3.2.2 If $(a_n)_n$ is a null sequence $(a_n \to 0)$
and $(b_n)_n$ is a bounded sequence
then $(a_nb_n)_n$ is a null sequence

Proof Exercise on Week 5 sheet.

Ex 3.2.3 If $p>0$ $p \in \mathbb{R}$ then the sequence
$1/n^p \to 0$ as $n \to \infty$

Proof Let $\varepsilon > 0$.
Then $\frac{1}{n^p} < \varepsilon$ iff $\frac{1}{\varepsilon} < n^p$ iff $\frac{1}{\varepsilon^{1/p}} < n$
So, if we set $N = [\frac{1}{\varepsilon^{1/p}}]+1$, then $N > \frac{1}{\varepsilon^{1/p}}$
so, if $n \geq N$, we'll have $\frac{1}{n^p} < \varepsilon$ (since the steps all reverse "iff")
So the sequence $1/n^p \to 0$ as $n \to \infty$ //

Ex 3.2.4 Claim $\frac{n^2+n+1}{n^2-n+1} \to 1$ as $n \to \infty$

$$= \frac{1+\frac{1}{n}+\frac{1}{n^2}}{1-\frac{1}{n}+\frac{1}{n^2}} \longrightarrow \frac{1+0+0}{1-0+0}$$ by 3.2.3 (with $p=1,2$) and AoL (3.2.1)

$$=1$$

Ex 3.2.5 If $p>0$ then the sequence n^p is unbounded
Given any $l \in \mathbb{R}^{\geq 0}$, there is N such that
$N \geq l^{1/p}$ so $N^p \geq l$ as required

Ex 3.2.6 (i) If $(a_n)_n$ is unbounded and $(b_n)_n$ is convergent
then $(a_n+b_n)_n$ is unbounded
(ii) If $(a_n)_n$ is unbounded and $k>0$ then $(ka_n)_n$ is unbounded

Proof (i) Choose B such that $B \geq |b_n| \, \forall n$.
~~Choose~~ Given any $l \geq 0$, $l \in \mathbb{R}$, there is
N such that $|a_N| \geq l+B$ (since $(a_n)_n$ is unbound
Then $|a_N+b_N| = |a_N-(-b_N)| \geq ||a_N|-|b_N||$ [by 2.2.(1b]
$$\geq l+B-B = l$$
So $|a_N+b_N| \geq l$ and ~~[illegible]~~ ev
so $(a_n+b_n)_n$ is unbounded //

Chapter 4 Special Sequences

26/2/19 (2)

We'll show:

4.1.1 If $c>0$ then $c^{1/n} \to 1$ as $n \to \infty$.

4.1.2 If $0<c<1$ then $c^n \to 0$ as $n \to \infty$

4.1.4 For any c, $\frac{c^n}{n!} \to 0$ as $n \to \infty$

4.1.5 $n^{1/n} \to 1$ as $n \to \infty$

4.1.6 If $0<c<1$, k fixed, then $n^k c^n \to 0$ as $n \to \infty$

4.1.1 If $c>0$ then $c^{1/n} \to 1$ as $n \to \infty$

Proof (1) If $c=1$ this is obvious.

(2) If $c>1$ write $c = 1+y$; ~~for~~ note $y>0$

So we want to show $(1+y)^{1/n} \to 1$ as $n \to \infty$.

Note for any $x \geq 0$ $(1+x)^n = 1+nx +$ positive terms ← Binomial Expansion

$\geq 1+nx$

So $1+x \geq (1+nx)^{1/n}$

Apply this with $x = \frac{y}{n}$: $1+\frac{y}{n} \geq (1+y)^{1/n} \geq 1$

Since $\frac{1}{n} \to 0$ as $n \to \infty$ and y is constant,

$1+y/n \to 1+0 = 1$ as $n \to \infty$ (by AoL)

So our sequence $c^{1/n} = (1+y)^{1/n}$ is sandwiched between two sequences each of which converges to 1. Hence, by the Sandwich Theorem,

$c^{1/n} \to 1$ as $n \to \infty$ //

(3) If $0<c<1$ then $\frac{1}{c} > 1$ so, by (2),

$\left(\frac{1}{c}\right)^{1/n} \to 1$ as $n \to \infty$

That is $\frac{1}{c^{1/n}} \to 1$ as $n \to \infty$

So by AoL, $\frac{1}{\frac{1}{c^{1/n}}} \to \frac{1}{1} = 1$ as required

4.1.2 If $0 < c < 1$ then $c^n \to 0$ as $n \to \infty$

Proof Note $\frac{1}{c} > 1$ so write $\frac{1}{c} = 1 + x$ with $x > 0$

so $\frac{1}{c^n} = (1+x)^n = 1 + nx +$ positive terms

$> 1 + nx > nx$

Let $\varepsilon > 0$.

Then there is N such that, for all $n \geq N$,

$n > \frac{1}{x\varepsilon}$.

Then, for all $n \geq N$,

$\frac{1}{c^n} > nx > \frac{1}{\varepsilon}$, Hence $c^n < \varepsilon$.

So the sequence $(c^n)_n \to 0$ as $n \to \infty$.

4.1.3 If $(a_n)_n$ is convergent with limit l, and if $M \in \mathbb{Z}^{\geq 0}$ and define a new sequence $b_n = a_{n+M}$, then $(b_n)_n$ converges, with limit l.

Proof [eg $M = 2$, $b_1 = a_3$, $b_2 = a_4$, $b_3 = a_5$, ---]

Given $\varepsilon > 0$,

There is N such that $\forall n \geq N$

$|a_n - l| < \varepsilon$. (since $a_n \to l$)

Consider $|b_n - l| = \underbrace{|a_{n+M} - l|}_{\text{will be } < \varepsilon \text{ if, eg, } n \geq N}$

so for all $n \geq N$,

we have $|b_n - l| < \varepsilon$

and so the sequence $(b_n)_n$ converges to l //

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Exactly one of these statements is true.

If $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ are convergent sequences with limits a, b, c respectively, and if, for all n, we have $a_n < b_n < c_n$, then $a < b < c$.

The sequence $7, 7^{1/2}, 7^{1/3}, 7^{1/4}, \ldots, 7^{1/n}, \ldots$ converges to 1.

If $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ are convergent sequences with limits a, b, c respectively, then $\left(\dfrac{a_n}{b_n + c_n}\right)_{n\in\mathbb{N}}$ is convergent and has limit $\dfrac{a}{b+c}$.

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Exactly one of these statements is true. — True or False?

If $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ are convergent sequences with limits a, b, c respectively, and if, for all n, we have $a_n < b_n < c_n$, then $a < b < c$. False eg $a_n = \frac{1}{n^3}$ $b_n = \frac{1}{n^2}$ $c_n = \frac{1}{n}$

$\leq \leq$ ✓

The sequence $7, 7^{1/2}, 7^{1/3}, 7^{1/4}, \ldots, 7^{1/n}, \ldots$ converges to 1. True

If $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ are convergent sequences with limits a, b, c respectively, then $\left(\dfrac{a_n}{b_n + c_n}\right)_{n\in\mathbb{N}}$ is convergent and has limit $\dfrac{a}{b+c}$. False – eg if $b = -c$

4.1.4 For any $c \in \mathbb{R}$ $\frac{c^n}{n!} \to 0$ as $n \to \infty$ 27/2/19 (1)

Proof [eg $c=5$: $\frac{5}{1}, \frac{5^2}{2!}, \frac{5^3}{3!}, \frac{5^4}{4!}, \dots \to 0$]

↳ **Note** if $a_n = \frac{c^n}{n!}$, then $a_{n+1} = \frac{c^{n+1}}{(n+1)!} = \frac{c}{n+1} a_n$

So, if we choose N so that $N \geq 2c$, $\rightsquigarrow \frac{c}{N} \leq \frac{1}{2}$

then, for $n \geq N$, $a_{n+1} = \frac{c}{n+1} a_n \leq \frac{1}{2} a_n$

so $a_{n+2} \leq \frac{1}{2} a_{n+1} \leq \frac{1}{2} \cdot \frac{1}{2} a_n = \frac{1}{2^2} a_n$

and, by induction, $a_{n+t} \leq \frac{1}{2^t} a_n$ for $n \geq N$.

In particular $a_{N+t} \leq \frac{1}{2^t} a_N$

But the sequence $\frac{1}{2^t} \to 0$ as $t \to \infty$

hence so does the sequence $\frac{1}{2^t} a_N$

But $0 \leq a_{N+t} \leq \frac{1}{2^t} a_N$

So, by the Sandwich Rule, $a_{N+t} \to 0$ as $t \to \infty$

Hence, by 4.1.3, the sequence $a_n \to 0$ as $n \to \infty$.

4.1.5 The sequence $n^{1/n} \to 1$ as $n \to \infty$.

[i.e. $1 \quad \sqrt{2} \quad 3^{1/3} \quad 4^{1/4} \quad 5^{1/5} \dots \to 1$]

Proof Each term $n^{1/n} \geq 1$. Consider $k_n = n^{1/n} - 1 \geq 0$

Then $n^{1/n} = 1 + k_n$

so $n = (1+k_n)^n = 1 + nk_n + \frac{n(n-1)}{2} k_n^2 +$ positive terms

So $n \geq \frac{n(n-1)}{2} k_n^2$

and hence $k_n^2 \leq \frac{2n}{n(n-1)} = \frac{2}{n-1}$

Therefore $k_n \leq \frac{\sqrt{2}}{\sqrt{n-1}} \leq \frac{\sqrt{2}}{\sqrt{\frac{n}{2}}} \leq \frac{2}{\sqrt{n}}$ 27/2/19 (2)

But by 3.2.3, $\frac{1}{n^{1/2}} \to 0$ as $n \to \infty$

so $\frac{2}{n^{1/2}} \to 0$ as $n \to \infty$

So, since $0 \leq k_n \leq \underbrace{\frac{2}{n^{1/2}}}_{\searrow 0}$ $\forall n$,

by the Sandwich Rule, $k_n \to 0$ as $n \to \infty$

That is $n^{1/n} - 1 \to 0$ as $n \to \infty$

Hence $n^{1/n} \to 1$ as $n \to \infty$

4.1.6 If $0 < c < 1$, k fixed, then $\lim_{n\to\infty} n^k c^n = 0$

[Writing $d = \frac{1}{c}$, so $d > 1$, then $\frac{n^k}{d^n} \to 0$ as $n \to \infty$]

Proof See notes or use l'Hopital [later]

Examples

4.1.7 (1) $\lim_{n\to\infty} (3n)^{1/n} = ?$ $= 1$

$(3n)^{1/n} = 3^{1/n} n^{1/n}$ + AoL

$3^{1/n} \xrightarrow{4.1.1} 1$, $n^{1/n} \xrightarrow{4.1.5} 1$

(2) $\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n$ By 4.1.2 $\left(\frac{1}{2}\right)^n \to 0$ as $n \to \infty$

hence by 3.1.4 $\left(-\frac{1}{2}\right)^n \to 0$ as $n \to \infty$

(3) $\frac{n!}{n^n} = \frac{n(n-1)\cdots 2.1}{n.n.\cdots n.n} = \underbrace{\frac{n}{n}\cdot\frac{(n-1)}{n}\cdot\cdots\cdot\frac{2}{n}}_{\leq 1}\cdot\frac{1}{n} \leq \frac{1}{n}$

Since $0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$ $\forall n$,

and since $\frac{1}{n} \to 0$ as $n \to \infty$

By the Sandwich Rule, $\frac{n!}{n^n} \to 0$ as $n \to \infty$

So n^n has a "higher order of growth" than $n!$, which has a higher order of growth than (eg) 2^n, which has a higher order of growth than any polynomial in n^k 27/2/19 ③

and $\log n$ ~~grows~~ has an even slower order of growth than eg. n

ie ~~$\lim_{n\to\infty} \frac{n}{\log n} =$~~

$$\lim_{n\to\infty} \frac{\log n}{n} = 0$$

(by L'Hopital)

See the tables in Section 4.2

§4.2 Computing examples ...

Ex 4.26 Define the sequence $(a_n)_n$ by

$$a_1 = 2 \qquad a_{n+1} = \frac{a_n^2+2}{2a_n+1} \qquad \text{eg } a_2 = \frac{2^2+2}{2\times2+1} = \frac{6}{5}$$

Does this sequence converge? and to what limit?

(c) If this sequence does converge, say with limit l then, by the Algebra of Limits

$$l = \lim_{\substack{(n\to\infty)\\ n+1\to\infty}} a_{n+1} = \lim_{n\to\infty} \left(\frac{a_n^2+2}{2a_n+1}\right) = \frac{(\lim a_n)^2+2}{2\lim a_n+1} = \frac{l^2+2}{2l+1}$$

So $2l^2+l = l^2+2$

$\Rightarrow l^2+l-2=0 \Rightarrow (l+2)(l-1)=0$

$\Rightarrow l=-2$ or 1

But all $a_n \geq 0$, so $l \neq -2$
hence $l = 1$

So if the limit exists, then it must be $= 1$

(a)(b) To prove $(a_n)_n$ converges we show:

$a_n \geq 1 \quad \forall n$
$a_n \geq a_{n+1} \quad \forall n$
} by induction on n

and then, this being a decreasing sequence, bounded below, it must converge (by the Monotone Convergence Theorem)