§2.5 Recall definitions of (strictly) increasing/decreasing sequences

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2.5.3 <u>Monotone Convergence Theorem</u> = (MCT)

If a sequence $(a_n)_n$ is increasing and bounded above [decreasing and bounded below] then it is convergent.

Proof We have $a_n \leq a_{n+1}$ $\forall n$ and $\exists B \in \mathbb{R}$ s.t. $\forall n$ $a_n \leq B$

Since $\{a_n\}_n$ is bounded above, $\sup\{a_n\}_n$ – the least upper bound of $\{a_n : n \in \mathbb{N}\}$ exists (by Completeness of $\mathbb{R}$) – say $\sup\{a_n : n \in \mathbb{N}\} = l$

We show $a_n \to l$ as $n \to \infty$

Given $\varepsilon > 0$ we know $l - \varepsilon$ is not an upper bound for $\{a_n\}_n$ (2.4.3) so there is N with $l - \varepsilon < a_N \leq l$.

Then, since $\forall n \geq N$, $a_N \leq a_n \leq l$ (because increasing)

we have $\forall n \geq N$, $l - \varepsilon < a_n \leq l$

and hence $|a_n - l| < \varepsilon$ $\forall n \geq N$.

Hence $(a_n)_n$ converges, with limit l.

Ex 2.5.4 $a_n = \frac{n^2-1}{n^2} = 1 - \frac{1}{n^2}$ is an increasing sequence, bounded above by 2, therefore this sequence converges (to 1)

§2.6 is <u>optional</u> How do we define eg 2^{π}?

We can define rational powers eg $2^{3/5} = (2^3)^{1/5}$ in terms of ordinary arithmetic

We can choose an increasing sequence 3, 3·1, 3·14, ... of rationals with limit π and then define

$2^{\pi} = \lim_{n\to\infty} 2^{(\text{these rationals})}$ $[2^3, 2^{3\cdot 1}, 2^{3\cdot 14}, \ldots]$ limit exists by MCT

Chapter 3 Calculating limits

17/2/20 (2)

§3.1 The Sandwich Rule

3.1.1 (Sandwich Theorem) Suppose $(a_n)_n$ $(c_n)_n$ are sequences converging to the same limit l.
Suppose $(b_n)_n$ is a sequence such that there is $N \in \mathbb{N}$ with $a_n \leq b_n \leq c_n$ $\forall n \geq N$.
Then $(b_n)_n$ converges to l.

__Proof__ Given $\varepsilon > 0$
there is N_1 s.t. $\forall n \geq N_1$, $|a_n - l| < \varepsilon$
and there is N_2 s.t. $\forall n \geq N_2$, $|c_n - l| < \varepsilon$
So if $n \geq \max\{N_1, N_2\}$ then $l - \varepsilon \leq a_n \leq l + \varepsilon$
and $l - \varepsilon \leq c_n \leq l + \varepsilon$
So, if $n \geq \max\{N_1, N_2, N\}$ then

$$l - \varepsilon \leq a_n \leq b_n \leq c_n \leq l + \varepsilon$$

and hence $|b_n - l| < \varepsilon$ $\forall n \geq \max\{N_1, N_2, N\}$
So $b_n \to l$ as $n \to \infty$.

__Note__ This is often used with one of the sequences $(a_n)_n$ $(c_n)_n$ being constant.
eg. if $c_n \to a$ as $n \to \infty$ and if $\forall n$ we have $c_n \geq b_n \geq a$
then $b_n \to a$ as $n \to \infty$. (Apply the Sandwich Rule with $a_n = a$ $\forall n$)

__Terminology__ a sequence that converges to 0 is said to be a __null__ sequence

Theorem 3.1.4 (i) If $(a_n)_n$ is a null sequence, then so are the sequences $(-a_n)_n$ and ~~...~~, $(|a_n|)_n$

(ii) If $(a_n)_n$ is a null sequence and if the sequence $(b_n)_n$ is such that, for sufficiently large n, we have $0 \le |b_n| \le |a_n|$, then $(b_n)_n$ is a null sequence.

Proof (i) Let $\varepsilon > 0$. Then there is N s.t. $\forall n \ge N$, $|a_n - 0| = |a_n| < \varepsilon$

So $||a_n| - 0| = |a_n| < \varepsilon$ for $n \ge N$

so $|a_n| \to 0$ as $n \to \infty$

and $|(-a_n) - 0| = |a_n| < \varepsilon$ for $n \ge N$

so $-a_n \to 0$ as $n \to \infty$

(ii) Given $\varepsilon > 0$, there is N s.t. $\forall n \ge N$ $|a_n| < \varepsilon$.

So by (i) $(|a_n|)_n$ is null

and hence by (i) $(-|a_n|)_n$ is null.

But $0 \le |b_n| \le |a_n|$ implies $-|a_n| \le b_n \le |a_n|$

so by the Sandwich Theorem, since both sequences $(-|a_n|)_n$ and $(|a_n|)_n$ converge to 0, the sequence $(b_n)_n$ converges to 0 //

Ex $(c_n)_n$ where $c_n = e^{-n} = \frac{1}{e^n}$.

Claim $c_n \to 0$ as $n \to \infty$

– this follows since ~~(we know)~~ we have

$e^n \ge n$ for all $n \ge 1$, hence

$0 \le e^{-n} \le \frac{1}{n}$ and we know ~~...~~ $\frac{1}{n} \to 0$ as $n \to \infty$

So by Sandwich Thm, $e^{-n} \to 0$ as $n \to \infty$

Recall Sandwich Rule:

I) $a_n \le b_n \le c_n \quad \forall n \ge N$

and $\left.\begin{matrix} a_n \to l \\ c_n \to l \end{matrix}\right\}$ as $n \to \infty$, then $b_n \to l$ as $n \to \infty$

Null sequence = sequence that converges to 0

Ex $a_n = \dfrac{e^n+1}{e^{n+1}+1} \le \dfrac{e^n+1}{e^{n+1}} = \underset{\text{constant}}{\dfrac{1}{e}} + \dfrac{1}{e^{n+1}}$ $\to 0$ as $n \to \infty$

so the sequence $\dfrac{e^n+1}{e^{n+1}} = \dfrac{1}{e} + \dfrac{1}{e^{n+1}}$ converges to $\dfrac{1}{e}$ as $n \to \infty$

So, if we can show that
$\frac{1}{e} \le a_n \ \forall n$, then
we can use the Sandwich Rule
to deduce that $a_n \to \frac{1}{e}$ as $n \to \infty$

$\left[\frac{1}{e} \le a_n \le \frac{1}{e} + \frac{1}{e^{n+1}} \right]$

show

$$a_n - \frac{1}{e} = \frac{e^n+1}{e^{n+1}+1} - \frac{1}{e} = \frac{e^{n+1}+e-e^{n+1}-1}{e(e^{n+1}+1)} = \frac{e-1}{e(e^{n+1}+1)} \ge 0 \checkmark$$

§3.2 Algebra of Limits

21/2/20 (2)

Theorem 3.2.1 (Algebra of Limits AoL Theorem)
Suppose $(a_n)_n$, $(b_n)_n$ are convergent sequences with limits a, b respectively. Then:

i) $(|a_n|)_n$ is convergent to $|a|$
ii) $\forall k \in \mathbb{R}$ $(ka_n)_n$ converges to ka
iii) $(a_n+b_n)_n$ converges to $a+b$
iv) $(a_n-b_n)_n$ converges to $a-b$
v) $(a_nb_n)_n$ converges to ab
vi) If $b_n \neq 0$ $\forall n$ and if $b \neq 0$ then $(a_n/b_n)_n$ converges to $\frac{a}{b}$
vii) If $b_n \neq 0$ $\forall n$ and if $b \neq 0$ then $\left(\frac{1}{b_n}\right)_n$ converges to $\frac{1}{b}$

Proof i) Given $\varepsilon > 0$, note $\big||a_n| - |a|\big| \leq |a_n - a|$ 2.2.3
So chose N s.t. $\forall n \geq N$ $|a_n - a| < \varepsilon$, and hence $\big||a_n| - |a|\big| < \varepsilon$ as required//

ii) [to prove $(ka_n)_n$ converges to ka]

rough work: Given $\varepsilon > 0$, we want $|\underbrace{ka_n - ka}_{|k||a_n-a|}| < \varepsilon$
so it's enough to get $|a_n - a| < \frac{\varepsilon}{|k|}$ if $k \neq 0$

~~If $k=0$ The result is obvious~~

Formal proof (Assume $k \neq 0$): Given $\varepsilon > 0$ we chose, since $a_n \to a$, $N \in \mathbb{N}$ s.t. $\forall n \geq N$ $|a_n - a| < \frac{\varepsilon}{|k|}$
Then $|k||a_n - a| < \varepsilon$ i.e. $|ka_n - ka| < \varepsilon$ as required

iii) [to prove $a_n+b_n \to a+b$]   21/2/20 (3)

Rough work [We want $|(a_n+b_n)-(a+b)| < \varepsilon$

$$\| \quad |\underbrace{(a_n-a)} + \underbrace{(b_n-b)}| \leq |a_n-a|+|b_n-b|$$

we can make these, by choosing $n \geq N$ large enough, as small as we like, say $< \frac{\varepsilon}{2}$]

Formal proof

Given $\varepsilon > 0$

choose N_1 s.t. $\forall n \geq N_1$ $|a_n-a| < \frac{\varepsilon}{2}$ N_1 exists since $a_n \to a$

choose N_2 s.t. $\forall n \geq N_2$ $|b_n-b| < \frac{\varepsilon}{2}$ N_2 exists since $b_n \to b$

Set $N = \max\{N_1, N_2\}$

Then, if $n \geq N$ we have both and

and hence $|(a_n+b_n)-(a+b)| = |(a_n-a)+(b_n-b)|$

$\underset{\Delta \text{ inequality}}{\leq} |a_n-a|+|b_n-b| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$

as required

iv) $a_n-b_n \to a-b$ – follows immediately from (ii) and (iii) (take $k=-1$ in (ii))

v) [to prove $a_nb_n \to ab$]

Rough work [We want $|a_nb_n-ab| < \varepsilon$

$= |a_nb_n - ab_n + ab_n - ab|$

~~$= |a_nb_n$~~ $= |\underbrace{(a_n-a)}_{\text{can make small}} \underbrace{b_n}_{\text{bounded since convergent}} + \underbrace{a}_{\text{constant}} \underbrace{(b_n-b)}_{\text{can make small (choose n big enough)}}|$

So if we choose

$C = \max\{B, |a|+1\}$ when $|b_n| \leq B \; \forall n$

then $|a_nb_n-ab| \leq |a_n-a||b_n|+|a||b_n-b|$

$\leq |a_n-a|C + C|b_n-b|$

so, if we choose n s.t. $|a_n-a|, |b_n-b| < \frac{\varepsilon}{2C}$ then $< \varepsilon$]

Formal proof that $a_n b_n \to ab$: 21/2/20 (4)

Given $\varepsilon > 0$,

Choose $B \in \mathbb{R}$ with $|b_n| \leq B \ \forall n$ (B exists since (b_n) is convergent hence bounded)

and let $C = \max\{B, |a|+1\}$

and choose N_1 so that $\forall n \geq N_1$ $|a_n - a| < \frac{\varepsilon}{2C}$

and choose N_2 so that $\forall n \geq N_2$ $|b_n - b| < \frac{\varepsilon}{2C}$

Let $N = \max\{N_1, N_2\}$ and

let $n \geq N$.

Then $|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab|$

$= \cancel{|(a_n - a)b_n|} \ |(a_n - a)b_n + a(b_n - b)|$

$\leq |(a_n - a)b_n| + |a(b_n - b)| = \underbrace{|a_n - a|}_{< \frac{\varepsilon}{2C}} \underbrace{|b_n|}_{\leq B \text{ hence } \leq C} + \underbrace{|a|}_{\leq C} \underbrace{|b_n - b|}_{< \frac{\varepsilon}{2C}}$

$\triangle$ ineq.

$< \frac{\varepsilon}{2C} \cdot C + C \cdot \frac{\varepsilon}{2C} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

i.e. $a_n b_n \to ab$ as $n \to \infty$. as required