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4.1.4 $c^n/n! \to 0$ as $n \to \infty$ c is a constant

Proof Let $a_n = \frac{c^n}{n!}$, note $a_{n+1} = \frac{c}{n+1} a_n$.

If we chose n large enough we can make $\frac{c}{n+1} \le \frac{1}{2}$. So set N so that $N \geq 2c$

so $\frac{c}{N+1} \le \frac{1}{2}$

So $a_{N+1} = \frac{c}{N+1} a_N \le \frac{1}{2} a_N$

and $a_{N+2} = \frac{c}{N+2} a_{N+1} \le \frac{1}{2} a_{N+1} \le \left(\frac{1}{2}\right)^2 a_N$

and so (by induction) $a_{N+t} \le \left(\frac{1}{2}\right)^t a_N$ for $t \geq 0$

So compare $(a_{N+t})_t$ with $\left(\frac{1}{2}\right)^t a_N$ and use the Sandwich Theorem:

We have $0 \le a_{N+t} \le \underbrace{\left(\frac{1}{2}\right)^t a_N}_{\text{converges to } 0}$ $(t \geq 0)$ all

So (Sandwich Thm)

$a_{N+t} \to 0$ as ~~$n \to \infty$~~ $t \to \infty$.

This is the original sequence a_n with ~~the~~ $a_1, a_2, \ldots, a_{N-1}$ removed.

So, by 4.1.3, $a_n \to 0$ as $n \to \infty$

4.1.5 The sequence $n^{1/n} \to 1$ as $n \to \infty$.

Proof Note $n^{1/n} \geq 1 \ \forall n \geq 1$

Set $k_n = n^{1/n} - 1$ (≥ 0)

So $n^{1/n} = k_n + 1$, hence

$$n = (k_n+1)^n = 1 + k_n n + \frac{n(n-1)}{2} k_n^2 + \text{positive terms}$$
$$\geq \frac{n(n-1)}{2} k_n^2$$

and hence $k_n^2 \leq \frac{2n}{n(n-1)} = \frac{2}{n-1}$ $(n \neq 1)$ 2/3/20 2

so $k_n \leq \frac{\sqrt{2}}{\sqrt{n-1}} \leq \frac{\sqrt{2}}{\sqrt{\frac{n}{2}}} = \frac{1}{\sqrt{n}}$

But $\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$ and $k_n \geq 0$

and hence (Sandwich Thm.)

$k_n \to 0$ as $n \to \infty$

So $n^{1/n} = k_n + 1 \xrightarrow[(AoL)]{} 0 + 1 = 1$ as $n \to \infty$ //

4.1.6 If $0 < c < 1$, k fixed then $\lim_{n\to\infty} n^k c^n = 0$

Equivalently if $d > 1$ $(d = 1/c)$

$\lim_{n\to\infty} \frac{n^k}{d^n} = 0$ (eg $\frac{n^{100}}{1.1^n} \to 0$ as $n \to \infty$)

Proof See proof in notes (or use l'Hopital's Rule)

Exs 4.1.7 (i) $\lim_{n\to\infty} (3n)^{1/n} = \lim_{n\to\infty} 3^{1/n} \lim_{n\to\infty} n^{1/n}$

$\lim_{n\to\infty} 3^{1/n} \to 1$ by 4.1.1, $\lim_{n\to\infty} n^{1/n} \to 1$ by 4.1.5

$= 1$

(ii) $\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n$ By 4.1.2 $\left(\frac{1}{2}\right)^n \to 0$ as $n \to \infty$

hence 3.14 $\left(-\frac{1}{2}\right)^n \to 0$ as $n \to \infty$

(iii) $\frac{n!}{n^n} = \frac{n(n-1) \cdots 1}{n \; n \cdots n} = \underbrace{\frac{n}{n} \cdot \frac{n-1}{n} \cdot \cdots \cdot \frac{2}{n}}_{\leq 1} \cdot \frac{1}{n} \leq \frac{1}{n}$

$\frac{1}{n} \to 0$ as $n \to \infty$

so $\frac{n!}{n^n} \to 0$ as $n \to \infty$ by Sandwich Thm

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Exs ~~4.1.7~~ ~~4.2.6~~

(i) $\frac{n^3}{2^n} \to 0$ by (4.1.6)

(ii) $\frac{3^n}{n^3} \to \infty$ does not converge by 4.1.6

consider $1/\left(\frac{3^n}{n^3}\right) = n^3 \cdot \left(\frac{1}{3}\right)^n$

(iii) $\frac{3^n + n!}{n!} = \frac{3^n}{n!} + 1$
$\to 0 + 1 = 1$
(4.1.4)

(iv) $\frac{3!}{n^3} \to 0$ (it's a constant times $\frac{1}{n^3}$)

(v) $\frac{1.1^n + n^{110}}{n^5 + 5^n}$

÷ top and bottom by 5^n:

$$\frac{\left(\frac{1.1}{5}\right)^n + \frac{n^{110}}{5^n}}{\frac{n^5}{5^n} + 1} \to \frac{0+0}{0+1} = \frac{0}{1} = 0$$

(vi) $\frac{n^{28} + 5n^7 + 1}{2^n}$

$= \frac{n^{28}}{2^n} + \frac{5n^7}{2^n} + \frac{1}{2^n}$

$\to 0 + 0 + 0 = 0$ as $n \to \infty$

Ex 4.2.6 Define the sequence $(a_n)_n$ by:

$a_1 = 2$ and $a_{n+1} = \frac{a_n^2 + 2}{2a_n + 1}$

(so $a_2 = \frac{a_1^2 + 2}{2a_1 + 1} = \frac{4+2}{4+1} = \frac{6}{5}$) etc $a_3 = \text{--}$)

Q Does this sequence have a limit?
If so, what is the limit?

If it has a limit l say then,

$$l = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim \frac{a_n^2 + 2}{2a_n + 1}$$

$$\underset{\text{AoL}}{=} \frac{(\lim a_n)^2 + 2}{2 \lim a_n + 1} = \frac{l^2 + 2}{2l + 1}$$

so $2l^2 + l = l^2 + 2$ so $l^2 + l - 2 = 0$ so $(l-1)(l+2) = 0$

so $l = 1$ or -2
But $a_n \geq 0$ ∀
so $l = 1$

Exs $\quad \dfrac{7^n+n^7}{9^n+n^9} = \dfrac{\left(\frac{7}{9}\right)^n + \frac{n^7}{9^n}}{1+\frac{n^9}{9^n}} \to \dfrac{0+0}{1+0} = 0$ as $n \to \infty$ $\qquad$ 3/3/20 ①

$\dfrac{(-1)^n + n^{110}}{n^5+5^n} = \dfrac{(-1)^n/5^n + \frac{n^{110}}{5^n}}{\frac{n^5}{5^n}+1} \to \dfrac{0+0}{0+1} = 0$ as $n \to \infty$

$\dfrac{10^{6n}+n!}{3\cdot n! - 2^n} = \dfrac{\frac{10^{6n}}{n!}+1}{3-\frac{2^n}{n!}} \to \dfrac{0+1}{3-0} = \dfrac{1}{3}$ as $n \to \infty$

Ex 4.2.6 Define the sequence $(a_n)_n$ by

$$a_1 = 2 \qquad a_{n+1} = \frac{a_n^2+2}{2a_n+1} \quad \text{for } n \geq 1$$

We show $a_n \geq 1 \ \forall n$

and $(a_n)_n$ is decreasing i.e. $a_n - a_{n+1} \geq 0 \ \forall n$

First show, by induction on n, that $a_n \geq 1 \ \forall n$

Proof Base case $n=1$ $\quad a_1 = 2 \geq 1$ ✓

Induction step Assume $a_n \geq 1$ and show $a_{n+1} \geq 1$

$a_{n+1} \geq 1$ iff ~~$a_{n+1}-1 \geq 0$~~ iff $\dfrac{a_n^2+2}{2a_n+1} \geq 1$

iff $a_n^2+2 \geq 2a_n+1$ because $2a_n+1 > 0$ (Ind Hyp)

iff $a_n^2 - 2a_n + 1 \geq 0$ iff $(a_n-1)^2 \geq 0$ which is true

Hence $a_{n+1} \geq 1$ as wanted

Second Show, by ~~induction on n~~, that $a_n - a_{n+1} \geq 0 \ \forall n$ 3/3/20 (2)

~~a_n Base case $n=1$ $a_1 = 2$, $a_2 = \frac{2^2+2}{4+1} = \frac{6}{5}$~~

~~$a_1 - a_2 \geq 0$ ✓~~

~~Induction step~~ Assume ~~a~~

Consider $a_n - a_{n+1} = a_n - \dfrac{a_n^2+2}{2a_n+1}$

$$= \frac{2a_n^2 + a_n - a_n^2 - 2}{2a_n+1} = \frac{a_n^2 + a_n - 2}{2a_n+1}$$

$$= \frac{(a_n+2)(a_n-1)}{2a_n+1} \geq 0$$ by what we showed above

So The sequence $(a_n)_n$ is decreasing and bounded below, hence, by the Monotone Convergence Theorem, it converges.

Let $l = \lim_{n\to\infty} a_n$. (what is l?)

We have $l = \lim_{n\to\infty} a_n = \lim_{\substack{n\to\infty \\ (n+1\to\infty)}} a_{n+1} = \lim \dfrac{a_n^2+2}{2a_n+1}$

$$\underset{\text{AoL}}{=} \frac{(\lim a_n)^2 + 2}{2\lim a_n + 1} = \frac{l^2+2}{2l+1}$$

So $2l^2 + l = l^2 + 2$ i.e. $l^2 + l - 2 = 0$

i.e. $(l+2)(l-1) = 0$

So $l = -2$ or 1

But $a_n \geq 1 \ \forall n$ so l cannot $= -2$

Hence $\underline{\underline{l = 1}}$ i.e. $\underline{\underline{\lim_{n\to\infty} a_n = 1}}$

Lemma 4.2.5 Suppose $(a_n)_n$ is convergent with limit l. Suppose $r, s \in \mathbb{R}$ s.t. $\exists N$ s.t. $\forall n \geq N$ $\quad r \leq a_n \leq s$. 3/3/20 (3)

Then $r \leq l \leq s$

(we used this to discard $l = -2$ above)

Chapter 5 Divergent Sequences

Recall we say $(a_n)_n$ is <u>divergent</u> if it is not convergent.

So eg $(-1)^n$ is divergent.

Defn 5.1.2 A sequence $(a_n)_n$ <u>goes to ∞</u> (or tends to ∞, or $a_n \to \infty$ as $n \to \infty$) if $\forall K \in \mathbb{R}^{>0}$ $\exists N$ s.t. $\forall n \geq N$ $\quad a_n > K$

eg $\quad a_n = n^2 \to \infty$ as $n \to \infty$

Because, given $K \in \mathbb{R}^{>0}$, choose $N = \lfloor \sqrt{K} \rfloor + 1$ (e.g.)

Then $\forall n \geq N$, $n^2 > K$, as claimed

eg $\quad a_n = (-1)^n n^2 \not\to \infty$ as $n \to \infty$

Also define $a_n \to -\infty$ to mean $\forall L \in \mathbb{R}^{<0}$ $\exists N$ s.t. $\forall n \geq N$

$a_n < L$

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(a) If $a_n \neq 0 \ \forall n$ and $a_n \to \infty$ as $n \to \infty$
Then $\frac{1}{a_n} \to 0$ as $n \to \infty$

Proof Given $\varepsilon > 0$, choose N s.t.
$\forall n \geq N \quad a_n > \frac{1}{\varepsilon}$ (so take "K" to be $1/\varepsilon$)
so then $\left|\frac{1}{a_n}\right| = \frac{1}{a_n} < \varepsilon$ as required
$\forall n \geq N$

(b) Sps $a_n \neq 0 \ \forall n$. and suppose eventually $a_n > 0$
If $\frac{1}{a_n} \to 0$ then $a_n \to \infty$ as
$n \to \infty$.