§2.3 10/12/20 (1)

2.3.6 A subset $S \subseteq \mathbb{R}$, $S \neq \emptyset$, is **bounded above** (by B)
if there is some $B \in \mathbb{R}$ such that

$$\forall s \in S \quad s \leq B$$

and, in this case, we say B is an **upper bound** for S

(e.g. $\mathbb{N} \subseteq \mathbb{R}$ is not bounded above
if $B \in \mathbb{R}$, then $\lfloor B \rfloor + 1 \in \mathbb{N}$ ~~which~~
~~is not~~ and $B \not\geq \lfloor B \rfloor + 1$
so B is not an upper bound for $\mathbb{N}$)

Similarly, $S \neq \emptyset$, $S \subseteq \mathbb{R}$ is **bounded below**,
by L say, if $\forall s \in S$, $s \geq L$ — then
say L is a **lower bound** for S.

A set S is **bounded** if S is bounded
above and bounded below.

e.g. $\mathbb{N}$ is bounded below (eg by 0,
and by -0.5, -1000)

e.g. $\mathbb{Z}$ is not bounded above or
bounded below.

e.g. any finite set is bounded

eg $S = (-1, 1)$ is bounded above by eg 2, below by eg -5

Note $S \subseteq \mathbb{R}$ is bounded iff $\exists M \in \mathbb{R}$ s.t.
$$|s| \leq M \quad \forall s \in S$$
$$(\text{ie } -M \leq s \leq M \quad \forall s \in S)$$

2.3.8 A sequence $(a_n)_{n \in \mathbb{N}}$ is bounded /above/below iff $\{a_n : n \in \mathbb{N}\}$ is ___

2.3.9 **Theorem** Every convergent sequence is bounded.

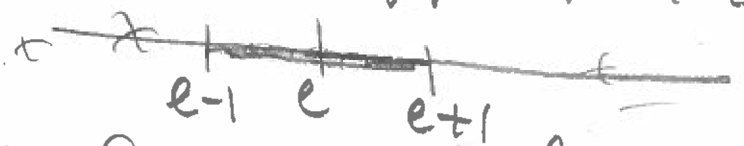
Proof Suppose $(a_n)_n$ is a convergent sequence. So $a_n \to l$ as $n \to \infty$ for some $l \in \mathbb{R}$.

Then choose $N \in \mathbb{N}$ s.t. (take $\varepsilon = 1$)
$$\forall n \geq N \quad |a_n - l| < 1$$
i.e. $\quad l-1 < a_n < l+1 \quad \forall n \geq N$

So if we set $B = \max\{l+1, a_1, a_2, \dots a_{N-1}\}$

Then $B \geq a_n \; \forall n$ $\begin{cases} B \geq a_1, \; B \geq a_2 \dots B \geq a_{N-1} \\ B \geq l+1 \geq a_N, a_{N+1}, \dots \end{cases}$

So B is an upper bound for $(a_n)_n$



Similarly for getting a lower bound (or use that $(a_n)_n$ has an upper bound iff $(-a_n)_n$ has a lower bound).

§2.4 Completeness for $\mathbb{R}$

Definition 2.4.1 Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$.
Then $M \in \mathbb{R}$ is a supremum (or least upper bound) for S if:

1) M is an upper bound for S
2) if $M' < M$ then M' is not an upper bound for S.
   (in which case $\exists x \in S$ with $x > M'$)

Note 2.4.2 If S has a supremum (a "sup") then it is unique.
To ~~see~~ this, if both M_1, M_2 are suprema for S and $M_1 \neq M_2$ say $M_2 < M_1$.
But then there would be, because M_1 is a supremum, some $x \in S$ with $M_2 < x$ — but then M_2 is not an upper bound for S, so not a supremum for S — contradiction

We write $\sup S$ for the supremum of S if it exists.

2.4.3 If $M = \sup(S)$ and if $\varepsilon > 0$ then there is $x \in S$ with $M - \varepsilon < x \le M$

Proof Since $M - \varepsilon < M$, $M - \varepsilon$ is not an upper bound for S

So there is $x \in S$ with $x \not\le M - \varepsilon$

$x > M - \varepsilon$

Note (2.4.5) $\sup(S)$ might or might not be in S

eg $(0,1)$ and $[0,1]$ both have $\sup = 1$; $1 \notin (0,1)$ but $1 \in [0,1]$

2.4.6 Completeness for $\mathbb{R}$

Every nonempty subset of $\mathbb{R}$ which is bounded above has a supremum

Note $\mathbb{Q}$ does not have this property

eg $S = \mathbb{Q} \cap (-\infty, \sqrt{2})$ is a subset of $\mathbb{Q}$ which is bounded above but with no supremum in $\mathbb{Q}$

(given $r \in S$, $r < \sqrt{2}$, then for some large enough $N \in \mathbb{N}$, $r + \frac{1}{N} < \sqrt{2}$)

~~Proof~~ Because, if $q \in \mathbb{Q}$ 10/2/20 5

were a supremum for S

then $\forall r \in \mathbb{Q}$ with $r < \sqrt{2}$, $r \leq q$

That implies $\sqrt{2} \leq q$ (by argument above)

But then, for some large enough $M \in \mathbb{N}$,

we'd have $q - \frac{1}{M} \geq \sqrt{2}$

so $q - \frac{1}{M}$ would also be an upper bound for S.

$\times$ contradiction

Completeness for $\mathbb{R}$: 11/2/20 (1)

every $S \subseteq \mathbb{R}$ with an upper bound has a least upper bound = supremum

$\mathbb{Q}$ does not have this property

eg $\mathbb{Q} \cap (-\infty, \sqrt{2})$ has no sup in $\underline{\underline{\mathbb{Q}}}$

But, by completeness of $\mathbb{R}$, the set $S = \{r \in \mathbb{R} : r^2 < 2\}$ has a sup.

– it's bounded above by, e.g. 1·5

Claim the sup of this set S is $\sqrt{2}$ (i.e. some $s \in \mathbb{R}$ with $s^2 = \sqrt{2}$)

Let $s = \sup\{r \in \mathbb{R} : r^2 < 2\}$

Claim $s^2 = 2$

Because, if not, then either

$$\underbrace{s^2 < 2}_{\Downarrow \text{ exercise}} \quad \text{or} \quad \underbrace{s^2 > 2}_{\Downarrow \text{ contradiction (now)}}$$

Assuming (for a contradiction) 11/2/20 2

that $s^2>2$, we'll produce another upper bound for $\{r\in\mathbb{R} : r^2<2\}$ which is $<s$ — contradicting that s is the least upper bound for

Set $t = \frac{s^2-2}{4s} > 0$. Consider $s-t$ $(<s)$

- $s-t = s - \frac{s}{4} + \frac{1}{2s} > \frac{s}{2} + \frac{1}{2s} > 0$
- $(s-t)^2 = s^2 - 2st + t^2 \geq s^2 - 2st$

$$= s^2 - \frac{s^2}{2} + 1 = \frac{s^2}{2} + 1 > 1+1 = 2$$

So any $r\in\mathbb{R}$ with $r^2<2$ has $r<s-t$

So $s-t$ is also an upper bound for $\{r\in\mathbb{R} : r^2<2\}$ so s is not the least upper bound — contradiction

2-4-8 $\mathbb{N}$ $(\subseteq\mathbb{R})$ has no upper bound.

Proof By contradiction — if $\mathbb{N}$ did have ~~an upper bound, say~~ an upper bound then, by Completeness, it would have a least upper bound, s say.

Then, by 2.4.5, taking $\varepsilon = \frac{1}{2}$, There would be some $x \in \mathbb{N}$ with $5 - \frac{1}{2} < x \leq 5$ 11/2/20 (3)

Add 1 to each term:

$$5 + \tfrac{1}{2} < x + 1 \; (\leq 5 + 1)$$

So $x+1 \in \mathbb{N}$, $x+1 > 5 + \frac{1}{2} > 5$

contradicting that 5 is an upper bound for $\mathbb{N}$

2.4.11 Similarly we have notions of lower bounds for a set, greatest lower bound = infimum

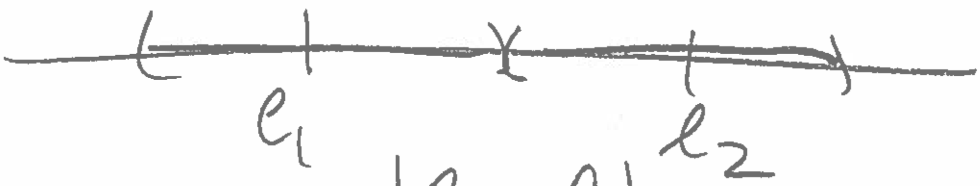
...

every nonempty subset of $\mathbb{R}$ with a lower bound (= bounded below) has a greatest lower bound = ~~infumor~~ infimum.

§2.5

2.5.1 If a sequence $(a_n)_n$ converges then it converges to a unique limit.

Proof Suppose $a_n \longrightarrow l_1$ and $a_n \longrightarrow l_2$ as $n \longrightarrow \infty$
and, for a contradiction, that $l_1 \neq l_2$



Given $\varepsilon = \frac{|l_2 - l_1|}{2}$ there is $N \in \mathbb{N}$ such that $\forall n \geq N$ $|a_n - l_1| < \varepsilon$
and $|a_n - l_2| < \varepsilon$

In particular

$$\left.\begin{aligned} l_1 - \varepsilon < a_N < l_1 + \varepsilon \\ l_2 - \varepsilon < a_N < l_2 + \varepsilon \end{aligned}\right\}$$

But $(l_1 - \varepsilon, l_1 + \varepsilon) \cap (l_2 - \varepsilon, l_2 + \varepsilon) = \emptyset$

$\cancel{\circ\circ}$

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11/2/20 5

Definition A sequence $(a_n)_n$ is (strictly) increasing if $a_n \le a_{n+1}\ \forall n$ $(a_n < a_{n+1}\ \forall n)$

("monotone increasing" = "increasing")

Similarly $(a_n)_n$ is (strictly) decreasing if $a_n \ge a_{n+1}\ \forall n$

$(a_n > a_{n+1}\ \forall n)$

2.5.3 Monotone Convergence Theorem

Any increasing sequence which is bounded above converges.

Any decreasing sequence which is bounded below converges

Proof next time

Ex 2.5.5 $a_n = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$

In fact (not so easy) this sequence is bounded above and increasing B — so converges (in fact to e) by the Monotone Convergence Thm