MATH10242 Sequences and Series: Solutions to Coursework Test

10th March 2020 11:05-11:40 (35 minutes duration)
ALL questions should be attempted No calculators may be used
Weighting within course unit: 20%

- *Fill in your details and answers on this cover sheet.*
- *Please put away all books, calculators, laptops, phones, etc.*
- *If you need more paper, we will provide it.*

Name: ....

ID Number: ....

Put your answers in the boxes provided.

	a	*b*	*c*	*d*	*e*	Total Mark
1	Yes	No	Yes	No	Yes	
2	$\frac{23}{3}$	X	X	X	X	
3	$\frac{-5}{4}$	X	X	X	X	
4	Yes	No	Yes	Yes	X	
5	X	X	X	X	X	
					Overall Mark	

Answer to Question 5 (continue on the back of this page if necessary):

Write your answers to Questions 1–4 in the appropriate boxes on the answer grid. This will be "Yes" or "No" for Questions 1 and 4, a certain number or "No" for Question 2 and a certain number for Question 3.
Answer Question 5 in the space below the grid.

Question 1. **5 marks**
(a) Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are convergent sequences. Does it follow that the sequence $(a_n^2 b_n)_{n\in\mathbb{N}}$ is convergent?

Solution: Yes. This follows directly from the Algebra of Limits Theorem.

(b) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with $a_n \neq 0$ for all n. Suppose that the sequence $(\frac{1}{a_n})_{n\in\mathbb{N}}$ is convergent.
Is the sequence $(a_n)_{n\in\mathbb{N}}$ necessarily bounded?

Solution: No. For instance take $a_n = n$.

(c) Suppose that $(a_n)_{n\in\mathbb{N}}$ is a null sequence and that $(b_n)_{n\in\mathbb{N}}$ is a bounded sequence. Is the sequence $(a_n b_n)_{n\in\mathbb{N}}$ necessarily convergent?

Solution: Yes, in fact this sequence $(a_n b_n)_{n\in\mathbb{N}}$ must be null.

(d) Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are strictly increasing sequences, with $\frac{b_n}{2} < a_n \leq b_n$ for all n. Is the sequence $\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}$ necessarily convergent?

Solution: No. For example, put $b_n = 2^n$ and $a_n = 2^{n-1} + 1$ if n is odd and $a_n = 2^n$ if n is even.

(e) Suppose that $(a_n)_{n\in\mathbb{N}}$ is an increasing sequence of negative real numbers and that $(b_n)_{n\in\mathbb{N}}$ is a decreasing sequence of positive real numbers. Is the sequence $(a_n + b_n)_{n\in\mathbb{N}}$ necessarily convergent?

Solution: Yes. Each sequence must be convergent - by the Monotone Convergence Theorem, so their sum is convergent (by the Algebra of Limits Theorem).

(very close to) seen [ILO 3, Med/High]

Question 2. Find the limit (if it exists) of the sequence $\left(\frac{(5n+3)^2(5n-3)^2 - 4n^4}{(3n+5)^2(3n-5)^2}\right)_{n\in\mathbb{N}}$.
If the limit does not exist, enter "No" in the box.

3 marks

Solution: Divide top and bottom by n^4 to get $\frac{(5n+3)^2(5n-3)^2 - 4n^4}{(3n+5)^2(3n-5)^2} = \frac{(5+\frac{3}{n})^2(5-\frac{3}{n})^2 - 4}{(3+\frac{5}{n})^2(3-\frac{5}{n})^2}$.
Using that each of the fractional terms has limit 0 as $n \to \infty$ (3.2.3) and applying the Algebra of Limits Theorem, we deduce that the limit is $\frac{5^2 5^2 - 4}{3^2 3^2} = \frac{621}{81} = \frac{23}{3}$ (but it's ok to leave the fraction unsimplified).

similar to seen [ILO 4, Easy]

Question 3. Calculate $\lim_{n\to\infty}\left(\sqrt{4n^2 - n} - \sqrt{(2n-1)(2n+3)}\right)$. **3 marks**

Solution: We have
$$\sqrt{4n^2 - n} - \sqrt{(2n-1)(2n+3)} = (\sqrt{4n^2 - n} - \sqrt{(2n-1)(2n+3)}) \cdot \frac{(\sqrt{4n^2 - n} + \sqrt{(2n-1)(2n+3)})}{\sqrt{4n^2 - n} + \sqrt{(2n-1)(2n+3)}} =$$

$$\frac{(4n^2-n)-(2n-1)(2n+3)}{\sqrt{4n^2-n}+\sqrt{(2n-1)(2n+3)}} = \frac{(4n^2-n)-(4n^2+4n-3)}{\sqrt{4n^2-n}+\sqrt{(2n-1)(2n+3)}} = \frac{-5n+3}{\sqrt{4n^2-n}+\sqrt{(2n-1)(2n+3)}}.$$

Divide top and bottom by n to get $\dfrac{-5+\frac{3}{n}}{\sqrt{4-\frac{1}{n}}+\sqrt{(2-\frac{1}{n})(2+\frac{3}{n})}}$ which $\to \dfrac{-5}{\sqrt{4-0}+\sqrt{(2-0)(2+0)}} = \dfrac{-5}{4}$.

similar to seen [ILO 4, Med]

Question 4. Do the following sequences converge? **4 marks**

(a) $\left(\dfrac{n^5+2^n}{n^2+5^n}\right)_{n\in\mathbb{N}}$

Solution: The fastest-growing term is 5^n so divide top and bottom by it to get $\dfrac{\frac{n^5}{5^n}+\frac{2^n}{5^n}}{\frac{n^2}{5^n}+1}$. Since each of $\frac{n^5}{5^n}$, $\frac{2}{5}^n$ and $\frac{n^2}{5^n}$ goes to 0 as $n\to\infty$, we deduce that the limit is $\dfrac{0+0}{0+1} = 0$ as $n\to\infty$. So this sequence converges.

(b) $\left(\dfrac{(n+2)!}{n!+n+2}\right)_{n\in\mathbb{N}}$

Solution: We have $\dfrac{(n+2)!}{n!+n+2} = \dfrac{(n+2)(n+1)n!}{n!+n+2}$ divide top and bottom by $n!$ to see that this equals $\dfrac{(n+2)(n+1)}{1+\frac{n}{n!}+\frac{2}{n!}}$ which $\to\infty$ as $n\to\infty$ so the sequence is not convergent.

(c) $\left(n^{-\frac{2}{n}}\right)_{n\in\mathbb{N}}$

Solution: $n^{-\frac{2}{n}} = \left(\dfrac{1}{n^{\frac{1}{n}}}\right)^2 \to \dfrac{1}{1}^2 = 1$. So the sequence converges.

(d) $\left(\dfrac{3\ln(n)}{\ln(2n)}\right)_{n\in\mathbb{N}}$

Solution: We have $\dfrac{3\ln(n)}{\ln(2n)} = \dfrac{3\ln(n)}{\ln 2+\ln n} = \dfrac{3}{\frac{\ln 2}{\ln n}+1}$ which $\to \dfrac{3}{0+1} = 3$ as $n\to\infty$. So the sequence is convergent.

similar to seen [ILO 4, Med]

Question 5. **5 marks, distributed as shown**

(a) Define what it means for a sequence $(a_n)_{n\in\mathbb{N}}$ to converge to a limit ℓ.

Solution: $(a_n)_{n\in\mathbb{N}}$ converges to ℓ iff, for every $\epsilon > 0$ there is $N\in\mathbb{N}$ such that, for all $n\geq N$, we have $\mid a_n-\ell\mid < \epsilon$ 1 mark

(b) Suppose that $a_n\geq 0$ for all n and that the sequence $(a_n)_{n\in\mathbb{N}}$ converges to 0. Prove, using your definition from part (a) that the sequence $(\sqrt{a_n})_{n\in\mathbb{N}}$ is convergent.

Solution: Given $\epsilon > 0$, choose N such that $|a_n| < \epsilon^2$ for all $n\geq N$. Then, for $n\geq N$ we have $|\sqrt{a_n}| = \sqrt{|a_n|} < \sqrt{\epsilon^2} = \epsilon$. Therefore the sequence $(\sqrt{a_n})_n$ is convergent (to 0). 3 marks

(c) Give an example of two non-convergent sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ such that the sequence $(a_n b_n)_{n\in\mathbb{N}}$ is convergent.

Solution: For instance take $a_n = b_n = (-1)^n$. 1 mark

(very close to) seen [ILOs 1,2, Med/High]

ILOs Tested

On successful completion of this course unit students will be able to:

1. express correctly the definitions of the basic concepts from the course unit, for example the definition of the limit of a sequence;

2. write short simple proofs involving those definitions and apply the Completeness property of the Reals where needed;

3. decide on the correctness or otherwise of statements, providing justifications or counterexamples as appropriate;

4. find the limit of a wide class of sequences;

Comments on student solutions

Q1:

Q2, Q3:

Q4:

Q5(a):

Q5(b):

Q5(c):