as** *$n \to \infty$, if for all $K < 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n < K$. This is written: $a_n \to -\infty$ as $n \to \infty$.*

[This is easily seen to be equivalent to saying that $-a_n \to \infty$ as $n \to \infty$.]

Example 5.1.10. The sequence $((-1)^n \cdot n)_{n\in\mathbb{N}}$ is unbounded (so does not converge) but neither tends to ∞ nor to $-\infty$ as $n \to \infty$.

Similarly $(8^n - n!) \to -\infty$ as $n \to \infty$. Why? Because this is exactly the same statement as saying that $-(8^n - n!) \to +\infty$ as $n \to \infty$. Which we proved in Examples 5.1.7.

Questions 6A:

(*These will appear on the Exercise sheet for Week 7, but are also relevant to the Coursework Exam.*)

Do the following sequences converge/diverge/tend to infinity or tend to minus infinity?

(a) $\big(\cos(n\pi)\sqrt{n}\big)_{n\in\mathbb{N}}$ (b) $\big(\sin(n\pi)\sqrt{n}\big)_{n\in\mathbb{N}}$

c) $\left(\dfrac{\sqrt{n^2+2}}{\sqrt{n}}\right)_{n\in\mathbb{N}}$ (d) $\left(\dfrac{n^3+3^n}{n^2+2^n}\right)_{n\in\mathbb{N}}$

e) $\left(\dfrac{n^2+2^n}{n^3+3^n}\right)_{n\in\mathbb{N}}$ (f) $\left(\dfrac{1}{\sqrt{n}-\sqrt{2n}}\right)_{n\in\mathbb{N}}$

Chapter 6

Subsequences

Looking at subsequences of a given sequence gives a practical test, Theorem 6.1.3, for non-convergence. These also feature in two of the most important ideas and results in analysis, namely Cauchy sequences and the Bolzano-Weierstrass theorem.

6.1 The Subsequence Test for Non-Convergence

Definition 6.1.1. *Suppose that* $1 \leq k_1 < k_2 < \cdots < k_n < \cdots$ *is a strictly increasing sequence of natural numbers. Then, given any sequence* $(a_n)_{n\in\mathbb{N}}$ *of real numbers we can form the sequence* $(a_{k_n})_{n\in\mathbb{N}}$*. Such a sequence is called a* **subsequence** *of the sequence* $(a_n)_{n\in\mathbb{N}}$*. In other words, a subsequence of* $(a_n)_{n\in\mathbb{N}}$ *is any sequence obtained by leaving out terms (as long as infinitely many are left in).*

Example 6.1.2. The sequence $(4^n)_{n\in\mathbb{N}}$ is a subsequence of the sequence $(n^2)_{n\in\mathbb{N}}$: we take $k_n = 2^n$. Then with $a_n = n^2$, we have $a_{k_n} = (k_n)^2 = (2^n)^2 = 2^{2n} = 4^n$.

Theorem 6.1.3. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence. For any subsequence* $(a_{k_n})_{n\in\mathbb{N}}$ *of the sequence* $(a_n)_{n\in\mathbb{N}}$ *we have:*
(i) if $a_n \to l$ *as* $n \to \infty$*, then* $a_{k_n} \to l$ *as* $n \to \infty$*;*
(ii) if $a_n \to \infty$ *as* $n \to \infty$*, then* $a_{k_n} \to \infty$ *as* $n \to \infty$*;*
(iii) if $a_n \to -\infty$ *as* $n \to \infty$ *then* $a_{k_n} \to -\infty$ *as* $n \to \infty$*.*

Proof. (i) Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ so that for all $n \geq N$, $|a_n - l| < \epsilon$. Now an easy induction shows that for all $n \in \mathbb{N}$ we have $k_n \geq n$. Hence, if $n \geq N$ then $k_n \geq N$, so $|a_{k_n} - l| < \epsilon$. So the same N works for the sequence $(a_{k_n})_{n\in\mathbb{N}}$.
The proofs of (ii) and (iii) are similar. $\square$

As mentioned above, the practical importance of Theorem 6.1.3 is that it gives us a method of proving that certain sequences do not converge. We can either find two subsequences of the given sequence that converge to different limits, or else one subsequence that converges to either ∞ or $-\infty$.

Example 6.1.4. The sequence $\left((-1)^n + \dfrac{1}{n^2}\right)_{n\in\mathbb{N}}$ does not converge. For suppose it does, to l say. Consider the subsequence with $k_n = 2n$. This is the sequence $\left((-1)^{2n} + \dfrac{1}{(2n)^2}\right)_{n\in\mathbb{N}}$, which is $\left(1 + \dfrac{1}{(2n)^2}\right)_{n\in\mathbb{N}}$ and this converges to 1. Hence, by Theorem 6.1.3(i), we must have $l = 1$.

But now consider the subsequence with $k_n = 2n+1$, i.e. $\left((-1)^{2n+1} + \dfrac{1}{(2n+1)^2}\right)_{n\in\mathbb{N}}$, which is $\left(-1 + \dfrac{1}{(2n+1)^2}\right)_{n\in\mathbb{N}}$, and this converges to -1. Hence by 6.1.3(i) we must have $l = -1$, a contradiction!

Example 6.1.5. The sequence $\left(\dfrac{n}{4} - \left[\dfrac{n}{4}\right]\right)_{n\in\mathbb{N}}$ does not converge. For suppose it does, to l say. Consider the subsequence with $k_n = 4n$, i.e. $\left(\dfrac{4n}{4} - \left[\dfrac{4n}{4}\right]\right)_{n\in\mathbb{N}}$. This is the sequence with all terms equal to 0 which of course converges to 0 and hence by 6.1.3(i), $l = 0$. However, consider now the subsequence with $k_n = 4n+1$, i.e. $\left(\dfrac{4n+1}{4} - \left[\dfrac{4n+1}{4}\right]\right)_{n\in\mathbb{N}}$. Note that $\left[\dfrac{4n+1}{4}\right] = \left[n + \dfrac{1}{4}\right] = n$. So this subsequence has all terms equal to $\dfrac{1}{4}$, and therefore it converges to $\dfrac{1}{4}$. So by 6.1.3(i), $l = \dfrac{1}{4}$, a contradiction.

Example 6.1.6. The sequence $\left(n\sin\left(\dfrac{n\pi}{2}\right)\right)_{n\in\mathbb{N}}$ neither converges, nor tends to ∞ nor tends to $-\infty$. For consider the subsequence with $k_n = 4n+1$, i.e. $\left((4n+1)\sin\left(\dfrac{(4n+1)\pi}{2}\right)\right)_{n\in\mathbb{N}}$. Now $(4n+1)\sin\left(\dfrac{(4n+1)\pi}{2}\right) = (4n+1)\sin\left(2n\pi + \dfrac{\pi}{2}\right) = (4n+1)\sin\left(\dfrac{\pi}{2}\right) = 4n+1$. So this subsequence is $(4n+1)_{n\in\mathbb{N}}$ which tends to ∞ as $n \to \infty$.

But now consider the subsequence with $k_n = 4n$, i.e. the sequence $\left(4n\sin\left(\dfrac{4n\pi}{2}\right)\right)_{n\in\mathbb{N}}$. Every term here is 0 (since $\sin(2n\pi) = 0$ for all $n \in \mathbb{N}$). So this subsequence converges to 0. So by 6.1.3 (i), (ii) and (iii), the original sequence does not converge and nor does it tend to ∞ or $-\infty$.

Example 6.1.7. Does $\lim\limits_{n\to\infty}\left(\left[\sqrt{n}\right] - \sqrt{n}\right)$ exist?

Hint: you may assume that $[\sqrt{m^2+m}] = m$.

Answer: Set $a_n = [\sqrt{n}] - \sqrt{n}$. We should use subsequences. One subsequence is pretty obvious—if we take $k_n = n^2$ then $a_{k_n} = [\sqrt{n^2}] - \sqrt{n^2} = n - n = 0$, so this subsequence clearly has limit 0.

For the second one we use the hint and take $k_n = n^2 + n$; thus

$$a_{k_n} = \left[\sqrt{n^2+n}\right] - \sqrt{n^2+n} = n - \sqrt{n^2+n}.$$

This is now one of those cases where we use the trick $(x-y) = \dfrac{(x-y)(x+y)}{(x+y)}$. So, in this case

$$\begin{aligned} n - \sqrt{n^2+n} = & \frac{(n-\sqrt{n^2+n})(n+\sqrt{n^2+n})}{(n+\sqrt{n^2+n})} = \frac{n^2-(n^2+n)}{n+\sqrt{n^2+n}} \\ & = \frac{-n}{n+\sqrt{n^2+n}} = \frac{-1}{1+\sqrt{1+1/n}}. \end{aligned}$$

Now, for example from the last Homework set, we know that $\lim_{n\to\infty}(\sqrt{1+1/n}) = \sqrt{1} = 1$. Hence by the AoL the last display has limit $\dfrac{-1}{1+1} = \dfrac{-1}{2}$.

Since we have two subsequences of the original sequence with different limits the original sequence cannot have a limit.

(Here is the proof of the hint: Simply observe that

$$[\sqrt{m^2+m}] = m \iff \sqrt{m^2+m} < m+1 \iff (m^2+m) < (m+1)^2 = m^2+2m+1,$$

which is certainly true!)

6.2 Cauchy Sequences and the Bolzano-Weierstrass Theorem

We conclude this chapter with two fundamental results of analysis. You are not required to know their proofs in this course. However, since the proofs do not require any further knowledge on your part, and since they are such useful results, the detailed proofs are given in the Appendix to this chapter.

Theorem 6.2.1 (The Bolzano-Weierstrass Theorem (1817))**.** *Every bounded sequence (a_n) has a convergent subsequence.*

Remarks: (1) It is important to note that the Theorem is *not* saying that (a_n) is convergent—which is false in general. For instance take our favourite bad example ($a_n = (-1)^n$). So, for this example one would have to take a genuine subsequence. Two obvious examples would be $(a_2 = 1, a_4 = 1, a_6 = 1, ...)$ and $(a_1 = -1, a_3 = -1, a_5 = -1, ...)$
(2) The way to think about the theorem is as a generalisation of the Monotone Convergence Theorem—which says that if one has an *increasing* bounded sequence then it is convergent. So, you can probably guess how one might try to prove this theorem: given a bounded sequence (a_n) then start with a_1, and look for some $a_{k_2} \geq a_1$ and induct. If this is not possible then try to do the same with descending sequences. In fact the proof needs to be a bit more subtle, but it develops from this idea.

We now come to the problem of giving a necessary and sufficient condition on a sequence for it to converge, without actually knowing what the limit might be. The following definition is crucial and will feature in many different guises in your future Analysis and Topology courses.

Definition 6.2.2. *A sequence $(a_n)_{n\in\mathbb{N}}$ is called a* **Cauchy sequence** *if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $i, j \in \mathbb{N}$ with $i, j \geq N$, we have $|a_i - a_j| < \epsilon$.*

So this is saying that the terms of the sequence $(a_n)_{n\in\mathbb{N}}$ are getting closer and closer together. Then we have a central theorem.

Theorem 6.2.3. *A sequence converges if and only if it is a Cauchy sequence.*

It is important in the definition of a Cauchy sequence that we are saying that *all* the terms get close together. For example if one defined a sequence inductively by $a_1 = 1$ and $a_n = a_{n-1} + \frac{1}{n}$, then certainly for all $\epsilon > 0$ we can find N such that $|a_n - a_{n+1}| < \epsilon$ for $n \geq N$. But, in fact this sequence is not convergent (we have seen an informal proof of this already and will see it again when we consider infinite series). So, the last theorem would not be true if we just assumed that *successive* terms got close together.

6.2.1 Proofs for the section - *optional*

Here are detailed proofs of the Bolzano-Weierstrass Theorem and the resulting theorem on Cauchy sequences. As mentioned above, *these proofs are not required in this course.* But the results are so useful that, if you have the time do go through them, it will set you up nicely for when you study real and complex analysis and metric spaces next year.

Theorem 6.2.4. (The Bolzano-Weierstrass Theorem). *Any bounded sequence (a_n) possesses a convergent subsequence. In other words there exist a sequence of numbers*

$$k_1 < k_2 \cdots < k_r < k_{r+1} < \cdots$$

such that the subsequence $(a_{n_k})_{k\in\mathbb{N}}$ is convergent.

Proof. Since (a_n) is bounded, it has a supremum $M = \sup\{a_n : n \in \mathbb{N}\}$ and an infimum $N = \inf\{a_n : n \in \mathbb{N}\}$. (see Chapter 2). The proof is quite sneaky; what we will do is construct a chain of real numbers

$$M_1 = M \geq M_2 \cdots \geq M_r \cdots \geq N$$

and a second sequence

$$N_1 = M_1 - 1,\ N_2 = M_2 - \frac{1}{2},\ N_3 = M_3 - \frac{1}{3},\ \cdots,\ N_r = M_r - \frac{1}{r} \cdots$$

such that some element a_{k_r} (with $k_r > k_{r-1}$) is squeezed between them: $N_r \leq a_{k_r} \leq M_r$. This will do. The reason is that by the Monotone Convergence Theorem, the $\{M_r\}$ have a limit, say μ. Then one shows that $\lim\limits_{r\to\infty} N_r = \mu$ as well and hence by the Sandwich Theorem $\lim\limits_{r\to\infty} a_{k_r} = \mu$ as well.

OK, let's see the details. As we said, we take $M_1 = M$ and $N_1 = M_1 - 1$. The point here is that N_1 is *not* an upper bound for $\{a_n\}$ and so there exists some a_{k_1} with $N_1 < a_{k_1} \leq M_1$. Now we let $M_2 = \sup\{a_n : n > k_1\}$. We note here that any upper bound for the $\{a_n\}$ must also be an upper bound for the smaller set $\{a_n : n > k_1\}$. Thus $M_2 \leq M_1$. Once again we set $N_2 = M_2 - \frac{1}{2}$. Then since N_2 is not an upper bound for $\{a_n : n > k_1\}$, there exists some a_{k_2} with $k_2 > k_1$ such that $N_2 < a_{k_2} \leq M_2$.

Now we induct. Suppose that we have found $(M_1, N_1, a_{k_1}, \cdots M_r, N_r, a_{k_r})$ in this way; thus they satisfy $N_i = M_i - \frac{1}{i} < a_{k_i} \leq M_i = \sup\{a_j : j > k_{i-1}\} \leq M_{i-1}$ and $k_i > k_{i-1}$ for each $1 < i \leq r$.

Then we set $M_{r+1} = \sup\{a_j : j > k_r\}$ and $N_{r+1} = M_{r+1} - \dfrac{1}{r+1}$. Then as N_{+1} is not an upper bound of $\{a_j : j > k_r\}$ there exists $N_{r+1} < a_{k_{r+1}} \leq M_{r+1}$ with $k_{r+1} > k_r$. This completes the inductive step.

Now, the $\{M_\ell\}$ is a descending sequence and, as each M_ℓ is an upper bound of some collection of the a_p, we see that the $M_\ell \geq a_p$ for some such a_p. Thus the M_ℓ are bounded below by N. Hence the Monotone Convergence Theorem from Q3 of the Examples for Week 3 implies that the limit $\lim\limits_{r\to\infty} M_r$ exists, say $\lim\limits_{r\to\infty} M_r = \mu$.

We next claim that $\lim\limits_{r\to\infty} N_r = \mu$ as well. To see this pick $\epsilon > 0$ and pick P such that if $p \geq P$ then $M_p - \mu = |M_p - \mu| < \epsilon/2$. We also know that (for $q \geq Q = \left[\dfrac{2}{\epsilon}\right] + 1$), we have $N_q \geq M_q - \dfrac{1}{q} \geq M_q - \dfrac{\epsilon}{2}$. By the triangle inequality

$$|N_q - \mu| \leq |N_q - M_q| + |M_q - \mu| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \text{for all } q \geq \max\{P, Q\}.$$

Thus, $\lim\limits_{r\to\infty} N_r = \mu$, as claimed.

Now we are done: since $N_r < a_{k_r} \leq M_r$ for each r, the Sandwich Theorem ensures that $\lim\limits_{r\to\infty} a_{k_r} = \mu$, as well. $\square$

Examples 6.2.5. The Theorem tells us nothing about what is the subsequence $\{a_{k_r}\}$ or its limit μ. Some exercises might give you a feel for the vagaries inherent in the argument.

For the ascending chain $a_n = 1 - \dfrac{1}{n}$ then $M_r = 1$ for all r and the $\{a_{k_r}\}$ is just some (indeed any) subsequence of the $\{a_n\}$. For a sequence like $\left(a_n = (-1)^n \dfrac{1}{n}\right)$ there are lots of possible subsequences— you could take the descending chain $\left(a_{2n} = \dfrac{1}{2n}\right)_{n\in\mathbb{N}}$ for all n or the ascending chain $\left(a_{2n+1} = -\dfrac{1}{2n+1}\right)_{n\in\mathbb{N}}$ for all n or some messier combination of the two. The limit is also not uniquely determined; just think about the case of $(a_n = (-1)^n)$.

In fact one can extend the proof of this theorem a bit to prove the following:

Challenging Exercise: *Prove that every sequence $(a_n)_{n\in\mathbb{N}}$ has a subsequence $(a_{k_n})_{n\in\mathbb{N}}$ which is either increasing (i.e. for all $n \in \mathbb{N}$, $a_{k_n} \leq a_{k_{n+1}}$) or decreasing (i.e. for all $n \in \mathbb{N}$, $a_{k_n} \geq a_{k_{n+1}}$).*

We now consider Cauchy sequences as defined in Definition 6.2.2. We begin with:

Lemma 6.2.6. *Every Cauchy sequence (a_n) is bounded.*

Proof. Take N such that, for $i, j \geq N$ we have $|a_i - a_j| < 1$. In particular, taking $i = N$ we see that

$$a_N - 1 < a_j < a_N + 1 \qquad \text{for all } j \geq N.$$

Thus $a_n \leq \max\{a_1, a_2, \ldots, a_{N-1}, a_N + 1\}$ for all n. The lower bound is similar. $\square$

We know that an increasing sequence (a_n) has a limit if and only if it is bounded (combine the Monotone Convergence Theorem with Theorem 2.3.9). Using Cauchy sequences we can get a general analogue:

Theorem 6.2.7. (The Cauchy Sequence Theorem) *A sequence (a_n) converges if and only if it is a Cauchy sequence.*

Proof. Suppose first that (a_n) is a Cauchy sequence and let $\epsilon > 0$ be given. As so often, we will use our various rules for convergence using $\eta = \dfrac{\epsilon}{2}$. By the Bolzano-Weierstrass Theorem we can at least find a convergent subsequence; say $(a_{k_r} : r \in \mathbb{N})$ with $\lim\limits_{r\to\infty} a_{k_r} = \mu$. If we just had a bounded sequence then there would be no reason why μ would equal $\lim\limits_{n\to\infty} a_n$; just think of our favourite bad sequence $((-1)^n)$. So somehow we have to use the Cauchy condition.

So, let $\epsilon > 0$ be given. In particular, there exists N_0 such that $|a_{k_r} - \mu| < \eta = \dfrac{\epsilon}{2}$ for all $k_r \geq N$. Also, by the Cauchy condition pick N_1 such that $|a_i - a_j| < \eta = \dfrac{\epsilon}{2}$ for all $i, j \geq N_1$. Set $N = \max\{N_0, N_1\}$ Then there is some $k_r > N$ and so, for this k_r we get $|a_{k_r} - a_j| < \dfrac{\epsilon}{2}$ for all $j \geq N$. But we also know that $|a_{k_r} - \mu| < \dfrac{\epsilon}{2}$ for this k_r. By the triangle inequality these combine to give

$$|a_j - \mu| \leq |a_j - a_{k_r}| + |a_{k_r} - \mu| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \text{for all } j \geq N.$$

Thus $\lim\limits_{n\to\infty} a_n = \mu$ as we wanted.

Conversely, suppose that (a_n) converges; say with $\lim_{n\to\infty} a_n = \nu$. Let $\epsilon > 0$ be given. Once again we use $\eta = \dfrac{\epsilon}{2}$. Thus there exists N such that $|a_j - \mu| < \frac{\epsilon}{2}$ for all $j \geq N$. But now we find that for $i, j \geq N$ we have

$$|a_i - a_j| \;\leq\; |a_i - \nu] + |a_j - \nu] \;<\; \frac{\epsilon}{2} + \frac{\epsilon}{2} \;= \epsilon.$$

So we have a Cauchy sequence. $\square$

Chapter 7

L'Hôpital's Rule

We will use L'Hôpital's Rule, which some of you will have seen before. We can't mathematically justify the rule in this course unit, simply because it involves differentiation of functions and we have not yet given the rigorous definition of when a function is differentiable nor of what the derivative is. That will be done in the second year course on Real Analysis. But the rule is very useful, so we will use it.

We consider two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, both tending to infinity. L'Hôpital's Rule gives a method for calculating $\lim_{n\to\infty} \dfrac{a_n}{b_n}$ under certain circumstances.

7.1 L'Hôpital's Rule

Assume that $f : [1,\infty) \to \mathbb{R}$ and $g : [1,\infty) \to \mathbb{R}$ are two functions which can be differentiated at least twice and for some $N > 0$ that $g'(x) \neq 0$ for $x > N$. Set $a_n = f(n)$ and $b_n = g(n)$ for $n \in \mathbb{N}$.

$$\textbf{Assume that} \qquad \lim_{n\to\infty} a_n = \infty = \lim_{n\to\infty} b_n. \qquad (\dagger)$$

Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$, *assuming that the Right Hand Side exists. Here* $g' = \dfrac{dg}{dx}$.

Second Version: *L'Hôpital's Rule also holds if you replace* $(\dagger)$ *by*

$$\textbf{Assume that} \qquad \lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n. \qquad (\dagger\dagger)$$

Since we do not yet have a rigorous definition of the derivative, we're not in a position to have a rigorous proof of the validity of L'Hôpital's Rule, though you may have seen some arguments for it when studying Calculus. Anyway, we will take it as correct (which it is) and see some examples of its use.

Example 7.1.1. Say $a_n = 2n + 3$ and $b_n = 3n + 2$. Take $f(x) = 2x + 3$ and $g(x) = 3x + 2$, so that clearly $\frac{a_n}{b_n} = \frac{f(n)}{g(n)}$ and the hypotheses of the rule are satisfied. Thus L'Hôpital's Rule tells us that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)} = \lim_{n\to\infty} \frac{2}{3} = \frac{2}{3}.$$

Example 7.1.2. Let $a_n = 3n^2 - 4n + 2$ and $b_n = (n + 1)^2$. Here we apply the rule to the functions $f(x) = 3x^2 - 4x + 2$ and $g(x) = (x + 1)^2$, noting that they satisfy the conditions (twice-differentiability and the latter having nonzero derivative for large enough x). We obtain

$$\lim_{n\to\infty} \frac{3n^2 - 4n + 2}{(n + 1)^2} = \lim_{n\to\infty} \frac{6n - 4}{2(n + 1)}$$

.

Still top and bottom go to ∞ as $n \to \infty$. But we may apply the rule again to obtain

$$\lim_{n\to\infty} \frac{6n-4}{2(n+1)} = \lim_{n\to\infty} \frac{6}{2} = 3.$$

Of course, in these examples one may also use the previously discussed method of dividing top and bottom by the fastest-growing term. However, L'Hôpital's Rule really comes into its own when applied to examples like the following. (They do all satisfy the differentiability condition and the condition, on the denominator, of being nonzero for large enough x, though we do not note this explicitly.)

Example 7.1.3. Consider the sequence $\left(\frac{\log n}{n}\right)_{n\in\mathbb{N}}$. Before applying L'Hôpital's Rule we should note that $\log n \to \infty$ as $n \to \infty$.

So we may indeed apply the Rule in this example (with $f(x) = \log x$ and $g(x) = x$) and we see that

$$\lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Example 7.1.4. Similarly (with $f(x) = \log x$ and $g(x) = \log(x+1)$):

$$\lim_{n\to\infty} \frac{\log n}{\log(n+1)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{(n+1)}} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1.$$

Example 7.1.5. And a more complicated example:

$$\lim_{n\to\infty} \frac{\log(4n^5-1)}{\log(n^2+1)} = \lim_{n\to\infty} \left(\frac{20n^4}{4n^5-1} \cdot \frac{n^2+1}{2n}\right) = \lim_{n\to\infty} \frac{10n^3(n^2+1)}{4n^5-1}$$

$$= \lim_{n\to\infty} \frac{10n^5+10n^3}{4n^5-1} = \lim_{n\to\infty} \frac{10+\frac{10}{n^2}}{4-\frac{1}{n^5}} = \frac{10+0}{4-0} = \frac{5}{2}.$$

However, before applying L'Hôpital's Rule you have to be careful to ensure that (†) *holds, since otherwise you may get rubbish:*

Example 7.1.6. Take $a_n = 1 - \frac{1}{n}$ and $b_n = 2 - \frac{1}{n}$. then using the Rule we get

$$\lim_{n\to\infty} \frac{1-\frac{1}{n}}{2-\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{-2}}{n^{-2}} = 1.$$

But even more obviously

$$\lim_{n\to\infty} \frac{1-\frac{1}{n}}{2-\frac{1}{n}} = \frac{1-0}{2-0} = \frac{1}{2}.$$

What went wrong? The problem is that (†) does not hold, since $\lim_{n\to\infty} a_n \neq \infty$. So, the rule should not have been applied.

Finally, L'Hôpital's Rule gives easy proofs of Lemma 4.1.6 and the related result for logs mentioned after the Tables on the Supplement to Section 5:

Example 7.1.7. For any $0 < c < 1$ and k we have $\lim_{n\to\infty} n^k c^n = 0$.

Proof: First, pick a natural number $K \geq k$. Since $n^K c^n \geq n^k c^n$, the Sandwich Rule says we need only prove that $\lim_{n\to\infty} n^K c^n = 0$. Secondly, we rewrite this as $\lim_{n\to\infty} \frac{n^K}{d^n} = 0$ (for $d = \frac{1}{c} > 1$). This ensures that we are in a situation where we can apply L'Hôpital's Rule and gives

$$\lim_{n\to\infty} \frac{n^K}{d^n} = \lim_{n\to\infty} \frac{K}{\ln(d)} \frac{n^{K-1}}{d^n}.$$

By induction on K the RHS has limit 0 (where one starts the induction at $K - 1 = 0$). $\square$

The rule for logs is similar: We are discussing $\lim_{n\to\infty} \frac{\ln(n)}{n^c}$, for any $c > 0$. Again we can apply the rule to get

$$\lim_{n\to\infty} \frac{\ln(n)}{n^c} \;=\; \lim_{n\to\infty} \frac{\frac{1}{n}}{cn^{c-1}} \;=\; \lim_{n\to\infty} \frac{1}{c}\frac{1}{n^c} \;\to\; 0.$$

Part II

Series

Chapter 8

Introduction to Series

In this chapter we give a rigorous definition of *infinite* sums of real numbers. We are interested in expressions of the form

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n + \cdots$$

and need to make sense of such expressions. This is closely related to sequences since this "infinite sum" is (by definition! as you will see) the limit of the *sequence of partial sums* (s_n) where

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots a_n.$$

You have probably seen that $\sum_{i=0}^{\infty} \frac{1}{2^i} = 2$. To see that: an easy induction shows that $s_n = \sum_{i=0}^{n} \frac{1}{2^i} = 2 - \frac{1}{2^n}$, then the limit of the sequence $(s_n)_n$ of partial sums is 2.

However, there are some surprising subtleties in these series. One of the basic tricky ones is the *Harmonic Series*: $\sum_{i=1}^{\infty} \frac{1}{n}$ which equals ∞. That might seem counterintuitive but, to see it, collect terms together to get:

$$\begin{array}{rcccccccccc} \sum_{i=1}^{\infty} \frac{1}{n} & = & 1 & + & \left(\frac{1}{2}\right) & + & \left(\frac{1}{3}+\frac{1}{4}\right) & + & \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) & + \cdots \\ & \geq & 1 & + & \left(\frac{1}{2}\right) & + & \left(\frac{1}{2}\right) & + & \left(\frac{1}{2}\right) & + \cdots . \end{array} \tag{8.1}$$

So this gives an infinite sum of halves, which is therefore infinite. In contrast, we will see that

$$\sum_{i=1}^{\infty} \frac{1}{n^{1.001}} < \infty.$$

So, we are going to need some new techniques!

8.1 The Basic Definitions

For real numbers a_n, an **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n \quad \text{(also written } a_1 + a_2 + a_3 + \cdots + a_n + \cdots\text{)}.$$

This may also be written $\sum\limits_{n\geq 1} a_n$ or even just $\sum a_n$.

Given such a series, we form **the sequence of partial sums** $(s_n)_{n\in\mathbb{N}}$ defined by setting

$$s_n = a_1 + a_2 + \cdots a_n \;=\; \sum_{i=1}^{n} a_i.$$

If $s_n \to s$ as $n \to \infty$, we then say that the series $\sum\limits_{n=1}^{\infty} a_n$ is *convergent with sum* s, and write $\sum\limits_{n=1}^{\infty} a_n = s$. If there is no real number s with this property, that is if the sequence (s_n) does not have a limit, then we say that the series $\sum\limits_{n=1}^{\infty} a_n$ is **divergent**. If the series is divergent but $\lim\limits_{n\to\infty} s_n = \infty$ then we say that $\sum\limits_{n=1}^{\infty} a_n = \infty$ (similarly with $-\infty$).

The Harmonic Series, revisited. Let's first check that the harmonic series (8.1) really does have sum ∞. So, what we have seen is that for any K there exists M (in fact $M = 2^{K+1}$ works) such that $\sum_{n=1}^{M} \dfrac{1}{n}$ gives (more than) a sum of $2K$ copies of $\frac{1}{2}$. In other words, if $m \geq M$ then $s_m = \sum_{n=1}^{m} \dfrac{1}{n} \geq \sum_{n=1}^{M} \dfrac{1}{n} \geq K$. Which is just what one needs to prove to see that $\sum_{i=1}^{\infty} \dfrac{1}{n} = \infty$.

Remark. Sometimes (as in the next example), it is more convenient to start the series/sequence at 0, thus looking at series of the form $\sum_{n=0}^{\infty} = a_0 + a_1 + \cdots$.

Example 8.1.1 (Geometric series). Let r be a real number with $|r| < 1$. Then the series $\sum\limits_{n=0}^{\infty} r^n$ converges with sum $\dfrac{1}{1-r}$.

Proof. Here, $a_n = r^n$ and $s_n = a_0 + a_1 + \cdots + a_n = 1 + r + \cdots + r^n$. So

$$s_n = 1 + r + \cdots + r^n \quad \Rightarrow \quad rs_n = r + r^2 + \cdots + r^{n+1}.$$

Subtracting we obtain

$$(1-r)s_n = 1 - r^{n+1}$$

because all the other terms cancel. Hence

$$s_n = \frac{1 - r^{n+1}}{1-r}.$$

Now since $-1 < r < 1$, $r^{n+1} \to 0$ as $n \to \infty$. (Use 4.1.2.) Hence $\dfrac{1-r^{n+1}}{1-r} \to \dfrac{1-0}{1-r} = \dfrac{1}{1-r}$ (by AoL). Thus $s_n \to \dfrac{1}{1-r}$ as $n \to \infty$. So, by definition, $\sum\limits_{n=0}^{\infty} r^n = \dfrac{1}{1-r}$. $\square$

Remark: This proof only works for $|r| < 1$. It might be tempting to put in other values of r since the right hand side is well defined for any r with $r \neq 1$. But this gives nonsense, e.g. "$\sum_{n=0}^{\infty} 2^n = \frac{1}{1-2} = -1$" is definitely *not* correct.

Example 8.1.2 (Using partial fractions). Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Here $a_n = \dfrac{1}{n(n+1)}$ and $s_n = a_1 + a_2 + \cdots + a_n = \dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots + \dfrac{1}{n(n+1)}$.

A trick that sometimes works is to use partial fractions on the terms. Here we use the identity

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

valid for $x \neq 0, -1$. Hence

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Now all the terms cancel apart from the first and the last, so we obtain

$$s_n = 1 - \frac{1}{n+1} \to 1 - 0 = 1 \text{ as } n \to \infty.$$

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Remark: It is very important in Example 8.1.2 that one only rearranges terms in a *finite* sum rather than rearranging terms in the infinite sum $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. The reason is that, with infinite sums, you can get all sorts of strange different answers by arranging things in different ways—the details can be found in Section 12.2.

It is quite rare for one to be able to find an exact formula for the nth partial sum (i.e. for the number s_n) as we did in these two examples. So, just as we did with the study of sequences, we need to build up a stock of general theorems that will help us to tell when a series does and does not converge without explicitly computing the partial sums. Our first result along these lines states that a series cannot converge unless the terms (i.e. the a_n) form a null sequence. Note, however, that the *converse* of this statement is false in general, as we saw with $\sum_{n=1}^{\infty} \frac{1}{n}$. Here is a similar type of example.

Example 8.1.3 (of non-convergence). Consider the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Here, $a_n = \dfrac{1}{\sqrt{n}}$ and we have shown (see Example 3.2.3) that $(a_n)_{n\in\mathbb{N}}$ is a null sequence. However, the series $\sum_{n=1}^{\infty} a_n$ does not converge. For $s_n = \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}}$, from which we deduce that $s_n \geq$ (number of terms) $\cdot$ (smallest term) $= n \cdot \dfrac{1}{\sqrt{n}} = \sqrt{n}$.

Thus $s_n \geq \sqrt{n}$ for all $n \in \mathbb{N}$, so $s_n \to \infty$ as $n \to \infty$ by the Infinite Sandwich Rule (since $\sqrt{n} \to \infty$ as $n \to \infty$). Hence, by definition, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ does not converge.

Theorem 8.1.4. (The nth term Test.) *If $\sum_{n=1}^{\infty} a_n$ is a convergent series, then $(a_n)_{n\in\mathbb{N}}$ is a null sequence.*

Proof. Suppose that $\sum_{n=1}^{\infty} a_n = s$. Let $s_n = a_1 + \cdots + a_n$ be the nth partial sum. So $s_n \to s$ as $n \to \infty$. Hence $\lim_{n\to\infty} s_{n-1} \to s$ as well (see Lemma 4.1.3), and so by AoL, $\lim_{n\to\infty}(s_n - s_{n-1}) = s - s = 0$. But $s_n - s_{n-1} = a_n$. So $a_n \to 0$ as $n \to \infty$, i.e. $(a_n)_{n\in\mathbb{N}}$ is a null sequence. $\square$

Theorem 8.1.5. (Algebra of Infinite Sums)

(i) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent series, with sums A and B respectively, then the series $\sum_{n=1}^{\infty}(a_n + b_n)$ is also convergent with sum $A + B$.

(ii) If $\sum_{n=1}^{\infty} a_n$ is a convergent series with sum A, and λ is any real number, then the series $\sum_{n=1}^{\infty} \lambda a_n$ is convergent with sum λA.

Proof. (i) For $n \in \mathbb{N}$, let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$ be the partial sums and similarly let $u_n = \sum_{k=1}^{n}(a_k + b_k)$ be the partial sums for the series $\sum_{n=1}^{\infty}(a_n + b_n)$.

Then, rearranging terms,

$$u_n = (\sum_{k=1}^{n} a_k) + (\sum_{k=1}^{n} a_k) = s_n + t_n \quad \text{for } n \in \mathbb{N}.$$

But $s_n \to A$ and $t_n \to B$ as $n \to \infty$ by the definition of convergence of the original series. Thus, by AoL for sequences, $u_n \to A + B$ as $n \to \infty$, i.e. $\sum_{n=1}^{\infty}(a_n + b_n) = A + B$, as required.

The proof of (ii) is left as an exercise. $\square$

Chapter 9

Series with Non-Negative Terms

In this chapter we establish some key facts about series with non-negative terms. The theory of such series is very much easier than the general case where negative terms are allowed.

9.1 The Basic Theory of Series with Non-Negative Terms

Theorem 9.1.1. *Suppose that $a_n \geq 0$ for all $n \in \mathbb{N}$ and let $s_n = \sum_{k=1}^{n} a_k$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the sequence $(s_n)_{n\in\mathbb{N}}$ is bounded above.*

Proof. Note that $s_{n+1} = s_n + a_{n+1} \geq s_n$, since $a_{n+1} \geq 0$. Hence the sequence $(s_n)_{n\in\mathbb{N}}$ is increasing. So, by the Monotonic Convergence Theorem, if $(s_n)_{n\in\mathbb{N}}$ is bounded above, then it converges and thus, by definition, the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Conversely, if $\sum_{n=1}^{\infty} a_n$ is convergent, then (s_n) is convergent (definition!) and so Theorem 2.3.9 says that $(s_n)_{n\in\mathbb{N}}$ is bounded above. $\square$

Theorem 9.1.2 (The Comparison Test)**.** *If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

Equivalently, if $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$ and $t_n = b_1 + b_2 + \cdots + b_n$. Since $a_n \leq b_n$ for all n it follows that $s_n \leq t_n$ for all n. But as $\sum_{n=1}^{\infty} b_n$ is a convergent series of non-negative terms there is, by Theorem 9.1.1, some $M > 0$ such that $t_n \leq M$ for all n. Hence $s_n \leq M$ for all n, and so, again by 9.1.1, $\sum_{n=1}^{\infty} a_n$ is convergent. $\square$

Exercise 9.1.3. Let $(a_n)_{n\in\mathbb{N}}$ be any sequence and let N be any positive integer. Show that the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the series $\sum_{n\geq N} a_n$ (which is the same as the series $\sum_{n=1}^{\infty} a_{n+N-1}$) is convergent.

Remark: This exercise is done in the Exercise sheet for Week 9, and essentially means that for any given test, you need only apply the test for "large n". Let's make this precise with:

Slightly Improved Comparison Test: *Let $N \in \mathbb{N}$. If $0 \le a_n \le b_n$ for all $n \ge N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

Proof. By (9.1.3), $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n\ge N} b_n = \sum_{m\ge 1} b_{N+m-1}$ converges $\Rightarrow \sum_{n\ge N} a_n = \sum_{m\ge 1} a_{N+m-1}$ converges (by the comparison test) $\Rightarrow \sum_{n\ge 1} a_n$ converges (by 9.1.3 again) $\square$

Example 9.1.4. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. For we have that $\frac{1}{(n+1)^2} \le \frac{1}{n(n+1)}$ for all $n \in \mathbb{N}$, and hence by Example 8.1.2 and the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ is convergent. Since this is just the series $\sum_{n\ge 2} \frac{1}{n^2}$, the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ follows from Exercise 9.1.3.

This example shows the value of the Comparison Test: it was straightforward to calculate the partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ and hence calculate explicitly the sum of this series (thereby establishing its convergence). But it is impossible to give a neat formula for the partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, so we resort to comparing its terms with those of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ in order to establish its convergence. In fact, as mentioned in the introduction to this course,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

but the proof of this requires methods from next year's complex analysis course.

Example 9.1.5. For any $p \ge 2$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent. To see this we simply observe that for all $n \in \mathbb{N}$, $\frac{1}{n^p} \le \frac{1}{n^2}$ (since $p \ge 2$) and apply the Comparison Test to the previous example.

Similarly we have:

Example 9.1.6. For any $p \le 1$ the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent. For suppose that $p \le 1$ and, for a contradiction, that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent. Since $\frac{1}{n} \le \frac{1}{n^p}$ for all $n \in \mathbb{N}$ we would have, by the Comparison Test, that $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent. But this contradicts Equation 8.1.

Remark: We've left the case $1 < p < 2$ open but, later, we will show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ but diverges if $p \le 1$.

We now come to an important test for the convergence of series.

Theorem 9.1.7 (The Ratio Test for series.)**.** *Suppose that $a_n > 0$ for all $n \in \mathbb{N}$ and assume that $\frac{a_{n+1}}{a_n} \to l$ as $n \to \infty$.*

(i) If $l < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $l > 1$, then $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

Remark: If $l = 1$, no conclusion can be drawn: For example, if $a_n = \dfrac{1}{n^2}$ or $a_n = \dfrac{1}{n}$, then in both cases we have that $\dfrac{a_{n+1}}{a_n} \to 1$ as $n \to \infty$. But in the first case $\sum\limits_{n=1}^{\infty} a_n$ converges, whereas in the second case it diverges.

Proof. (i) Suppose that $l < 1$. We want to compare our series to the geometric series $\sum r^n$ for some r. We need a little room to manoeuvre and taking $l = r$ wouldn't give us that, so we will take a slightly bigger number for r.

So, choose r so that $l < r < 1$ (e.g. take $r = \dfrac{1+l}{2}$ to be the average of l and 1) and set $\epsilon = r - l$. Then $\epsilon > 0$ so we may choose $N \in \mathbb{N}$ so that $l - \epsilon < \dfrac{a_{n+1}}{a_n} < l + \epsilon$ for all $n \geq N$.

But $\epsilon = r - l$ says that $l + \epsilon = r$ and so, as $a_n > 0$ we get $0 < a_{n+1} < r \cdot a_n$ for $n \geq N$. In particular

$$a_{N+1} < r \cdot a_N, \qquad \textit{and} \qquad a_{N+2} < r \cdot a_{N+1} < r^2 \cdot a_N$$

and so on. So (by induction) we obtain

$$\forall n \geq N, \qquad 0 < a_n < r^{n-N} \cdot a_N$$

Collecting terms we see that

$$\sum_{n \geq N} a_n \;\leq\; \sum_{n \geq N} r^{n-N} \cdot a_N \;=\; \sum_{n \geq 0} r^n \cdot a_N \;=\; a_N \sum_{n \geq 0} r^n. \qquad (*)$$

But as $0 < r < 1$ it follows from Example 8.1.1 that $\sum_{n \geq 0} r^n$ converges. Now we are basically done: by The Algebra of Infinite Sums 8.1.5(ii), $a_N \sum_{n \geq 0} r^n$ converges and so from (*) and the Slightly Improved Comparison Test, $\sum_{n \geq 1} a_n$ converges, as required.

(ii) Now suppose that $l > 1$. In this case we choose r so that $1 < r < l$ and, by following the same procedure as above, we obtain an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n > r^{n-N} \cdot a_N = r^n \cdot (\dfrac{a_N}{r^N})$. But, since $r > 1$, this implies that $a_n \to \infty$ as $n \to \infty$ and, in particular, it implies that $(a_n)_{n \in \mathbb{N}}$ is not a null sequence. So $\sum\limits_{n=1}^{\infty} a_n$ diverges by Theorem 8.1.4. $\square$

Example 9.1.8. Consider the series $\sum\limits_{n=1}^{\infty} \dfrac{n^2}{2^n}$.

Let $a_n = \dfrac{n^2}{2^n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2} \cdot \left(1 + \frac{1}{n}\right)^2.$$

So, by AoL,

$$\frac{a_{n+1}}{a_n} \to \frac{1}{2} \cdot (1+0)^2 = \frac{1}{2} \qquad \text{as } n \to \infty.$$

Since $\frac{1}{2} < 1$, it follows from the Ratio Test that $\sum\limits_{n=1}^{\infty} \dfrac{n^2}{2^n}$ converges.

Example 9.1.9. Let x be any positive real number and consider the series $\sum_{n\geq 0} \dfrac{x^n}{n!}$.

Here, $a_n = \dfrac{x^n}{n!}$, so

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1}.$$

But $\dfrac{x}{n+1} \to 0$ as $n \to \infty$ and $0 < 1$. So by the Ratio Test, $\sum_{n\geq 0} \dfrac{x^n}{n!}$ converges.

Aside: We can define e^x to be the sum $\sum \dfrac{x^n}{n!}$. One can then show (directly from this definition) that $e^{x+y} = e^x \cdot e^y$. This justifies the notation, and also implies its consistency with our previous discussion of exponentiation (see Section 2.6).

Example 9.1.10. (The Bouncing Ball.) Suppose that a rubber ball is dropped from a height of 1 metre and that each time it bounces it rises to a height of (2/3) of the previous height. How far does it travel before it stops bouncing (and yes it does stop)?

Answer: First it drops $1m$. Then it rises up and drops $2/3m$. Then it rises up and drops $(2/3)^2 m$, etc etc. So the total distance travelled is

$$\begin{aligned} & 1 + 2 \times \tfrac{2}{3} + 2 \times \left(\tfrac{2}{3}\right)^2 + 2 \times \left(\tfrac{2}{3}\right)^3 + \cdots \\ = \; & 1 + 2 \times \tfrac{2}{3}\left(1 + \tfrac{2}{3} + \left(\tfrac{2}{3}\right)^2 + \cdots\right) \;=\; 1 + \tfrac{4}{3} \cdot \tfrac{1}{1-2/3} \;=\; 5. \end{aligned}$$

9.2 The Integral Test

We consider a function $f : [1, \infty) \to \mathbb{R}$ which is:

(i) positive (at least, non-negative), so $f(x) \geq 0 \;\; \forall x \geq 1$;

(ii) decreasing, so $f(x) \leq f(y) \;\; \forall \; x \geq y \geq 1$;

(iii) continuous.

Here continuity (meaning being continuous) is a condition you may have seen before, but will be made precise in your analysis course next year. For the record (although you need not remember this) the definition is as follows. A function f is **continuous on** $X = [1, \infty)$ if for all $x \in X$ and $\epsilon > 0$ there exists $\delta > 0$ such that if $y \in R$ with $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. That is, if we choose $x \in X$ and a (small) interval I around $f(x)$, then there is some (perhaps very small) interval around x which is sent inside I by f. Roughly speaking it means that "there are no breaks in the graph of f". This condition rules out functions such as

$$f(x) = \begin{cases} 1 & \text{if } x \leq 2, \\ \frac{1}{2} & \text{if } x > 2. \end{cases}$$

(In this example, try taking $x = 2$ and $\epsilon = \frac{1}{4}$.) Examples of functions $f(x)$ satisfying (i), (ii) and (iii) are $\dfrac{1}{x}$, $\dfrac{1}{x^2}$, $\dfrac{1}{x \log(1+x)}$, 2^{-x}

Theorem 9.2.1 (The Integral Test)**.** *Let $f : [1, \infty) \to \mathbb{R}$ be a function satisfying the three conditions above. Then the series*

$$\sum_{n=1}^{\infty} f(n)$$

converges if and only if the sequence

$$\left(\int_1^n f(x)dx\right)_{n\in\mathbb{N}}$$

converges as $n \to \infty$.

This in turn happens if and only if the sequence $\left(\int_1^n f(x)dx\right)_{n\in\mathbb{N}}$ *is bounded.*

Before giving the proof here are some examples of how the test works.

Example 9.2.2. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$. We shall show that it diverges.

Let $f : [1, \infty) \to \mathbb{R}$ be the function $f(x) = \frac{1}{x}$, which clearly satisfies our three conditions. Now

$$\int_1^n f(x)dx = \int_1^n \frac{1}{x}dx = [\log x]_1^n = \log n - \log 1 = \log n.$$

But (as in Example 7.1.3) $(\log n)_{n\in\mathbb{N}}$ is a divergent sequence, so by the Integral Test, $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series.

Here is the important example which was left partly unresolved in the previous chapter:

Example 9.2.3. If p is a real number with $p > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a convergent series.

Here we take $f(x) = \frac{1}{x^p}$. This clearly satisfies the three conditions for the Integral Test to be applicable. We have

$$\int_1^n f(x)dx \;=\; \int_1^n x^{-p}dx \;=\; \left[\frac{x^{-p+1}}{(-p+1)}\right]_1^n \;=\; \frac{n^{-p+1}}{-p+1} - \frac{1}{-p+1} \;=\; \frac{1}{p-1}\left(1 - \frac{1}{n^{p-1}}\right).$$

Now since $p > 1$, we have that $\frac{1}{n^{p-1}} \to 0$ as $n \to \infty$. So

$$\int_1^n \frac{1}{x^p}dx \to \frac{1}{p-1}(1-0) = \frac{1}{p-1} \quad \text{as } n \to \infty.$$

In particular, the sequence $\left(\int_1^n \frac{1}{x^p}dx\right)_{n\in\mathbb{N}}$ is convergent (with limit $\frac{1}{p-1}$). Hence, by the Integral Test, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent (for $p > 1$).

Exercise 9.2.4. Use the Integral test to prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if $p < 1$. (We also proved this in Example 9.1.6.)

Proof of the Integral Test: First of all, it is clear that $\int_1^n f(x)dx \le \int_1^{n+1} f(x)dx$ and so the sequences of integrals is an increasing sequence. Hence the second paragraph of the Theorem is nothing more than the Monotone Convergence test.

So we look at the first assertion. The basic idea is that the integral $\int_{x=1}^n f(x)dx$ defines the area under that curve for $1 \le x \le n$. This we can approximate by summing the areas of a series of rectangles with width 1. The point of the proof is that this is (close to) the partial sum we want. To make this precise we can either draw it (which I will do in the lectures since it is probably easier to understand) or a give more algebraic proof, as follows.

Let $n \in \mathbb{N}$ and consider the two rectangles A, B in the xy-plane given by

$$A = \{< x, y > \in \mathbb{R}^2 : n \leq x \leq n+1,\ 0 \leq y \leq f(n+1)\},$$

$$B = \{< x, y > \in \mathbb{R}^2 : n \leq x \leq n+1,\ 0 \leq y \leq f(n)\}.$$

If you draw the graph of f, bearing in mind that it is a positive and decreasing function, you will see that the rectangle A is contained in the region between the x-axis and the graph of f, and that B contains this region. So it follows that

$$\text{area}(A) \leq \int_n^{n+1} f(x)dx \ \leq \text{area}(B).$$

But $\text{area}(A) = 1 \cdot f(n+1)$ and $\text{area}(B) = 1 \cdot f(n)$, so

$$f(n+1) \leq \int_n^{n+1} f(x)dx \ \leq f(n) \quad .....(*)$$

and this holds for all $n \in \mathbb{N}$.

We now let $N \in \mathbb{N}$, and we sum the inequalities $(*)$ for $n = 1, \ldots, N-1$ to obtain

$$\sum_{n=1}^{N-1} f(n+1) \leq \int_1^2 f(x)dx + \int_2^3 f(x)dx + \cdots + \int_{N-1}^N f(x)dx \ \leq \sum_{n=1}^{N-1} f(n)$$

which gives, for all $N \in \mathbb{N}$,

$$\sum_{n=2}^{N} f(n) \leq \int_1^N f(x)dx \ \leq \sum_{n=1}^{N-1} f(n) \quad .....(**).$$

After this preparatory work we now prove Theorem 9.2.1.

($\Rightarrow$): Suppose that the series $\sum\limits_{n=1}^{\infty} f(n)$ converges. Then the partial sums are bounded (by 9.1.1, since the terms $f(n)$ are positive). So there is a positive real number M such that for all $N \in \mathbb{N}$ we have that $\sum_{n=1}^{N-1} f(n) \leq M$. Hence by $(**)$, $\int_1^N f(x)dx \leq M$ for all $N \in \mathbb{N}$ and so the sequence $\left(\int_1^n f(x)dx\right)_{n\in\mathbb{N}}$ is a bounded sequence.

But since f is a positive function, an increase in the range over which we integrate f will result in a larger value for the integral. This implies that the sequence $\left(\int_1^n f(x)dx\right)_{n\in\mathbb{N}}$ is an increasing sequence. Thus, by the Monotone Convergence Theorem, it is a convergent sequence as required.

($\Leftarrow$): Suppose that the sequence $\left(\int_1^n f(x)dx\right)_{n\in\mathbb{N}}$ converges. Then it is bounded (by Theorem 2.3.9). Let L be a bound for it. So $\int_1^N f(x)dx \leq L$ for all $N \in \mathbb{N}$. Hence, by $(**)$, $\sum_{n=2}^{N} f(n) \leq L$ for all $N \in \mathbb{N}$. This means that the partial sums of the series $\sum_{n=2}^{\infty} f(n)$ are bounded and so, being a series of positive terms, it converges (by 9.1.1). But then so does the series $\sum_{n=1}^{\infty} f(n)$ (by Exercise 9.1.3), as required.

This completes the proof of the Integral Test.

Remark 9.2.5. Sometimes the integral $\int_{x=1}^n f(x)dx$ can be awkward to compute at the Left Hand End, but in that case, since it is always OK to compute integrals of the form $\int_{x=K}^n f(x)dx$ for some fixed $1 \leq K$ (and then $K \leq n$), it may be that there is a convenient choice for K.

Indeed, in view of 9.1.3, one only has to verify that that the three conditions (i), (ii) and (iii) hold for all *sufficiently large* x, i.e. for all $x \geq K$ (for some given $K \in \mathbb{N}$). The same proof shows that

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \sum_{n=K}^{\infty} f(n) \text{ converges} \iff \left(\int_K^n f(x)dx\right)_{n\geq K} \text{ converges.}$$

Example 9.2.6. The series $\sum_{n=2}^{\infty} \dfrac{n^2}{n^3-1}$ diverges.

Proof: Consider the function $f : [2, \infty) \to \mathbb{R}$ defined by $f(x) = \dfrac{x^2}{x^3-1}$.

Clearly $f(x) > 0$ for $x \geq 2$ and f is continuous. To see that f is decreasing, we can either use calculus or algebra. Using calculus is easier: if $f(x) = \dfrac{x^2}{x^3-1}$ then (after simplification) $f'(x) = (-x^4 - 2x)(x^3-1)^{-2}$. Clearly this is negative for $x > 1$ and so our function $f(x)$ decreases.

For an algebraic proof, suppose that $2 \leq x \leq y$. Then

$$\frac{x^2}{x^3-1} \geq \frac{y^2}{y^3-1} \iff x^2y^3 - x^2 \geq y^2x^3 - y^2$$

$$\iff y^2 - x^2 \geq y^2x^3 - x^2y^3 \iff (y-x)(y+x) \geq y^2x^2(x-y).$$

Since the last statement here is true (for $2 \leq x \leq y$) we can reverse the bi-implications. This shows that f is indeed decreasing.

So we may apply the Integral Test (in the form of 9.2.5). Now $\int_2^n f(x)dx = \int_2^n \dfrac{x^2}{x^3-1}dx$. In order to evaluate the integral we make the substitution $u = x^3 - 1$. Thus $du = 3x^2dx$ and the new limits are $u = 7$ to $u = n^3 - 1$. Thus

$$\int_2^n f(x)dx = \frac{1}{3}\int_7^{n^3-1} \frac{du}{u} = \frac{1}{3}\left[\ln u\right]_7^{n^3-1} = \frac{1}{3}(\ln(n^3-1) - \ln 7).$$

Since $\frac{1}{3}(\ln(n^3-1) - \ln 7) \to \infty$ as $n \to \infty$, the sequence $\left(\int_2^n f(x)\right)_{n\geq 2}$ diverges and hence so does the series $\sum_{n=2}^{\infty} \dfrac{n^2}{n^3-1}$ by the Integral Test.

Remark 9.2.7. *Assume that conditions (i)–(iii) from Theorem 9.2.1 hold. Let $K \in \mathbb{N}$ (usually I would have $K = 1$.) Then integrals of the form $\int_K^{\infty} f(x)dx$ are called* **improper** *integrals and in more detail we have the following.*

For any $K < r < s$ we have

$$0 \leq F(r) = \int_K^r f(x)dx \leq \int_K^r f(x)dxF(r) + \int_r^s f(x)dxF(r) = \int_K^s f(x)dx = F(s),$$

and so by the Monotone Convergence Theorem either

1. *The sequence $\left(\int_K^n f(x)dx\right)_{n\geq K}$ is bounded and hence convergent, in which case we write its limit as $\int_K^{\infty} f(x)dx$ and say that "$\int_K^{\infty} f(x)dx$ converges".*
2. *Alternatively the sequence $\left(\int_K^n f(x)dx\right)_{n\geq K}$ is unbounded and hence tends to infinity. In this case we say "$\int_K^{\infty} f(x)dx$ diverges" or indeed that $\int_K^{\infty} f(x)dx = \infty$.*

So we paraphrase Theorem 9.2.1 as saying that

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_1^{\infty} f(x) \text{ converges.}$$

(Finally, note that talking about improper integrals usually involves a somewhat different kind of limit, namely the limit of the $F(r)$ as the real number r tends to infinity. Also, if the conditions (i)–(iii) do not hold, then you have to be much more careful with the definitions and properties of improper integrals.)

Exercise 9.2.8. If f is a function to which the Integral Test applies and if it tells us that we have convergence, then we certainly **cannot** conclude that $\sum\limits_{n=1}^{\infty} f(n) = \int_1^{\infty} f(x)dx$.

Examine the proof of the Integral Test to show that

$$\sum_{n=2}^{\infty} f(n) \leq \int_1^{\infty} f(x)dx \leq \sum_{n=1}^{\infty} f(n)$$

whenever either side converges. Show, in fact, that we always have *strict* inequality here.

Example 9.2.9. Consider the series $\sum_{n=2}^{\infty} \dfrac{1}{n \ln n}$. To investigate convergence we take $f(x) = \dfrac{1}{x \ln x}$ in the Integral Test and consider the integral $\int_2^n \dfrac{1}{x \ln x} dx$. In order to evaluate it we make the substitution $u = \ln x$. Then $du = \frac{dx}{x}$ and the new range of integration is from $u = \ln 2$ to $u = \ln n$. Thus

$$\int_2^n \frac{1}{x \ln x} dx = \int_{\ln 2}^{\ln n} \frac{du}{u} = [\ln u]_{\ln 2}^{\ln n} = \ln(\ln n) - \ln(\ln 2).$$

Now (exercise) $\ln(\ln n) \to \infty$ as $n \to \infty$ and so the sequence $\left(\int_2^n \dfrac{1}{x \ln x} dx\right)_{n \in \mathbb{N}}$ diverges. Hence, by the Integral Test, the series $\sum_{n=2}^{\infty} \dfrac{1}{n \ln n}$ diverges.

Exercise 9.2.10. Let $p \in \mathbb{R}$. Prove that the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

converges if and only if $p > 1$. What about the series

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n \, (\ln \ln n)^p} \ ?$$

(We take the lower limit of summation to be 3 here because $\ln \ln 2$ is negative and so $(\ln \ln 2)^p$ might not be defined.)

Final Remark: In the integral test you do have to be careful to check that conditions (i - iii) hold. The trouble is that without them it is easy to get silly counterexamples. For example, suppose we take the function $y(x) = 1 + \cos((2n+1)\pi)$; this function has been chosen so that $f(n) = 0$ for all n and $f(x) \geq 0$ for all $x \geq 0$. So, here $\sum f(n) = 0 + \cdots + 0 = 0$ is certainly convergent, whereas the integral $\int_{x=1}^{\infty} f(x)dx = \infty$.

Exercises 9.2.11. No doubt when you first saw the techniques of integration it took a while to get the intuition about which technique to use for which example. The same goes for our techniques for testing the convergence of infinite series $\sum a_n$ (and sequences). The only way to develop this intuition is to do lots of examples. Here are some examples, with partial hints about how to approach them:

Remember the geometric series and the series $\sum n^p$. Can you easily compare the given sequence with these?

If there is a dominant term of the form $n!$ or c^n the Ratio test will probably work. But something like $\sum \frac{n!}{(n+1)!}$ or $\sum \frac{n!}{(n+1)!}$ simplify to $\sum \frac{1}{n+1}$ respectively $\sum \frac{1}{(n+2)(n+1)}$ so nothing is perfect. If $\lim_{n\to\infty} a_n \neq 0$ then the "Test for Divergence" (meaning 9.1.1) will work. If the function can easily be integrated, do so. If the terms have alternating positive and negative terms, see the next chapter.

So, what about:

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$. **Answer:** Diverges by 9.1.1 as $\lim_{n\to\infty} a_n \neq 0$.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$. **Answer:** Converges by comparison with $\sum b_n$ for $b_n = \frac{\sqrt{n^3}}{n^3} = n^{-3/2}$ which converges as $3/2 > 1$. Here you have to put a little work into the comparison test, but, since we want to get convergence we should show that $a_n = \frac{\sqrt{n^3+1}}{3n^3+4n^2+2} \leq \lambda b_n$ for some λ. So certainly

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2} \leq \sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3} = \frac{1}{3}\sum_{n=1}^{\infty} \sqrt{n^{-3}+n^{-6}} \leq \frac{2}{3}\sum_{n=1}^{\infty} \sqrt{n^{-3}}.$$

And, finally this converges by CT and 9.2.3.

(3) $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$. **Answer:** This converges, either by the ratio test or the integral test. By integrating by parts

$$\int_{x=1}^{n} xe^{-x}dx = -(x+1)e^{-x}\Big|_{x=1}^{n} = -(n+1)e^{-n} + 2e^{-1} \to 2e^{-1} < \infty$$

as $n \to \infty$.

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n}$. **Answer:** The Ratio Test will work to show the first converges and the second diverges. (There is an easier test to use for the second one—do you see it?)

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$. **Answer:** Since this is closely related to a geometric series we should use the Comparison Test to compare it to $\sum \frac{1}{3^n}$ and deduce that it converges.

Chapter 10

Series with Positive and Negative Terms

10.1 Alternating Series

We know that for a series $\sum_{n=1}^{\infty} a_n$ to converge it is definitely *not* sufficient that the sequence $(a_n)_{n\in\mathbb{N}}$ of terms be null. (It is necessary, but not sufficient.) For example, $\frac{1}{n} \to 0$ as $n \to \infty$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge. However, it turns out that if the terms a_n decrease in modulus and alternate in sign:

$$a_1 > 0,\ \ a_2 < 0,\ \ a_3 > 0, \ldots$$

then it is both necessary and sufficient for the convergence of $\sum_{n=1}^{\infty} a_n$ that $(a_n)_{n\in\mathbb{N}}$ be null.

For example, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *does* converge as the following argument suggests:

We have

$$\begin{aligned}
&1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n+1} - \frac{1}{2n+2} + \cdots \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) + \cdots \\
&= \frac{1}{2} + \frac{1}{12} + \cdots + \frac{1}{(2n+1)(2n+2)} \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)},
\end{aligned}$$

and this series converges by the Comparison Test ($\frac{1}{(2n+1)(2n+2)} \leq \frac{1}{(n+1)^2}$ for all n).

(I say "suggests" here because the argument is not completely rigorous. Why not?) For an argument which gives the sum exactly, see the end of the chapter.

Theorem 10.1.1 (The Alternating Series Test)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a decreasing sequence of positive terms such that $a_n \to 0$ as $n \to \infty$. Then the series*

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

Proof. Let

$$s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$$

be a typical partial sum. Then

$$s_{2n} = a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} - a_{2n}$$

$$= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n}.$$

Notice that all the terms in brackets here are non-negative since the sequence $(a_n)_{n\in\mathbb{N}}$ is decreasing. Also $a_{2n} > 0$. It follows that for all $n \in \mathbb{N}$, $s_{2n} \leq a_1$.

Also $s_{2(n+1)} = s_{2n} + a_{2n+1} - a_{2n+2}$. Since $a_{2n+1} \geq a_{2n+2}$, we obtain that $s_{2(n+1)} \geq s_{2n}$ for all $n \in \mathbb{N}$. Hence $(s_{2n})_{n\geq 1}$ is an increasing sequence which is bounded above (by a_1). Therefore it converges by the Monotone Convergence Theorem. Let its limit be ℓ. So $s_{2n} \to \ell$ as $n \to \infty$.

Now consider the sequence $(s_{2n+1})_{n\geq 1}$. Then $s_{2n+1} = s_{2n} + a_{2n+1}$, and both the series s_{2n} and a_{2n+1} converge. So, by the Algebra of Limits Theorem

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} a_{2n+1} = \ell - 0 = \ell.$$

It now follows easily in this situation that $s_n \to \ell$ as $n \to \infty$, as well, which (by definition) implies that $\sum\limits_{n=1}^{\infty} a_n$ converges, as required.

In more detail, let $\epsilon > 0$ be given. Choose N_1 so that $|s_{2n} - \ell| < \epsilon$ for all $2n \geq N_1$ and N_2 so that $|s_{2n+1} - \ell| < \epsilon$ for all $2n+1 \geq N_2$. Then clearly $|s_m - \ell| < \epsilon$ for all $m \geq N = \max\{N_1, N_2\}$. So, yes, $s_n \to \ell$ as $n \to \infty$ as required. $\square$

Remark: Sometimes it is useful to slightly modify the series in the Alternating series. The same argument will work (or we can use the version proved) to deduce either of the following slightly modified versions of the theorem:

(A) *Let $(a_n)_{n\in\mathbb{N}}$ be a decreasing sequence of positive terms such that $a_n \to 0$ as $n \to \infty$. Then the series*

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges.

(B) *Let $(a_n)_{n\geq 0}$ be a decreasing sequence of positive terms such that $a_n \to 0$ as $n \to \infty$. Then the series*

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

converges.

Remark: It might be tempting to say that any alternating sum converges. But that is false: *Let a_n be positive numbers such that $a_n \geq 0$ for all n and (a_n) is decreasing. Then $\sum_{n\geq 1}(-1)^n a_n$ converges* **if and only if** *(a_n) is null.*

Proof. The direction $\Leftarrow$ is The Alternating Series Test 10.1.1. The direction $\Rightarrow$ is the "Nullity Theorem 8.1.4." $\square$

Example 10.1.2. Let $p \in \mathbb{R}$. Then $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^p}$ converges if and only if $p > 0$.

Proof: If $p \leq 0$, then for all n, $\left|\dfrac{(-1)^{n+1}}{n^p}\right| = n^{-p} \geq 1$. So $\left(\dfrac{(-1)^{n+1}}{n^p}\right)_{n\in\mathbb{N}}$ is not a null sequence and hence the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^p}$ cannot converge, by 8.1.4.

On the other hand, if $p > 0$, then $\left(\frac{1}{n^p}\right)_{n\in\mathbb{N}}$ is a decreasing null sequence of positive terms. Hence the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ converges by the Alternating Series Test.

We end this section with a precise formula for the alternating sum $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. The last step in this argument requires the definition of an integral as a limit of sums of areas. If you have not seen this before, do not worry, since it is not something required for this course. But the computation (pointed out by Carolyn Dean) is fun and interesting. This computation is not something you need to remember for this course.

Example 10.1.3. The series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges to $\ln(2)$.

Proof. This follows form a more careful analysis of the partial sums

$$s_{2N} = \sum_{n=1}^{2N} a_n = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2N-1} - \frac{1}{2N}.$$

Claim: $s_{2N} = \frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{2N}$.

Proof of the Claim: It is clear that it holds for $N = 1$ since $s_2 = 1 - \frac{1}{2} = \frac{1}{2}$. So suppose that it holds for some N. Then, by definition

$$\begin{aligned}
s_{2(N+1)} &= 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2N+1} - \frac{1}{2N+2} \\
&= s_{2N} + \frac{1}{2N+1} - \frac{1}{2N+2} \\
&= \left(\frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{2N}\right) + \frac{1}{2N+1} - \frac{1}{2}\frac{1}{N+1} && \text{by } \textit{the inductive hypothesis} \\
&= \left(\frac{1}{N+2} + \cdots + \frac{1}{2N}\right) + \frac{1}{2N+1} + \frac{1}{2}\frac{1}{N+1} && \text{by } \textit{a simple manipulation.}
\end{aligned}$$

Thus, by induction, the Claim is proven.

This proves that for all n we have

$$s_{2n} = \frac{1}{n}\left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}}\right].$$

Finally, this is the *Riemann Sum* for the integral $\int_{x=0}^{1} \frac{1}{1+x}dx$ where one is dividing the region $[0,1]$ into n subdivisions. (You might not have seen this before, but it's a simple enough idea, and will be done properly next year.) So, by definition

$$\lim_{n\to\infty} s_{2n} = \int_{x=0}^{1} \frac{1}{1+x}dx = \ln(x)\Big|_1^2 = \ln(2). \qquad \square$$

10.2 Absolute Convergence

Suppose that we are given a series $\sum_{n=1}^{\infty} a_n$ where some of the a_n are positive and some negative. Then (apart from the Alternating Series Test) there are not so many good rules for deciding whether $\sum_{n=1}^{\infty} a_n$ converges or not. For example something like

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots$$

(where every third term is negative) is not something covered by one of our rules. The one other rule we will have is that the series $\sum_{n=1}^{\infty} a_n$ does converge provided the series $\sum_{n=1}^{\infty} |a_n|$ converges. Before stating the result we make a definition.

Definition 10.2.1. *Let $\sum_{n=1}^{\infty} a_n$ be any series. We say that $\sum_{n=1}^{\infty} a_n$ is* **absolutely convergent** *if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent. If $\sum_{n=1}^{\infty} a_n$ is convergent, but not absolutely convergent, then we say it is* **conditionally convergent**.

For example, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent because $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is convergent. However, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent because it converges (see Example 10.1.2) but $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges.

The definition above rather presupposes that absolute convergence is stronger than convergence and we next give a proof of this fact.

Theorem 10.2.2. *If the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.*

Proof. We are given that $\sum_{n=1}^{\infty} |a_n|$ converges. Let

$$p_n = \begin{cases} a_n & \text{if } a_n \geq 0, \\ 0 & \text{if } a_n \leq 0. \end{cases}$$

Then $\sum_{n=1}^{\infty} p_n$ is a series of positive terms and for all $n \in \mathbb{N}$, $p_n \leq |a_n|$. Hence $\sum_{n=1}^{\infty} p_n$ converges by the Comparison Test. Similarly, if

$$q_n = \begin{cases} |a_n| & \text{if } a_n < 0, \\ 0 & \text{if } a_n \geq 0 \end{cases}$$

then $\sum_{n=1}^{\infty} q_n$ converges.

Hence, by the Algebra of Infinite Sums Theorem 8.1.5, $\sum_{n=1}^{\infty} (p_n - q_n)$ is convergent. But for all $n \in \mathbb{N}$, $\quad p_n - q_n = a_n$ and we are done. $\square$

Example 10.2.3. $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{4}\right)$ converges absolutely and hence converges.

Proof: The sine terms are $\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0 - \frac{1}{\sqrt{2}}, , \cdots$, so the alternating series test does not help. However, as $\sin(x) \leq 1$, we do know that $0 \leq \left|\frac{1}{n^2} \sin\left(\frac{n\pi}{4}\right)\right| \leq \frac{1}{n^2}$ for all n. Hence $\sum_{n=1}^{\infty} \left|\frac{1}{n^2} \sin\left(\frac{n\pi}{4}\right)\right|$ converges by the comparison test. $\square$.

The same sort of argument means that we can modify some of our earlier tests to work for series with positive and negative terms. For example:

Theorem 10.2.4. (The Modified Ratio Test) *Suppose that a_n for all $n \in \mathbb{N}$ are any real numbers and assume that $\left|\frac{a_{n+1}}{a_n}\right| \to l$ as $n \to \infty$.*

(i) If $l < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and hence convergent.

(ii) If $l > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $l = 1$, then we still cannot conclude whether $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.

Proof. (i) In this case the Ratio Test 9.1.7 says that $\sum_{n\geq 1} |a_n|$ converges; i.e. that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) In this case we know from the proof of 9.1.7 that $|a_n| \to \infty$ as $n \to \infty$. So, certainly $a_n \not\to 0$ as $n \to \infty$. Thus, by Theorem 8.1.4, $\sum_{n=1}^{\infty} a_n$ is divergent. $\square$

Example 10.2.5. $\sum_{n=1}^{\infty} x^n$ (and indeed $\sum_{n=1}^{\infty} (-1)^n x^n$) converges if $|x| < 1$ and diverges if $|x| > 1$. In this case, one can also check that it diverges when $|x| = 1$.

Final Comments. In Example 9.2.11, we saw how one might approach testing series for convergence. Of course, there, the sequences all had positive terms. So how should one approach general series? In a sense the rules are easier, as there are really only three possibilities for a series $\sum_{n=1}^{\infty} a_n$:

1. Is $\lim_{n\to\infty} = 0$? If not then the series diverges by the n-th term test Theorem 8.1.4.
2. Does the Alternating Series Test 10.1.1 apply? If so, use it!
3. Otherwise you had better hope that the Absolute Convergence Test (Theorem 10.2.2) applies, in which case you are back to the ideas of Chapter 9. As we will see in the next chapter, one of the most important cases where this case applies is when one can use the Modified Ratio Test 10.2.4.

For examples of all these cases, see the next Exercise Sheet.

Chapter 11

Power Series

We now consider (the simplest case of) series where the nth term depends on a real variable x and we ask for which values of x does the series converge.

Definition 11.0.6. *A series of the form* $\sum\limits_{n=1}^{\infty} a_n x^n$ *(also written* $\sum_{n\geq 1} a_n x^n$*) is called a* **power series** *(in the variable* x*).*

For example, as we have mentioned before, we can write $e^x = \sum\limits_{n=1}^{\infty} \frac{1}{n!} x^n$ as a power series. Similarly, the geometric series $\sum\limits_{n=1}^{\infty} x^n$ is a power series (with $a_n = 1$ for all n). As we saw in Example 10.2.5, this converges absolutely for $|x| < 1$ and diverges otherwise. For another example, consider

Example 11.0.7. $\sum\limits_{n=1}^{\infty} \frac{1}{n} x^n$.

Let's use the (modified) Ratio Test to study this example. Since it is customary to write the series as $\sum\limits_{n=1}^{\infty} a_n x^n$ as we have done, when using the tests we should set (say) $c_n = \frac{1}{n} x^n$, and apply that rule to $\sum\limits_{n=1}^{\infty} c_n$. So,

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| = |x| \cdot \frac{n}{n+1} \quad \to |x| \qquad \text{as } n \to \infty.$$

Thus, if $|x| < 1$ the series converges absolutely, by the Modified Ratio Test 10.2.4, while it diverges if $|x| > 1$. When $|x| = 1$ we get two more cases:

If $x = 1$ the series is $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ which we know diverges.

If $x = -1$ the series is $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ which we know converges (by the Alternating Test 10.1.1 and the Remark just after that theorem).

The pattern we see in these last two examples is actually a general phenomenon: using the ratio test we find that for any series there exists some number $R \geq 0$ such that $\sum\limits_{n=1}^{\infty} a_n x^n$ converges absolutely if $|x| < R$, it diverges if $|x| > R$ and may or may not converge when $|x| = R$.

(In the last two examples $R = 1$ but we will see other examples where it can be any positive number. We will also see examples where $R = 0$ and $R = \infty$)

11.1 The Radius of Convergence of a Power Series

The main result concerning convergence of power series is the following.

Theorem 11.1.1. *Let* $\sum_{n=1}^{\infty} a_n x^n$ *be a power series. Then there are three possibilities concerning convergence.*

Case (i): $\sum_{n=1}^{\infty} a_n x^n$ *converges only for* $x = 0$.

Case (ii): $\sum_{n=1}^{\infty} a_n x^n$ *converges for all* $x \in \mathbb{R}$.

Case(iii): There exists a unique positive real number R *such that*

(a) $\sum_{n=1}^{\infty} a_n x^n$ *converges absolutely for all* x *with* $|x| < R$, *and*

(b) $\sum_{n=1}^{\infty} a_n x^n$ *diverges for all* x *with* $|x| > R$.

Before we go through the proof we require a lemma.

Lemma 11.1.2. *If the power series* $\sum_{n=1}^{\infty} a_n x^n$ *converges when* $x = r$ *(where* r *is a nonzero real number) and* u *is a real number satisfying* $0 < |u| < |r|$, *then it converges absolutely for* $x = u$.

Proof. We are given that $\sum_{n=1}^{\infty} a_n r^n$ converges. In particular $a_n r^n \to 0$ as $n \to \infty$ (see 8.1.4) and hence (by Theorem 2.3.9) there is some $M > 0$ such that $|a_n r^n| \leq M$ for all $n \in \mathbb{N}$.

Let $t = \dfrac{|u|}{|r|}$. Then $0 < t < 1$ and

$$|a_n u^n| = |a_n r^n| t^n \leq M t^n$$

for all $n \in \mathbb{N}$.

But $\sum_{n=1}^{\infty} M t^n$ is convergent (since the geometric series $\sum_{n=1}^{\infty} t^n$ is convergent as $0 < t < 1$) and hence $\sum_{n=1}^{\infty} |a_n u^n|$ is convergent by the Comparison Test. Thus $\sum_{n=1}^{\infty} a_n u^n$ is absolutely convergent as required. $\square$

Proof. [of 11.1.1]

Assume that neither case (i) nor case (ii) in the statement of Theorem 11.1.1 holds. Then there exists some $b > 0$ such that $\sum_{n=1}^{\infty} a_n b^n$ is convergent, and some $c > 0$ such that $\sum_{n=1}^{\infty} a_n c^n$ is divergent.

Now consider the set

$$S = \{\gamma > 0 : \sum_{n=1}^{\infty} |a_n x^n| \text{ converges for all } x \text{ such that } |x| < \gamma\}.$$

Now S is not empty because $b \in S$ by Lemma 11.1.2. Also S is bounded above by c. (Since if $c' \in S$ for some $c' > c$ then by definition of S, $\sum_{n=1}^{\infty} |a_n c^n|$ would converge, and hence so would $\sum_{n=1}^{\infty} a_n c^n$, by 10.2.2 - a contradiction.)

Thus S is a non-empty set of real numbers which is bounded above. Hence by the Completeness Property 2.4.6 it has a supremum, R say. We now show that this R has the desired properties.

(a) Suppose that $x \in \mathbb{R}$ and $|x| < R$. Then there is some $y \in S$ with $|x| < y$ (by Definition 2.4.1). But then $\sum_{n=1}^{\infty} |a_n x^n|$ converges by the definition of S. That is, the series $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely.

(b) Suppose that $x \in \mathbb{R}$ and $|x| > R$. Choose any y with $|x| > y > R$. Suppose, for a contradiction, that $\sum_{n=1}^{\infty} a_n x^n$ converges. Then by 11.1.2, $\sum_{n=1}^{\infty} |a_n u^n|$ converges for all u with $|u| < |x|$. In particular, this series converges for all u with $|u| < y$, which implies that $y \in S$. But this is impossible since $y > R = \sup(S)$.

There is one last thing to prove, namely that the R above is unique. It is left as an exercise to show that there can be at most one real number R having properties (a) and (b). $\square$

Definition 11.1.3. *If the real number R has the property that the series $\sum_{n=1}^{\infty} a_n x^n$ converges for all x with $|x| < R$, and diverges for all x with $|x| > R$, then R is called the* **radius of convergence** *(***RoC** *for short) of the power series $\sum_{n=1}^{\infty} a_n x^n$.*

We also extend this definition by putting $R = 0$ if $\sum_{n=1}^{\infty} a_n x^n$ only converges for $x = 0$, and putting $R = \infty$ if $\sum_{n=1}^{\infty} a_n x^n$ converges for all $x \in \mathbb{R}$.

By the theorem this definition does cover all possible cases and $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely for $|x| < R$.

Finding the Radius of Convergence:

To calculate the Radius of Convergence of a given series we always use the Ratio Test. In the course of the calculations in the following examples we tacitly assume that $x \neq 0$. This is justified since a power series always converges for $x = 0$ (and since we have been making the assumption that our series begin at $n = 1$, the sum is always 0 when $x = 0$.)

The Interval of Convergence. Even after one has found the radius of convergence R of a power series $\sum_{n=1}^{\infty} a_n x^n$, you can also ask what happens when $|x| = R$.

Of course, there is nothing to check if $R = 0$ or $R = \infty$ so suppose that $0 < R < \infty$. In this case the Ratio Test will not help and you need to use some other rule to see if $\sum_{n=1}^{\infty} a_n x^n$ converges. The set

$$\{x : \sum_{n=1}^{\infty} a_n x^n \text{ converges}\}$$

is called the **interval of convergence**. Certainly this contains $(-R, R)$ and it may or may not contain the end points $\pm R$.

Example 11.1.4. $\sum\limits_{n=1}^{\infty} x^n$ - Here we already saw that the Radius of Convergence (RoC) is $R = 1$ and the interval of convergence is $(-1, 1)$.

Example 11.1.5. $\sum\limits_{n=1}^{\infty} \frac{1}{n} x^n$ - Here we already saw that the RoC is $R = 1$ and the interval of convergence is $[-1, 1)$; that is it converges if $x = -1$ but diverges if $x = +1$.

Example 11.1.6. Consider the series $\sum\limits_{n=1}^{\infty} nx^n$. To find the RoC we let $c_n = |nx^n|$. Then $\sum\limits_{n=1}^{\infty} c_n$ is a series of positive terms and

$$\frac{c_{n+1}}{c_n} = \frac{(n+1)|x|^{n+1}}{n|x|^n} = \left(1 + \frac{1}{n}\right) \cdot |x| \to (1+0) \cdot |x| = |x|, \text{ as } n \to \infty.$$

So the (Modified) Ratio Test now tells us that if $|x| < 1$ then $\sum\limits_{n=1}^{\infty} c_n$ converges absolutely, whereas if $|x| > 1$ then $\sum\limits_{n=1}^{\infty} c_n$ diverges. This shows that the RoC is 1. It is left for you to check that it diverges if $x = \pm 1$.

Example 11.1.7. Consider the series $\sum\limits_{n=1}^{\infty} \frac{(3n)!}{(n!)^3} x^n$. We shall show that its radius of convergence is $\frac{1}{27}$.

As above we let $c_n = \left| \frac{(3n)!}{(n!)^3} x^n \right|$ Then $c_n > 0$ and

$$\begin{aligned}
\frac{c_{n+1}}{c_n} &= \frac{(3(n+1))! \cdot |x|^{n+1}}{((n+1)!)^3} \cdot \frac{(n!)^3}{(3n)! \cdot |x|^n} \\
&= \frac{(3n)!(3n+1)(3n+2)(3n+3) \cdot |x|^{n+1}}{(n!)^3(n+1)^3} \cdot \frac{(n!)^3}{(3n)! \cdot |x|^n} \\
&= \frac{(3n+1)(3n+2)(3n+3)}{(n+1)(n+1)(n+1)} \cdot |x| \\
&= \frac{(3+\frac{1}{n})(3+\frac{2}{n})(3+\frac{3}{n})}{(1+\frac{1}{n})(1+\frac{1}{n})(1+\frac{1}{n})} \cdot |x| \\
&\to \frac{(3+0)(3+0)(3+0)}{(1+0)(1+0)(1+0)} \cdot |x| \text{ (as } n \to \infty) \\
&= 27|x|.
\end{aligned}$$

So by the Ratio Test, $\sum\limits_{n=1}^{\infty} c_n$ converges for $27|x| < 1$, i.e. for $|x| < \frac{1}{27}$, and diverges for $|x| > \frac{1}{27}$. Therefore $\sum\limits_{n=1}^{\infty} \left| \frac{(3n)!}{(n!)^3} x^n \right|$ converges for $|x| < \frac{1}{27}$ and diverges for $|x| > \frac{1}{27}$. Thus, as in the

previous example (and, indeed, all examples like this), we conclude that the RoC of the series $\sum\limits_{n=1}^{\infty} \frac{(3n)!}{(n!)^3} x^n$ is $\frac{1}{27}$.

Example 11.1.8. Consider the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. (It is often more natural to start the series at $n = 0$ rather than, as we have been doing, $n = 1$. This makes no difference at all to convergence issues. In particular, it does not affect the radius of convergence.)

Here we let $c_n = \left|\frac{x^n}{n!}\right|$ and, just as in Example 9.1.9, we see that $\frac{c_{n+1}}{c_n} \to 0$ as $n \to \infty$ no matter what x is. Since $0 < 1$ it follows that the radius of convergence of this series (the exponential series) is ∞.

Example 11.1.9. Finally, consider the series $\sum_{n\geq 0} n!\, x^n$. We have, for $c_n = |n!\, x^n|$, that

$$\frac{c_{n+1}}{c_n} = \frac{(n+1)!\, x^{n+1}}{n!\, x^n} = (n+1)x.$$

Notice that here *for any* $x \neq 0$, $(n+1)x \to \infty > 1$ as $n \to \infty$. Thus the Ratio Test says that the series diverges for any $x \neq 0$. Thus the RoC of the series $\sum_{n\geq 0} n!\, x^n$ is 0: the series converges for no value of x except $x = 0$.

Chapter 12

Further Results on Power Series - *further reading*

This chapter discusses some results that will be useful for next year even if they are getting a little outside the scope of this year's course. In particular, you are not required to know these results for the exams.

12.1 More General Taylor Series.

First, we have only been discussing power series of the form $\sum_{n\geq 0} a_n x^n$. However when working with Taylor series one often wants to work with expansions around values α other than zero, in which case one would get an expression like $\sum_{n\geq 1} a_n(x-\alpha)^n$. Using a substitution $y = (x-\alpha)$ allows one to reduce to the case we have been considering and so essentially all the same theorems will hold. For example, one gets the following variant of Theorem 11.1.1:

Theorem 12.1.1. *Consider the series* $\sum\limits_{n=1}^{\infty} a_n(x-\alpha)^n$*, for some fixed number* α*. Then there are three possibilities concerning convergence.*

Case (i): $\sum\limits_{n=1}^{\infty} a_n(x-\alpha)^n$ *converges only for* $x = \alpha$*.*

Case (ii): $\sum\limits_{n=1}^{\infty} a_n(x-\alpha)^n$ *converges for all* $x \in \mathbb{R}$*.*

Case(iii): There exists a unique positive real number R *such that*

(a) $\sum\limits_{n=1}^{\infty} a_n(x-\alpha)^n$ *converges absolutely for all* x *with* $|x-\alpha| < R$*, and*

(b) $\sum\limits_{n=1}^{\infty} a_n(x-\alpha)^n$ *diverges for all* x *with* $|x-\alpha| > R$*.*

Proof. Set $y = x - \alpha$. Then our series becomes $\sum\limits_{n=1}^{\infty} a_n y^n$. So, now apply Theorem 11.1.1 to that series and you will find the conclusion you get is exactly the present theorem. For example, if $\sum\limits_{n=1}^{\infty} a_n y^n$ has radius of convergence R then $\sum\limits_{n=1}^{\infty} a_n y^n$ converges if $|y| < R$, which is the same

as saying that $\sum\limits_{n=1}^{\infty} a_n(x-\alpha)^n = \sum\limits_{n=1}^{\infty} a_n y^n$ converges if $|x-\alpha| < R$. Similarly it diverges if $|x-\alpha| > R$. $\square$

12.2 Rearranging Series

Let us begin with the following remarkable example. Given a series $\sum_{n=1}^{\infty} a_n$, then a **rearrangement** of this series means a series of the form $\sum_{n=1}^{\infty} b_n$ which is got by rearranging the terms of the first one. More formally, there exists a bijection $\phi : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{\phi(n)}$ for each n.

Here is a famous example.

Example 12.2.1. Recall from Example 10.1.3 that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2).$$

Here is a rearrangement with sum $\dfrac{\ln(2)}{2}$. To do this we use the rearrangement where we always have two negative terms in succession but only one positive one:

$$\begin{aligned}
&1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \cdots \\
&= \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) \cdots \\
&= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{10} - \frac{1}{12}\right) \cdots \\
&= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots\right) = \frac{\ln(2)}{2}.
\end{aligned}$$

In fact we can get any number we want in this way.

Proposition 12.2.2. *Suppose that $\sum_{n\geq 1} a_n$ is a conditionally convergent series, and let α be* **any** *real number. Then there exists some rearrangement $\sum_{n\geq 1} b_n$ of this series for which $\sum_{n\geq 1} b_n = \alpha$.*

Proof. (Outline) We start with some notation essentially coming from Question 3 on the Exercise sheet for Week 11:

Notation 12.2.3. Given a series $\sum_{n\geq 1} a_n$, write

$$a_n^+ = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n \leq 0 \end{cases} \qquad \text{and} \qquad a_n^- = \begin{cases} 0 & \text{if } a_n \geq 0 \\ a_n & \text{if } a_n \leq 0 \end{cases}$$

By that exercise sheet, the series a_n^+ and a_n^- are divergent. Hence $\sum a_n^+ = \infty$, since all the terms are positive and, similarly, $\sum a_n^- = -\infty$.

Now, we construct the sequence $\{b_n\}$: Since $\sum a_n^+ = \infty$ we can therefore take the first few positive numbers $a_{k_1}, a_{k_2} \ldots, a_{k_r}$ so that $X_1 = a_{k_1} + a_{k_2} + \cdots + a_{k_r} > \alpha$. (More precisely we choose the smallest possible r with this property.) Then, as $\sum a_n^- = -\infty$, we can add to this the first few negative numbers $a_{k_{r+1}}, a_{k_{r+2}} \ldots, a_{k_s}$ so that

$$X_2 = (a_{k_1} + a_{k_2} + \cdots + a_{k_r}) + \left(a_{k_{r+1}} + a_{k_{r+2}} + \cdots + a_{k_s}\right) < \alpha.$$

(Again we take s minimal with this property.) Now keep going; adding positive terms a_j to get bigger than α then immediately adding more negative ones so that the sum becomes less than α, then immediately adding more positive terms *et cetera*.

Finally, we must check that the sum actually converges to α. This is the same idea as in the proof of the Alternating Series test, but the notation is messy. More formally, we have constructed the numbers $X_1 > \alpha$, then $X_2 < \alpha$, and $X_3 > \alpha$, etc. Let a_{ℓ_u} be the last number added to make X_u; thus in the last paragraph $a_{\ell_1} = a_{k_r}$ while $a_{\ell_2} = a_{k_s}$. The key point to notice is that, as each a_{ℓ_j} is chosen minimally, $|X_j - \alpha| \leq |a_{\ell_j}|$ in each case. But, by the nth term test Theorem 8.1.4, $\lim_{j\to\infty} |a_{\ell_j}| \to 0$ as $j \to \infty$. Therefore, by the Sandwich Theorem 3.1.4, $|X_j - \alpha| \to 0$ as $j \to \infty$. Which is exactly what we wanted to prove. $\square$

Strangely enough, for *absolutely convergent* series, taking rearrangements does not change the sum.

Theorem 12.2.4. *Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series, say with $\sum_{n=1}^{\infty} a_n = \ell$. Then for any rearrangement $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ we have $\sum_{n=1}^{\infty} b_n = \ell$, as well.*

Proof. Define the a_n^+, a_n^- and similarly b_n^+, b_n^- as in Notation 12.2.3. By Question 3(a) of the Exercise sheet for Week 11, $\sum_{n=1}^{\infty} a_n^+$ is convergent, say with $A = \sum_{n=1}^{\infty} a_n^+$. Also, obviously the partial sums $\{u_\ell = \sum_{n=1}^{\ell} a_n^+\}$ form an increasing sequence. Now, for any t, the terms $b_1^+, \ldots, b_t^+$ appear in $\{a_1^+, \ldots, a_\ell^+\}$ for some ℓ and hence the partial sums

$$v_t = \sum_{n=1}^{t} b_n^+ \leq \sum_{n=1}^{\ell} a_n^+ \leq \sum_{n=1}^{\infty} a_n^+.$$

Hence $\{v_t\}_{t\geq 1}$ is an ascending and bounded sequence. Thus, by the Monotone Convergence Theorem, it has a limit, say B. Notice also that $B \leq A$ by Lemma 4.2.5. In particular this also shows that $\sum b_n$ is absolutely convergent (if not then Question 3 of the Exercise sheet for Week 11 would imply that $\sum b_n^+ = \infty$). So we can reverse this argument and see that each $u_\ell \leq B$ and hence that $A \leq B$.

In other words $A = B$.

Now repeat this argument for the a_n^- (with decreasing sequences of partial sums) to show that $\sum_{n\geq 1} a_n^- = \sum_{n\geq 1} b_n^-$. Now $\sum_{n\geq 1} a_n = \sum_{n\geq 1} a_n^+ + \sum_{n\geq 1} a_n^-$ (use Question 3(a) of the Exercise sheet for Week 11, again), and similarly $\sum_{n\geq 1} b_n = \sum_{n\geq 1} b_n^+ + \sum_{n\geq 1} b_n^-$. Therefore we conclude that

$$\sum_{n\geq 1} a_n \;=\; \sum_{n\geq 1} a_n^+ + \sum_{n\geq 1} a_n^- \;=\; \sum_{n\geq 1} b_n^+ + \sum_{n\geq 1} b_n^- \;=\; \sum_{n\geq 1} b_n.$$

$\square$

One significant application of this result is that it tells us what happens when we multiply power series. To set this up, recall the equation $e^x \cdot e^y = e^{x+y}$. In terms of power series this should mean that

$$\left(\sum_{n\geq 0} \frac{x^n}{n!}\right)\left(\sum_{n\geq 0} \frac{y^n}{n!}\right) = \left(\sum_{n\geq 0} \frac{(x+y)^n}{n!}\right).$$

("should" because when you construct exponentials formally in next year's Real Analysis course you will proceed in the opposite direction: you *define* the exponential e^x as the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. This for example solves the problem of what exactly e is, but it does mean that rules like $e^x e^y = e^{x+y}$ are no longer "obvious".)

Anyway, this displayed equation is true and actually works much more generally. To prove this, we start with an analogous result for series. Here it is more natural to start summing our power series at $n = 0$ rather than $n = 1$ and so the next few results will be phrased in that way.

Theorem 12.2.5. *Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be absolutely convergent series, say with $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$.*

Write $\sum_{i,j\geq 0} a_i b_j$ for the infinite sum consisting of the product, in any order, of every term of the first series multiplied by every term of the second series. Then $\sum_{i,j\geq 0} a_i b_j$ is absolutely convergent (in the sense that the sum $\sum_{i,j\geq 1} |a_i b_j|$ is also convergent) and $\sum_{i,j\geq 0} a_i b_j = AB$.

Remark: One use of this theorem is when we need to make sense of a summation with two indices in an expression like $\sum_{i,j\geq 0} a_i b_j$. The problem is that one does not want to take something like $\sum_{j=0}^{\infty} (\sum_{i=0}^{\infty} a_i b_j)$, since now one has an infinite sum of infinite sums—which causes a whole new set of problems! We get around this by making it into a single infinite sum $\sum_{m\geq 0} c_m$. But then one runs into the problem that there are many different ways of choosing the sum. The theorem says this does not matter since whatever way you do it you get the same answer.

Proof. Recall from the Foundations of Pure Mathematics course that $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is countable, so pick your favourite bijection $\phi : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, and write it as $\phi(m) = (\phi_1(m), \phi_2(m))$. Now set $c_m = a_{\phi_1(m)} b_{\phi_2(m)}$; thus the definition of $\sum_{i,j\geq 0} a_i b_j$ in the statement of the theorem just means $\sum_{m\geq 0} c_m$. Then, by considering the partial sums as in the previous theorem, we see that the partial sums $\gamma_t = \sum_{n=0}^{t} |c_m|$ form an increasing sequence of non-negative terms and certainly

$$\gamma_t \;\leq\; X = \left(\sum_{n=0}^{\infty} |a_n|\right)\left(\sum_{n=0}^{\infty} |b_n|\right) \quad \text{for} \quad t \in \mathbb{N}.$$

Therefore, by the Monotone Convergence Theorem again, the sequence $\{\gamma_t\}$ has a limit bounded above by X. In other words

$$\gamma = \sum_{m=0}^{\infty} |c_m| \;\leq\; X \;=\; \left(\sum_{n=0}^{\infty} |a_n|\right)\left(\sum_{n=0}^{\infty} |b_n|\right).$$

But we can now apply Theorem 12.2.4 to conclude that, however we ordered the terms, $\sum_{i,j\geq 1} a_i b_j$ must give the same number γ. Now, one way of constructing the doubly infinite sum is for the $(N+1)^2$ term to be $\left(\sum_{n=0}^{N} a_n\right)\left(\sum_{n=0}^{N} b_n\right)$. Since

$$\left(\sum_{n=0}^{N} a_n\right)\left(\sum_{n=0}^{N} b_n\right) \to AB \qquad \text{as } N \to \infty,$$

it follows that in fact our sum $\sum_{i,j\geq 0} a_i b_j$ equals AB, as well.

Notice that since we only get one possible sum, Proposition 12.2.2 says the series must be absolutely convergent. □

As a special case of the theorem we get:

Corollary 12.2.6. (Cauchy Product) *Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be absolutely convergent series, say with $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$. Set $c_n = \sum_{r=0}^{n} a_r b_{n-r}$ for each n.*

Then $\sum_{n=0}^{\infty} c_n$ converges absolutely with $\sum_{n=0}^{\infty} c_n = AB$.

Proof. Once again, the given sequence of partial sums $\sum_{n=0}^{t} c_n$ is a subsequence of some sequence of partial sums constructed from $\sum_{i,j\geq 0} a_i b_j$. Therefore, by Theorem 6.1.3, it also

converges to AB. Once again, as we only get one possible sum, Proposition 12.2.2 says the series must be absolutely convergent. $\square$

Finally we get the result on exponentials that we wanted:

Corollary 12.2.7. *Let $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Then $E(x)$ is absolutely convergent for all x. Moreover for any $x, y \in \mathbb{R}$ one has $E(x)E(y) = E(x+y)$.*

Proof. Of course $E(x)$ is absolutely convergent for all x, by Example 11.1.8. The second sentence follows from the Corollary 12.2.6 once one notices that (by using the Binomial Theorem)

$$\frac{(x+y)^n}{n!} = \frac{1}{n!}\sum_{r=0}^{n} \frac{n!}{r!(n-r)!} x^r y^{n-r} = \sum_{r=0}^{n} \left(\frac{x^r}{r!}\right)\left(\frac{y^{n-r}}{(n-r)!}\right).$$

$\square$

As you might imagine, the Cauchy Product Theorem 12.2.6 and its variants can be used to prove lots of other product formulas.

12.3 Analytic Functions

This section gives an indication of where you will go with power series and related topics in future courses.

Let R be a positive real number or ∞ and suppose that the series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R. Then for each x in the interval $(-R, R)$ the series $\sum_{n=0}^{\infty} a_n x^n$ converges. Let us call its sum $f(x)$. Then $f : (-R, R) \to \mathbb{R}$ and

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Functions that can be obtained this way are called *analytic* (on $(-R, R)$).

One can now show that this analytic function f can be differentiated and that $f'(x)$ is given by the result of differentiating the series term by term:

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}.$$

A fundamental fact is that the differentiated series has the same radius of convergence as the original series, namely R. So we may differentiate again:

$$f''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and repeat this for any $k \in \mathbb{N}$ to obtain

$$f^{(k)}(x) = \sum_{n=0}^{\infty} n(n-1)(n-2)\cdots(n-k+1) a_n x^{n-k}.$$

Now the negative exponents of x do not actually appear (as their coefficients contain a 0, so vanish) and we usually rewrite this series (by changing the variable of summation from n to $n+k$) as

$$f^{(k)}(x) = \sum_{n=0}^{\infty} a_{n+k}(n+k)(n+k-1)(n+k-2)\cdots(n+1)x^n.$$

Now if we put $x = 0$ in this expression then all the terms except the first vanish, and we obtain the following formula for the coefficients of the original series in terms of the function f:

$$f^{(k)}(0) = k!a_k$$

or

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

So, to summarise, analytic functions behave very nicely with respect to differentiation (and integration and almost all other operations) and the coefficients a_n can easily be determined. These functions therefore are very convenient to work with.

In the other direction suppose now that, rather than a power series, we are given a function $f : (-R, R) \to \mathbb{R}$ which can be differentiated as many times as we please. Does it follow that it is an analytic function? Certainly we know what its power series must be, by the formula above, namely

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

This power series is called the *Taylor series* of the function f and one might guess that it converges to $f(x)$ for all $x \in (-R, R)$. If you have calculated examples of Taylor series before, like for e^x, $\sin x$ or $\ln(1 + x)$, then this has always been the case. And, indeed, it is true with any "reasonable" function.

However, one can easily write down functions for which the Taylor series does not converge (except at $x = 0$) and, rather more shockingly, there are also examples of functions $f : \mathbb{R} \to \mathbb{R}$ whose Taylor series converges for all x (i.e. the radius of convergence is ∞) but do not converge to $f(x)$ ...