

# Algebra of Limits Theorem, continued.

24/2/20  
①

~~(v)~~  $b_n \neq 0 \forall n$  and  $b = \lim_{n \rightarrow \infty} b_n$   $b \neq 0$

(vi) then  $\frac{a_n}{b_n} \rightarrow \frac{a}{b}$  as  $n \rightarrow \infty$  ( $a_n \rightarrow a$  as  $n \rightarrow \infty$ )

Proof of (vi) - it's enough to do the case where  $a_n = 1 \forall n$  i.e. show  $\left(\frac{1}{b_n}\right)_n$  is convergent to  $\frac{1}{b}$

because then, by (iv)  $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n}$  is convergent (to  $\frac{a}{b}$ ).

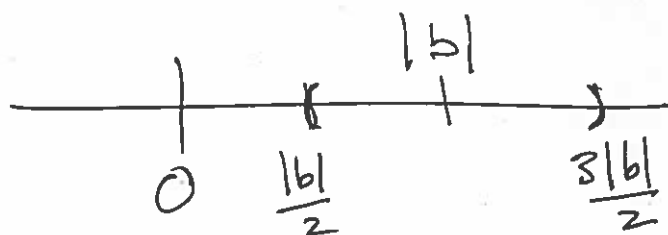
Rough work

We want to show  $\frac{1}{b_n} \rightarrow \frac{1}{b}$

So given  $\varepsilon > 0$ , we want to make

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| < \varepsilon \quad \text{i.e.} \quad \left| \frac{b - b_n}{b b_n} \right| < \varepsilon$$

i.e. make  $\underbrace{|b - b_n|}_{\text{can make as small as we like}} \cdot \underbrace{\frac{1}{|b|}}_{\text{constant}} \cdot \underbrace{\frac{1}{|b_n|}}_{\text{?}} < \varepsilon$  by choosing  $n$  large enough



we can bound  $\frac{1}{|b_n|}$  above

Because  $b_n \rightarrow b$ ,  $|b_n| \rightarrow |b|$  so there is  $N_1$  s.t.  $\forall n \geq N_1, ||b_n| - |b|| < \frac{1}{2}$

$|b - b_n| \frac{1}{|b|} \cdot \frac{1}{|b_n|}$  in particular, s.t.  $|b_n| > \frac{|b|}{2}$

$\left| \frac{1}{b} - \frac{1}{b_n} \right| \frac{2}{|b|}$  and hence  $\frac{1}{|b_n|} < \frac{2}{|b|}$

So if  $n \geq N_1$  and also since  $b_n \rightarrow b$ , there is  $N_2$  s.t.  $\forall n \geq N_2, |b_n - b| < \varepsilon |b|^2$

Then if  $n \geq N_1$  and  $n \geq N_2$   $\left| \frac{1}{b_n} - \frac{1}{b} \right| \leq \frac{\varepsilon |b|^2}{2} \cdot \frac{1}{|b|} \cdot \frac{2}{|b|} = \varepsilon$

Exercise write down the formal proof // 24/2/20

≠

3.2.2 I  $(a_n)_n$  is a null sequence ( $a_n \rightarrow 0$  as  $n \rightarrow \infty$ )  
and  $(b_n)$  is a bounded sequence  
then  $(a_n b_n)_n$  is a null sequence.

Proof Exercise on Week 5 sheet =

Ex 3.2.3 I  $p > 0$ ,  $p \in \mathbb{R}$  then the sequence

$$\frac{1}{n^p} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Proof (eg  $p = \frac{1}{10} \rightsquigarrow \frac{1}{1}, \frac{1}{2^{1/10}}, \frac{1}{3^{1/10}}, \frac{1}{4^{1/10}}, \dots \rightarrow 0$ )

Given  $\varepsilon > 0$ ,  $\frac{1}{n^p} < \varepsilon$  iff  $\frac{1}{\varepsilon} < n^p$  iff  $\frac{1}{\varepsilon^{1/p}} < n$

So, if we set  $N = \lceil \frac{1}{\varepsilon^{1/p}} \rceil + 1$  then  $N > \frac{1}{\varepsilon^{1/p}}$   
and hence, if  $n \geq N$ ,  $n > \frac{1}{\varepsilon^{1/p}}$  so  $\frac{1}{n^p} < \varepsilon$ .

Hence  $\frac{1}{n^p} \rightarrow 0$  as  $n \rightarrow \infty$  //

Ex 3.2.4 Claim  $\frac{n^2 + n + 1}{n^2 - n + 1} \rightarrow 1$  as  $n \rightarrow \infty$

$$= \frac{1 + \frac{1}{n} + \frac{1}{n^2}}{1 - \frac{1}{n} + \frac{1}{n^2}} \rightarrow \frac{1 + 0 + 0}{1 - 0 + 0} = 1 //$$

Ex 3.2.6 (i) I If  $(a_n)_n$  is unbounded and  $(b_n)$  converges  
then  $(a_n + b_n)_n$  is unbounded

(ii) I If  $(a_n)_n$  is unbounded and  $k \neq 0$  then

Proof (i) Choose  $B > 0$  s.t.  $|b_n| < B \forall n$   
( $k a_n)_n$  is unbounded)

Given  $l \geq 0$ ,  $l \in \mathbb{R}$ , there is  $N$   
such that  $|a_n| \geq l + B$  (because  $(a_n)_n$  is unbounded)  
Then  $|a_n + b_n| = |a_n - (-b_n)| \geq |a_n| - |b_n|$  (by 2.2.3)  
 $\geq |l + B - B| = l$  as required //

# Chapter 4 Special Sequences

24/2/20  
(3)

We'll show

4.1.1 If  $c > 0$  then  $c^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$

4.1.2 If  $0 < c < 1$  then  $c^n \rightarrow 0$  as  $n \rightarrow \infty$

4.1.4 For any  $c$ ,  $\frac{c^n}{n!} \rightarrow 0$  as  $n \rightarrow \infty$

4.1.5  $n^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$

4.1.6 If  $0 < c < 1$ ,  $k$  fixed, then  $n^k c^n \rightarrow 0$  as  $n \rightarrow \infty$

4.1.1 If  $c > 0$  then  $c^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$

Proof ① If  $c = 1$  then this is obvious

② If  $c > 1$  write  $c = 1 + y$ ; note  $y > 0$

So we want to show  $(1 + y)^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$

Note for any  $x \geq 0$   $(1 + x)^n = 1 + nx + \text{positive terms}$

$$\geq 1 + nx$$

$$\text{so } (1 + x)^n \geq 1 + nx$$

Apply this with  $x = \frac{y}{n}$ :  $(1 + \frac{y}{n})^n \geq (1 + y) \geq 1$

$$\text{so } 1 + \frac{y}{n} \geq (1 + y)^{1/n} \geq 1$$

Since  $y$  is fixed,  $\frac{y}{n} \rightarrow 0$  as  $n \rightarrow \infty$

$$\text{so } 1 + \frac{y}{n} \rightarrow 1 \text{ as } n \rightarrow \infty$$

So, by the Sandwich Theorem,  $(1 + y)^{1/n} = c^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$ .

③ If  $0 < c < 1$  then, by case ②, since  $\frac{1}{c} > 1$   
 $(\frac{1}{c})^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$

"  $\frac{1}{c^{1/n}} \rightarrow 1$  so, by AOL (v)  $c^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$

Which of the following definitions of  
 "the sequence  $(a_n)_n$  converges to limit  $l$ "  
 is/are correct?

1.  $\exists \epsilon > 0 \forall N \in \mathbb{N}$  such that  $\exists n > N, |a_n - l| < \epsilon$ .  $\times$  nonsense  $\Rightarrow$  not usable
2. The numbers  $a_n$  get closer and closer to  $l$ . rough idea but not precise ( $\times$ )
3. Given  $\epsilon > 0$  there is  $N \in \mathbb{N}$  such that  $|a_N - l| < \epsilon$ .  $\times$
4. Given  $\epsilon > 0$  there is  $N$  such that for all  $n \geq N$ , the distance between  $a_n$  and  $l$  is less than  $\epsilon$ .  $OK$
5. There is some integer  $N$  such that for all integers  $n \geq N, |a_n - l| < 10^{-6}$ .  $\times$
6. Given  $\epsilon > 0$  there is  $N \in \mathbb{N}$  such that  $|a_n - l| < \epsilon$ .  $\times$  incomplete
7. Given  $\epsilon > 0$  there is  $N$  such that the distance between  $a_N$  and  $l$  is less than  $\epsilon$ .  $\times$
8.  $\forall \epsilon > 0 \exists N \in \mathbb{N}$  such that  $\forall n > N, |a_n - l| < \epsilon$ .  $\checkmark$
9.  $\exists N$  such that  $\forall n \geq N, |a_n - l| < \epsilon$ .  $\times$
10. Given  $\epsilon > 0$  there is  $N \in \mathbb{N}$  such that for all  $n > N, |a_n - l| < \epsilon$ .  $\checkmark$  (same as 8)
11.  $\exists \epsilon > 0 \exists N \in \mathbb{N}$  such that  $\forall n > N, |a_n - l| < \epsilon$ .  $\times$

## The Algebra of Limits

### 3.2.1 - The Algebra of Limits Theorem

Let  $(a_n)_{n \in \mathbb{N}}$ ,  $(b_n)_{n \in \mathbb{N}}$  be sequences and let  $a, b$  be real numbers. Suppose that  $\lim_{n \rightarrow \infty} a_n = a$  and that  $\lim_{n \rightarrow \infty} b_n = b$ . Then:

- (i)  $\lim_{n \rightarrow \infty} |a_n| = |a|$ ;
- (ii) for any  $k \in \mathbb{R}$ ,  $\lim_{n \rightarrow \infty} ka_n = ka$ ;
- (iii)  $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$ ;
- (iv)  $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = a \cdot b$ ;
- (v) if  $b_n \neq 0$  (for all  $n$ ), and  $b \neq 0$ , then  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{a}{b}$ .
- (vi) in particular, if  $b_n \neq 0$  (for all  $n$ ), and  $b \neq 0$ , then  $\lim_{n \rightarrow \infty} \frac{1}{b_n} = \frac{1}{b}$ .

For example: if  $\lim_{n \rightarrow \infty} a_n = 2$  and  $\lim_{n \rightarrow \infty} b_n = -3$ , then:

$$\begin{aligned} \lim_{n \rightarrow \infty} (a_n + b_n) &= \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = 2 + (-3) = -1 \\ \lim_{n \rightarrow \infty} (a_n - b_n) &= 2 - (-3) = 5 \\ \lim_{n \rightarrow \infty} a_n \cdot b_n &= 2(-3) = -6 \\ \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} = \frac{2}{-3} = -\frac{2}{3} \quad (\text{ignoring any } b_n = 0) \\ \lim_{n \rightarrow \infty} \frac{a_n + b_n^2}{a_n - b_n} &= \frac{\lim_{n \rightarrow \infty} a_n + (\lim_{n \rightarrow \infty} b_n)^2}{\lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n} = \frac{2 + 9}{2 - (-3)} = \frac{11}{5} \end{aligned}$$

### 3.2.2 = Week 5 Sheet, Question 2:

(a) Let  $(a_n)$  be a null sequence and let  $(b_n)$  be a bounded sequence (not necessarily convergent). Then  $(a_n \cdot b_n)_{n \in \mathbb{N}}$  is a null sequence.

**Solution:** (a) *Proof.* We are given that there exists  $B > 0$  such that  $|b_n| \leq B$  for all  $n$ . Also, for any  $\eta > 0$  there exists  $N \in \mathbb{N}$  such that  $|a_n| < \eta$  for all  $n \geq N$ .

So, if  $\epsilon > 0$  is given, take  $\eta = \frac{\epsilon}{B}$ . Then with  $N$  as above we get  $n \geq N$

$$|a_n b_n| = |a_n| \cdot |b_n| < \eta B = \epsilon,$$

as required.

(b) True or false?

Let  $(a_n)$  be a convergent sequence and let  $(b_n)$  be a bounded sequence (not necessarily convergent). Then  $(a_n \cdot b_n)_{n \in \mathbb{N}}$  is a convergent sequence.

**Solution:** False eg  $a_n = 1 \forall n$ ,  $b_n = (-1)^n$  but  $a_n b_n = (-1)^n$  which is not convergent.

**Example 3.2.3** For any fixed positive real number  $p$ ,  $n^p \rightarrow \infty$ , equivalently  $\frac{1}{n^p} \rightarrow 0$ , as  $n \rightarrow \infty$ .

eg  $p = 100$       $1^{100}, 2^{100}, 3^{100}, 4^{100}, \dots \rightarrow \infty$  as  $n \rightarrow \infty$

$p = \frac{1}{100}$       $1^{\frac{1}{100}}, 2^{\frac{1}{100}}, 3^{\frac{1}{100}}, 4^{\frac{1}{100}}, \dots \rightarrow \infty$  as  $n \rightarrow \infty$

even though  $100^{\frac{1}{100}} \approx 1.047$

Chapter 4: Some Special Sequences - Comparisons between basic sequences.

→ 4.1.1 For any  $c > 0$ ,  $c^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$ .  
 eg  $c = 2$   $2^1, 2^{1/2}, 2^{1/3}, 2^{1/4}, \dots \rightarrow 1$  as  $n \rightarrow \infty$   
 eg  $c = 1/2$   $1/2, 1/2^{1/2}, 1/2^{1/3}, 1/2^{1/4}, \dots \rightarrow 1$  as  $n \rightarrow \infty$

→ 4.1.2 If  $0 < c < 1$ , then  $c^n \rightarrow 0$  as  $n \rightarrow \infty$  eg  $c = 0.5$   $0.5, (0.5)^2, (0.5)^3, \dots \rightarrow 0$  as  $n \rightarrow \infty$   
 (if  $c = 1$ , then  $c^n = 1$  for all  $n$ )  
 (if  $c > 1$ , then  $c^n \rightarrow \infty$  as  $n \rightarrow \infty$ )

**Lemma 4.1.3** Suppose that  $(a_n)$  is a convergent sequence with limit  $l$ . For any integer  $M$ , let  $b_n = a_{n+M}$  (if  $M$  happens to be negative, we just take  $b_n = 0$  or any other number for  $1 \leq n \leq -M$ ). Then  $\lim_{n \rightarrow \infty} b_n = l$ .

What does that mean??  
 eg  $b_1 = a_6, b_2 = a_7, b_3 = a_8, \dots$   
 $M = 5$

→ 4.1.4 Any  $c \in \mathbb{R}, c > 1$ . Which wins? (as  $n \rightarrow \infty$ ):  $c^n$  or  $n!$

Answer:  $n!$ , in the strong sense that  $\frac{c^n}{n!} \rightarrow 0$  as  $n \rightarrow \infty$ .

eg  $c = 1000$   $\frac{1000^1}{1!}, \frac{1000^2}{2!}, \frac{1000^3}{3!}, \dots, \frac{1000^n}{n!}, \dots \rightarrow 0$  as  $n \rightarrow \infty$

→ 4.1.5 What happens to  $n^{1/n}$  as  $n \rightarrow \infty$ ?  
 ( $n$  is increasing but the  $n$ th root is pulling in the other direction; and compare with 4.1.1)

Answer:  $\lim_{n \rightarrow \infty} n^{1/n} = 1$  (so the  $n$ th root process "wins").  
 $1, 2^{1/2}, 3^{1/3}, 4^{1/4}, \dots, n^{1/n}, \dots \rightarrow 1$  as  $n \rightarrow \infty$

→ 4.1.6 Any  $d > 1$ . Fix any  $k \in \mathbb{R}$  but the question is only interesting if  $k > 0$ . Which wins?  $n^k$  or  $d^n$

Answer:  $d^n$ , in the strong sense that  $\frac{n^k}{d^n} \rightarrow 0$  as  $n \rightarrow \infty$ . (To compare with 4.1.6 in the printed notes,  $d$  here is  $1/c$  there.)

eg  $d = 1.0001, k = 1000000$   
 $\frac{1^{1000000}}{1.0001}, \frac{2^{1000000}}{(1.0001)^2}, \frac{3^{1000000}}{(1.0001)^3}, \dots$

→ 4.1.7(3) Which wins?  $n^n$  or  $n!$   $\frac{1}{1}, \frac{2!}{2^2}, \frac{3!}{3^3}, \dots, \frac{n!}{n^n} \rightarrow 0$  as  $n \rightarrow \infty$

Answer:  $n^n$  in the sense that  $\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$ .

We can see this easily if we look at directly at the terms:

$$\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot n}{n \cdot n \cdot n \cdot \dots \cdot n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \dots \cdot \frac{n}{n} \leq \frac{1}{n}$$

But  $1/n \rightarrow 0$  as  $n \rightarrow \infty$   
 so (Sandwich Theorem)  
 $\frac{n!}{n^n} \rightarrow 0$  as  $n \rightarrow \infty$

Since all the terms are positive and  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$  we can use the Sandwich Theorem to get

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0.$$

→ 4.2.3 (last one) Any  $c > 0$ . Which wins:  $\ln(n)$  or  $n^c$ ?

Answer: For any  $0 < c$ , we have  $\lim_{n \rightarrow \infty} \frac{\ln(n)}{n^c} = 0$

eg  $\frac{\ln(n)}{n^{1/100}} \rightarrow 0$  as  $n \rightarrow \infty$

In summary: (see Table in notes)

$c^{1/n}$  (for  $0 < c < 1$ ) increases to 1

$c^{1/n}$  (for  $1 < c$ ) and  $n^{1/n}$  decrease to 1

increasing to  $\infty$ , slowest to fastest:  $\ln(n)$   $n^k$  ( $k > 0$ )  $d^n$  ( $d > 1$ )  $n!$   $n^n$