26/3/19
(1)

Ex Does $\sum_{n\geq 3} \frac{1}{n^2-n-2}$ converge? to what?

$n^2-n-2 = (n-2)(n+1)$ and $\frac{1}{n^2-n-2} = \frac{1}{3}\left(\frac{1}{n-2} - \frac{1}{n+1}\right)$

So nth partial sum

$$S_n = \sum_{i=3}^{n} \frac{1}{n^2-n-2} = \frac{1}{3}\left(\sum_{i=3}^{n} \frac{1}{i-2} - \sum_{i=3}^{n} \frac{1}{i+1}\right)$$

$$= \frac{1}{3}\left(\sum_{j=1}^{n-2} \frac{1}{j} - \sum_{j=4}^{n+1} \frac{1}{j}\right)$$

$$= \frac{1}{3}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{n-1} - \frac{1}{n} - \frac{1}{n+1}\right)$$

$$\longrightarrow \frac{1}{3}\left(\frac{11}{6}\right) \text{ as } n \to \infty$$

So the nth partial sums do converge, hence $\sum_{n\geq 3} \frac{1}{n^2-n-2}$ converges to $\frac{11}{18}$

Recall 8.1.4 (nth term test) If $\sum a_n$ converges then $a_n \to 0$ as $n \to \infty$

8.1.5 If $\sum a_n = a$, $\sum b_n = b$ are convergent series then $\sum (a_n + b_n) = a + b$, $\sum \lambda a_n = \lambda a$ $\forall \lambda \in \mathbb{R}$

Chapter 9 Series with ~~positive~~ nonnegative terms

9.1.1 If $a_n \geq 0$ then the series $\sum a_n$ converges iff the sequence of partial sums $s_n = \sum_{i=1}^{n} a_n$ is bounded above.

Proof ($\Rightarrow$) If $\sum a_n$ converges then, by definition, the sequence $(s_n)_n$ converges, hence (2.3.8) is bounded above.

($\Leftarrow$) Since $a_n \geq 0$ $\forall n$, the sequence $(s_n)_n$ is increasing, so, since bounded above, it's convergent by the Monotone Convergence Theorem.
So $\sum a_n$ is converg

9.1.2 (Comparison Test) 26/3/19

If $0 \le a_n \le b_n \ \forall n$

and if $\sum b_n$ converges, then $\sum a_n$ converges

Proof Compare the nth partial sums:

$$0 \le \sum_{i=1}^{n} a_i \le \sum_{i=1}^{n} b_i$$

this sequence of partial sums converges since $\sum b_n$ converges, hence bounded above

so this sequence of partial sums ~~converges (by the Sandwich Theorem)~~ is bounded above

Hence, by 9.1.1, $\sum a_n$ converges

9.1.3 Note $\sum_{n\ge1} a_n$ converges iff $\sum_{n\ge N} a_n$ converges

so we can weaken the hypothesis in 9.1.2 to $0 \le a_n \le b_n \ \forall n \ge N$ for some N.

Ex 9.1.5 If $p \ge 2$ then $\sum_{1}^{\infty} \frac{1}{n^p}$ converges

(eg $\sum \frac{1}{n^2}$, $\sum \frac{1}{n^3}$, $\sum \frac{1}{n^{2.5}}$ converge)

by comparison with $\sum \frac{1}{n^2}$ which we already showed to be convergent.

(since if $p \ge 2$ then $n^p \ge n^2$ so $\frac{1}{n^p} \le \frac{1}{n^2}$)

Ex 9.1.6 If $p \le 1$ then $\sum_{1}^{\infty} \frac{1}{n^p}$ diverges

(eg $\sum \frac{1}{n}$, $\sum \frac{1}{\sqrt{n}}$, diverge)

by comparison with $\sum \frac{1}{n}$ which we already showed to be divergent

since $p \le 1$, $n^p \le n$ so $\frac{1}{n^p} \ge \frac{1}{n}$

9.1.7 (Ratio Test) Suppose $a_n \geq 0 \ \forall n$ 26/3/19

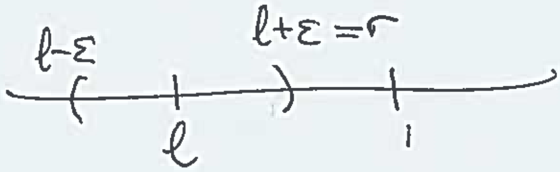
If $\frac{a_{n+1}}{a_n} \to l$ as $n \to \infty$ then:

(i) if $l < 1$ then $\sum a_n$ converges;

(ii) if $l > 1$ then $\sum a_n$ diverges;

[(iii) if $l = 1$ – no conclusion, try some other test]

Proof (i) Suppose $l < 1$



Choose $\varepsilon > 0$ so that $l + \varepsilon < 1$. Set $r = l + \varepsilon$.

Choose N so that $\forall n \geq N$, $\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon$

(since $\frac{a_{n+1}}{a_n} \to l$) ~~i.e.~~ so $\frac{a_{n+1}}{a_n} < l + \varepsilon = r$

That is (for $n \geq N$), $a_{n+1} < r a_n$

So
$$a_{N+1} < r a_N$$
$$a_{N+2} < r a_{N+1} < r^2 a_N$$
$$\vdots$$
$$a_{N+t} < r a_{N+t-1} < \cdots < r^t a_N$$

Hence $\cancel{\sum a_j < a_N \sum_{j=}}$ $\quad \sum_{j=1}^{t} a_{N+j} < a_N \sum_{j=1}^{t} r^j$

since $\sum_1^\infty r^j$ converges ($|r| < 1$)

the partial sums $\sum_1^t r^j$ converge,

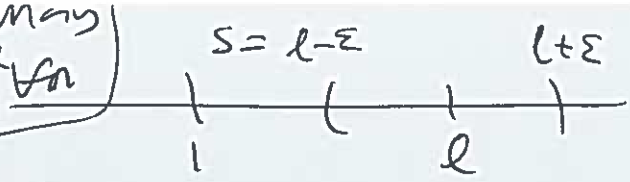
hence so do the $a_N \sum_{j=1}^{t} r^j$ and hence so do the $\sum_{j=1}^{t} a_{N+j}$

These are the partial sums of the series $\sum_{n \geq N+1} a_n$, so this series converges

~~hence~~ $\sum$ hence $\sum_{n=1}^{\infty} a_n$ converges

(ii) $l>1$

we may assume $a_n>0$ $\forall n$



Choose $\varepsilon>0$
so that $l-\varepsilon>1$

Since $\frac{a_{n+1}}{a_n}\to l$, there is N such that

$\forall n\geq N$, $\left|\frac{a_{n+1}}{a_n}-l\right|<\varepsilon$

In particular $\forall n\geq N$, $\frac{a_{n+1}}{a_n} > l-\varepsilon = s > 1$

So $\forall n\geq N$, $a_{n+1} > s a_n$

So $a_n>0$ and the a_{n+j} are increasing as $j\to\infty$

So $a_n \not\to 0$ as $n\to\infty$
and hence (nth term test) $\sum a_n$ diverges

Ex

Convergent or divergent? 27/3/19 (1)

(1) $\sum \frac{\cos^2(n)}{n^2}$ (2) $\sum \frac{e^n}{n^3}$ (3) $\sum \frac{2n!}{(2n)!}$ ~~$\sum \frac{2n+3}{2n+4}$~~

(4) $\sum \frac{2n+3}{3n-4}$ (5) $\sum \frac{1}{\ln(n)}$ (6) $\sum \frac{1}{2^n-2}$ ~~$\sum \frac{n!}{4}$~~

(1) $0 \leq \frac{\cos^2(n)}{n^2} \leq \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ cvgt

so $\sum \frac{\cos^2(n)}{n^2}$ is cvgt by Comparison Test

(2) divergent since $\frac{e^n}{n^3} \not\to 0$ as $n \to \infty$ (nth Term Test)

(3) Try Ratio Test: $\frac{a_{n+1}}{a_n} = \frac{2(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{2n!} = \frac{(n+1)}{(2n+2)(2n+1)}$

$\to 0$ as $n \to \infty$

and $l = 0 < 1$ so convergent

(4) $a_n = \frac{2n+3}{3n-4} = \frac{2+3/n}{3-4/n} \to \frac{2}{3} \neq 0$ as $n \to \infty$

so divergent (nth Term Test)

(5) $\ln(n) < n$ so $\frac{1}{\ln(n)} > \frac{1}{n}$ but $\sum \frac{1}{n}$ diverges

so $\sum \frac{1}{\ln(n)}$ diverges (Comparison Test)

(6) Convergent by comparison with $\sum \frac{1}{2^n}$

$\sum \frac{1}{2^n-2}$ since $2^n-2 > 2^{n-1}$ so $\frac{1}{2^n-2} < \frac{1}{2^{n-1}}$

So, since $\sum \frac{1}{2^{n-1}}$ is cvgt

so is $\sum \frac{1}{2^n-2}$

Ex 9.1.9 Fix a real number $x > 0$ and consider the series $\sum_{n\geq 0} \frac{x^n}{n!}$

2 + 15 ||[illegible]

Ratio test: $\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1} \longrightarrow 0$ as $n \to \infty$

So $l = 0 < 1$ hence convergent (in fact to e^x)

(see Power Series later)

§ 9.2 The Integral Test

Suppose $f : [1, \infty) \longrightarrow \mathbb{R}$ such that:

- $f(x) \geq 0 \quad \forall x \geq 1$
- f is decreasing ie $x < y$ then $f(x) \geq f(y)$
- f is "continuous" ("no breaks in the graph of f")

9.2.1 (Integral test) Suppose f is as above.

Then $\sum_1^\infty \underbrace{f(n)}_{\text{"}a_n\text{"}}$ converges iff $\int_1^\infty f(x)\,dx$ converges/exists

$$\int_1^\infty f(x)\,dx = \lim_{k \to \infty} \int_1^k f(x)\,dx$$

Proof – see later

Ex 9.2.3 If $p > 1$ then $\sum_1^\infty \frac{1}{n^p}$ converges

Regard $\frac{1}{n^p}$ as $f(n)$ where $f(x) = x^{-p}$

Consider $\int_1^\infty f(x)\,dx = \int_1^\infty x^{-p}\,dx = \lim_{k \to \infty} \int_1^k x^{-p}\,dx$

$$= \lim_{k \to \infty} \left[\frac{x^{-p+1}}{(-p+1)} \right]_1^k = \lim_{k\to\infty} \left(\frac{-1}{-p+1} + \frac{1}{k^{p-1}} \cdot \frac{1}{1-p} \right)$$

$$= \frac{-1}{1-p}$$

The integral converges hence the series converges

Ex 9.2.9 27/3/19 3

$$\sum_{n\geq 2} \frac{1}{n\ln(n)}$$ regard as $f(n)$

where $f(x) = \dfrac{1}{x\ln x}$

(f does satisfy the conditions)

so $\sum \frac{1}{n\ln(n)}$ converges iff $\displaystyle\int_2^\infty \frac{dx}{x\ln x}$ converges

$$\int_2^k \frac{dx}{x\ln x} = \Big[\ln\ln(x)\Big]_2^k = \ln\ln k - \ln\ln 2$$

$$\longrightarrow \infty \text{ as } k \to \infty$$

The $\int$ diverges hence so does the $\sum$.

Looking ahead.

a series $\sum a_n$ is alternating if successive terms have opposite signs, and $|a_n| \geq |a_{n+1}| \ \forall n$

insert

$a_n \to 0$ as $n \to \infty$

eg $\displaystyle\sum_{n=1} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots$

Thm: ~~the~~ any alternating series converges.

A series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

A series $\sum a_n$ converges conditionally if it converges but not absolutely

eg $\sum \frac{(-1)^n}{n}$ converges conditionally