STABILITY, SIMPLICITY AND THE MODEL THEORY OF BILINEAR FORMS

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Contents

Abstract

This thesis is concerned with the study of model theoretic properties of various classes of algebraic structures. We first investigate rings with stable theories. Then, in response to a problem posed by Chatzidakis and Pillay, we give an example of a complete, non-ω-stable theory of modules, T, such that $T \cup \{\sigma \text{ is an automorphism}\}$ has a model companion.

An infinite dimensional vector space over a finite field with a bilinear form has simple unstable theory, is $\aleph_0$-categorical and has quantifier elimination in a natural language. We investigate the model theory of vector spaces with bilinear forms over possibly infinite fields and prove quantifier elimination, completeness and model completeness results for these structures.

A countably infinite dimensional space with bilinear form can be considered as an ascending union of finite dimensional substructures. Motivated by the similarity between this situation and the definition of smoothly approximable structures we define the notion of a structure smoothly approximated by a given theory.

We note that a finite dimensional space over an algebraically closed field with a bilinear form is $\aleph_1$-categorical (and hence stable) and give an algebraic characterisation of non-forking in these structures. This non-forking relation can be extended to a meaningful notion of independence on the infinite dimensional space which is a 'smooth' union of these finite dimensional structures. We define a dimension and show that every definable set in such an infinite dimensional space has either finite dimension or finite codimension.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or institute of learning.

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Chapter 1

Introduction

The aim of this thesis is to investigate the model theoretic properties of certain classes of algebraic structures. In particular, we study rings (Chapter 3), modules (Chapter 4, Section 4.5) and vector spaces with bilinear forms (Chapters 5-12). Many of the model theoretic methods used come from the branch of model theory known as stability theory although here they are sometimes applied to unstable contexts.

The thesis falls roughly into four unequal parts. Chapter 2 on stable and simple theories, Chapter 3 on rings, Chapter 4 on structures with generic predicates or automorphisms and finally, Chapters 5-12 on vector spaces with bilinear forms (of which Chapters 5 and 6 are reviews of the relevant literature).

The first part (Chapter 2) explores some of the literature on simple and stable theories. The concepts of forking and dividing in an arbitrary complete theory are introduced and simple theories are defined. We then give a proof (due to Kim [38]) that forking and dividing are equivalent in a simple theory, and that the non-forking relation defines a notion of independence on simple theories which is symmetric and transitive. The class of stable theories is a subclass of the class of simple theories. We discuss various characterisations of this class of theories and of non-forking in these theories. We then consider how some properties of stable theories, such as the existence of canonical bases and the definability of types, generalise to simple theories. Except for some minor observations in Sections 2.5.1 and 2.5.2 (e.g. the definability part of Proposition 2.5.5, Corollary 2.5.6 and the proof of 2.5.8) none of the results of this chapter are original.

Chapter 3 investigates properties of rings which have stable theories. Many of the results on these rings apply to the larger class of rings without the strict order property, which includes the class of rings with simple theories. We demonstrate that a ring without the strict order property is semiprimary and from this deduce structure theorems (due to Cherlin and Reineke [18]) for stable rings which are central idempotent lifting (in particular stable commutative rings), from which it follows that such rings are built out of stable local rings. We then prove that if R is a stable local ring then, except in the case where the Jacobson radical has finite index, both the Abelian group $(R, +)$ and the unit group $U(R)$ are connected (Theorem 3.2.16).

Chapter 4 is concerned with a construction due to Chatzidakis and Pillay ([13]) which given a stable theory, T, produces a simple, usually unstable, theory T_A. The theory T_A is constructed as the model companion of a theory T_0 extending T, obtained by adding either a generic predicate or a generic automorphism to T. In the case of generic automorphisms it is difficult in general to establish whether the model companion T_A exists. In [13] it is shown that if T is an ω-stable theory of a module then T_A exists. It is then asked if there is a non-ω-stable theory of modules such that T_A exists. In Section 4.5 an example is given providing a positive solution to this question (Proposition 4.5.10). We also show that if it were possible to give necessary and sufficient conditions for T_A to exist for an arbitrary theory of modules this would then provide a complete classification of the simple von-Neumann regular rings which have coherent polynomial rings, a long standing problem in ring theory.

From Chapter 5 onwards we turn our attention to vector spaces with bilinear forms. Chapter 5 provides a survey of the classical theory of bilinear forms with particular emphasis on the less familiar case of infinite dimensional vector spaces with bilinear forms, where many of the well known results of finite dimensional spaces break down.

Chapter 6 explores some of the model theoretic motivation behind this work. An infinite dimensional bilinear space over a finite field is an example of a *lie geometry*. These structures were studied by Hrushovski, Macpherson and others as part of the classification of smoothly approximable structures. It was demonstrated that a smoothly approximable structure is essentially a structure built in a particular way out of lie geometries. We give some model theoretic properties of these lie geometries (such as quantifier elimination, $\aleph_0$-categoricity and simplicity) and sketch the process in which they are glued together to produce the class of smoothly approximable structures.

In Chapter 7 we begin to consider vector spaces with bilinear forms over infinite fields. We discuss possible choices of language for these structures; the difficulty being that, as opposed to the case of finite fields (or of vector spaces without bilinear forms), it does not make sense to add the field to the language. We show that if we require a language strong enough to interpret vector addition and orthogonality we are then forced into using a very expressive two-sorted language with separate sorts for vector and scalar elements in which it is possible to axiomatise the theory of the underlying field (Theorem 7.4.2). (This is essentially an application of Desargues' Theorem to the fact that linear span on a vector space with bilinear form is definable in terms of orthogonality.) We also prove (in contrast to the case of finite fields) that, in any sufficiently expressive language, the theory of an infinite dimensional vector space over an infinite field with bilinear form is never a simple (Proposition 7.4.1).

Before going on to study the model theory of vector spaces with bilinear forms in the two sorted language, a language is considered where the orthogonality relation is not definable and hence where the definability results of Chapter 7 do not apply. Suppose that K is an infinite field and F is a finite subfield of K. Then if V is a K-vector space with bilinear form Φ we can decompose Φ into infinitely many different bilinear forms with respect to some fixed basis for K as a vector space over F. We use a language where vector addition, K-scalar multiplication and the decomposed bilinear forms into F (but not the original bilinear form Φ) are definable and prove that if K is algebraic over F it is possible to chose an F-basis for K such that the resulting theory T has elimination of quantifiers in this language and is simple (unstable).

Chapter 9 begins the of study vector spaces with bilinear forms in the two sorted language resulting from the definability of orthogonality as described in Chapter 7. In this language linear dependence is definable and we show that if predicates for linear span are added to the language as new atomic formulas then these structures have quantifier elimination in this expanded language (Theorem 9.2.3). We also prove a model completeness result and deduce that under certain conditions on the field the theory of vector spaces with bilinear forms over that field are complete and the countable models are classified by classifying the countable models of the field. We relate this last result to the notion of (ω, ω)-categoricity, a weakening of the notion of $\aleph_0$-categoricity defined in [32]. We also characterise algebraic and definable closure in theories of vector spaces with bilinear forms (Proposition 9.5.1). From this point on we mostly restrict our attention to spaces over algebraically closed fields - model theoretically the most easily understood infinite fields. This is to avoid confusing the issues concerning the model theory of bilinear forms with the model theory of the underlying field.

A countably infinite dimensional vector space with bilinear form over an algebraically closed field can be considered as a union of finite dimensional non-degenerate subspaces. Whilst the infinite dimensional space does not have a simple theory (and therefore is unstable), the finite dimensional spaces are

ω-stable and in fact $\aleph_1$-categorical. In Chapter 10 we study these finite dimensional spaces. We prove that such a space is interpretable in its base field and if constants for an orthonormal (or normal) basis are added to the space then the expanded structure is bi-interpretable with its base field. An algebraic characterisation of non-forking on these finite dimensional structures (without a fixed basis) is given (Proposition 10.2.2) and Morley rank is related to a dimension function on definable sets (Proposition 10.3.8).

In Chapter 11, some of the results on vector spaces with bilinear forms are applied to theories of pure two-sorted vector spaces (using a much stronger language than the usual first order language used for theories of vector spaces). Originally we proved many of these results before the corresponding results for spaces with bilinear forms. However, it seemed easier to give these results as consequences of the more general case than to prove everything twice. Thus, in this section, we conclude that theories of two-sorted pure vector spaces have elimination of quantifiers modulo a set of formulas defining linear independence and an elimination set for the field sort. We also characterise algebraic and definable closure and give an algebraic characterisation of non-forking in these structures.

Chapter 12 returns to theories of vector spaces with bilinear forms. Using the results of Chapter 10 we study countably infinite dimensional spaces over algebraically closed fields as the unions of finite dimensional spaces and show, by the quantifier elimination, that these finite dimensional spaces are homogenous substructures. Motivated by the similarity between this situation and the definition of smoothly approximable structures (by finite substructures) we define a notion of a structure smoothly approximated by a given theory (Definitions 12.1.2 and 12.1.3). By exploiting the fact that a countably infinite dimensional space is a 'smooth' union of finite dimensional substructures, the non-forking relation on the (stable) finite dimensional substructures can be extended to a notion of independence on their union. This notion of independence behaves very much like a notion of independence in a stable theory (only the local character axiom fails) but of course the theory of the infinite dimensional space is not even simple.

We explore the properties of dividing and some related concepts on the infinite dimensional space and show that if dividing is equivalent to a concept of 'dividing uniformly' that forking and dividing are equivalent. We also investigate the existence of a dimension on definable sets of vector elements and show that in some sense every such definable set has finite dimension or finite codimension (Proposition 12.4.1).

We conclude by giving a list of some open questions and some ideas for extensions of this work.

Chapter 2

Simple and Stable Theories

2.1 Notation and Conventions

L will denote a first order language, which will always contain a symbol for equality. An L-theory T is a consistent set of L-sentences which is usually assumed to be complete. M and N etc. will denote models of T and also the domains of these models. $|L|$ and $|T|$ denote the cardinality of L, i.e. the number of logical symbols in L. Given a model $M \vDash T$, $|M|$ denotes the cardinality of the domain of M.

We assume familiar theorems of first order predicate logic such as compactness and the upwards and downwards Löwenheim-Skolem theorems. In this section we briefly outline some well known definitions and theorems from model theory. (See [12] or [31] for details.)

Definition 2.1.1 *Let L be a language, let T an L-theory and let κ be an infinite cardinal.*

1. *A model $M \vDash T$ is κ-*saturated *if for every $A \subseteq M$ with $|A| < \kappa$, every complete type $p \in S(A)$ is realised in M.*

2. *A model $M \vDash T$ is κ-*homogenous *if for every $A \subseteq M$ with $|A| < \kappa$, every elementary map $f : A \to M$ extends to an automorphism of M.*

(This is sometimes called strong κ-homogeneity.)

3. *A model $M \models T$ is* κ-universal *if for every model $N \equiv M$ with $|N| < \kappa$, N can be elementarily embedded in M.*

4. *A model $M \models T$ is* saturated *(respectively* homogenous*, respectively* universal*) if M is $|M|$-saturated (resp. $|M|$-homogenous, resp. $|M|^+$-universal).*

Note that a κ-saturated model is necessarily κ-homogenous and κ-universal. For a given theory T and cardinal κ, κ-saturated models of T of cardinality κ do not necessarily exist. However, if κ is an infinite cardinal with $|\kappa| \geq |L|$ then any theory T possesses a κ^+-saturated model of cardinality 2^κ. A (countable) theory T has a countable saturated model if and only if T is a small theory. (A theory T is small if there are only countably many n-types over $\emptyset$ for each $n \in \omega$.)

We work in a *monster model* $\tilde{M}$ of T which is a λ-saturated model of T for some very large cardinal λ (e.g. λ strongly inaccessible if such cardinals exist). This means that any model of T of cardinality $< \lambda$ is an elementary submodel of $\tilde{M}$ and that any type over a small subset of $\tilde{M}$ is realised in $\tilde{M}$. (A subset $A \subset M$ is said to be small if $|A| < |\tilde{M}|$.) If $\bar{a}$ is a tuple in $\tilde{M}$ and $\phi(\bar{x})$ is an L-formula we can then write $\models \phi(\bar{a})$ unambiguously to mean that $\tilde{M} \models \phi(\bar{a})$ and also that $M \models \phi(\bar{a})$ for any model M of T with $|M| < \lambda$ and $\bar{a}$ contained in M. Once the monster model $\tilde{M} \models T$, has been fixed, by a model of T we really mean an elementary submodel of $\tilde{M}$, an element will be an element of $\tilde{M}$ and a set will be a small subset of $\tilde{M}$. For $A \subseteq \tilde{M}$ we write $S_n(A)$ for the set of complete n-types over A and write $S(A)$ for the set $\bigcup_{n\in\omega} S(A)$.

Fix a complete theory T in a language L. Given a model $M \models T$, it is often useful to be able to regard equivalence classes of $\emptyset$-definable equivalence relations as elements of M. We can do this by expanding M to a many-sorted structure in the many-sorted language L^{eq}.

Let $\mathcal{E}$ be the set of all L-formulas $E(\bar{x}, \bar{y})$ which define equivalence relations on n-tuples of models of T for some $n \in \omega$. For each $E \in \mathcal{E}$ we add a new sort S_E to the language and a new function symbol f_E taking n-tuples from sort $S_=$ to elements of sort S_E. ($S_=$ is the sort consisting of equivalence classes of single elements under the trivial equivalence relation '='.) The resulting language is an expansion of L and is called L^{eq}. The theory T^{eq} contains the axioms of T restricted to the sort $S_=$, together with axioms expressing that for each $E \in \mathcal{E}$, f_E is a surjective map such that $\forall \bar{x} \forall \bar{y} (E(\bar{x}, \bar{y}) \leftrightarrow f_E(\bar{x}) = f_E(\bar{y}))$. This theory is complete and if $M^{eq} \models T^{eq}$ then $M^{eq} \upharpoonright S_=$ is isomorphic to some model M of T and then $M^{eq} \upharpoonright S_E = M^n/E$.

The process of passing from M to M^{eq} is relatively harmless: if X is a subset of M definable in M^{eq} then X is already definable in M. Also, if $\tilde{M}$ is a monster model for T then $\tilde{M}^{eq}$ is a monster model of T^{eq} and there is a natural one to one correspondence between elementary submodels of $\tilde{M}$ and elementary submodels of $\tilde{M}^{eq}$. The elements of $\tilde{M}^{eq}$ are referred to as imaginary elements. The elements of sort $S_=$, which correspond to the elements of the original structure M are called home-sort elements.

Working in L^{eq} gives us the very useful property that definable sets have canonical parameters. Suppose that $X = \phi(\bar{x}, \bar{a})$ is a definable set in $\tilde{M}$. Let $E(\bar{y}, \bar{y}')$ be the equivalence relation $\forall \bar{x} (\phi(\bar{x}, \bar{y}) \leftrightarrow \phi(\bar{x}, \bar{y}'))$. Then $\models E(\bar{a}, \bar{b}) \Leftrightarrow \phi(\tilde{M}, \bar{a}) = \phi(\tilde{M}, \bar{b})$ and hence if c is the element $\bar{a}/E$ of $\tilde{M}^{eq}$, an automorphism $f \in Aut(\tilde{M}^{eq})$ fixes the definable set X setwise if and only if $f(c) = c$. The (unique) element c is called the *code (or canonical parameter)* of the definable set X. This argument can also be carried out for a definable set X of $\tilde{M}^{eq}$, showing that the theory T^{eq} has the property of elimination of imaginaries.

Definition 2.1.2 *A theory T has* elimination of imaginaries *if for every definable set X (in $\tilde{M}$) there is a tuple $\bar{c}$ in $\tilde{M}$ such that for any $f \in Aut(\tilde{M})$, f fixes X pointwise if and only if $f(\bar{c}) = \bar{c}$.*

Equivalently, T has elimination of imaginaries if and only if for every

$a \in \tilde{M}^{eq}$ there is a tuple $\bar{b}$ in $\tilde{M}$ such that a and $\bar{b}$ are interdefinable ($a \in dcl(\bar{b})$ and $\bar{b} \in dcl(a)$).

As remarked above, T^{eq} has elimination of imaginaries for any theory T. This means that it is only necessary to carry out the T^{eq} construction once, i.e. $(M^{eq})^{eq} = M^{eq}$. However there are also examples of theories which have elimination of imaginaries in their natural (one-sorted) language. For instance the theory of algebraically closed fields (of some fixed characteristic) has elimination of imaginaries. (See e.g. [55].)

Another advantage of working in $\tilde{M}^{eq}$ is that tuples of elements in $\tilde{M}$ become single elements in $\tilde{M}^{eq}$. We usually assume that any model $M \models T$ is in fact a model of T^{eq} and make use of imaginary sorts where convenient. In particular if $\bar{b}$ is a tuple of $S_=$-sort elements then $\bar{b}$ can be considered as a single element in M^{eq} and so we can write $a \in M$ to mean that a is a tuple of elements from M or even an element of some other imaginary sort. We will therefore not distinguish between tuples and single elements except on a few occasions when it is clear we must work in $\tilde{M}$ rather than $\tilde{M}^{eq}$. In this case we slightly abuse notation by writing $\bar{b} \in M$ to mean that $\bar{b}$ is an n-tuple for some n and $\bar{b} \in M^n$. We also write Ab for $A \cup \{b\}$ for the union of a set and a tuple.

Since we will be working in $\tilde{M}^{eq}$, algebraic and definable closure will often be taken to mean algebraic and definable closure in the structure $\tilde{M}^{eq}$ (sometimes written $acl^{eq}(-)$ and $dcl^{eq}(-)$). Note that $\tilde{M}^{eq} = dcl^{eq}(\tilde{M})$. We also write $stp(a/A)$ to mean $tp(a/acl^{eq}(A))$ and here it is very important that the algebraic closure is taken in $\tilde{M}^{eq}$.

Definition 2.1.3 *Let L and L' be languages and let M be an L-structure and N an L'-structure.*

1. *An* interpretation *of M in N consists of the following data: firstly a definable subset U of N^{eq}, secondly, for each $\emptyset$-definable relation R on M a $\emptyset$-definable (in N^{eq}) relation $\bar{R}$ on U and finally an isomorphism of*

L-structures $f : (M, R, \dots) \to (U, \bar{R}, \dots)$. (We sometimes refer to the isomorphism f as the interpretation.)

2. *The structures M and N are* bi-interpretable *if there are interpretations f of M in N and g of N in M such that the map $g \circ f$ is $\emptyset$-definable in M and the map $f \circ g$ is $\emptyset$-definable in N.*

Note that in order to show that M is interpretable in N, as in part 1 of the definition, it is sufficient to consider the $\emptyset$-definable relations on M induced by unnested atomic formulas. (An unnested atomic formula is a formula of the form $x = y$, $x = c$, $F(\bar{x}) = y$ and $P(\bar{x})$ for c, F, P the constants, functions and predicates respectively, in the language L.) See [31] Section 5.3 for further details. Note also that it is possible to have structures M and N such that M is interpretable in N, N is interpretable in M but M and N are not bi-interpretable. In fact we later produce examples of such pairs of structures. See Proposition 10.1.3.

We conclude this section by stating two well known theorems on categoricity.

Theorem 2.1.4 (Ryll-Nardzewski) *For a (countable) theory T the following are equivalent:*

1. *T is $\aleph_0$-categorical (i.e. T has a unique countable model).*
2. *For each $n \in \omega$, $S_n(\emptyset)$ is finite.*
3. *For each $n \in \omega$, there are only finitely many formulas in n free variables (up to equivalence mod T).*

Theorem 2.1.5 (Morley) *Let T be a countable theory. Suppose that T is categorical in some uncountable cardinality then T is categorical in all uncountable cardinalities. (Such a theory is referred to as an* uncountably categorical theory *or an* $\aleph_1$-categorical theory*.)*

The above theorem was extended to theories of uncountable cardinality by Shelah: for any theory T, if T is categorical in some cardinal $\lambda > |T|$, then T is categorical in all cardinals $\mu > |T|$ ([61]).

2.2 Forking and Dividing

In this section we define the concepts of dividing and forking in an arbitrary theory and explore some of their basic properties. We also introduce the class of simple theories and give some equivalent characterisations of this class.

We work, throughout, in a complete theory T, with a monster model $\tilde{M}$. For the moment, there are no other assumptions on T. All the definitions and results of this section are due to Shelah and can be found in [61] or [62]. See also [38] and [28] for a more contemporary approach.

Definition 2.2.1 *Let $\phi(x, y) \in L$, let b be a tuple and let A be a set of parameters:*

1. *The formula $\phi(x, b)$* divides *over A* with respect to k *if there is an infinite A-indiscernible sequence $\{b_i : i \in \omega\}$ with $b_0 = b$, such that the set $\{\phi(x, b_i) : i \in \omega\}$ is k-inconsistent.*

   $\phi(x, b)$ divides *over A if $\phi(x, b)$ divides over A with respect to k for some $k \in \omega$. (Hence $\phi(x, b)$ divides over A if and only if there is an infinite A-indiscernible sequence $\{b_i : i \in \omega\}$ with $b_0 = b$, such that $\{\phi(x, b_i) : i \in \omega\}$ is inconsistent.)*

2. *A (partial) type $p(x)$* divides *over A if there is some formula $\phi(x, b)$ such that $p(x) \vdash \phi(x, b)$ and $\phi(x, b)$ divides over A.*

Lemma 2.2.2 *A formula $\phi(x, b)$* divides *over A with respect to k if and only if there is some set $\{b_i : i \in \omega\}$ of realisations of $tp(b/A)$ such that $\{\phi(x, b_i) : i \in \omega\}$ is k-inconsistent.*

Lemma 2.2.3 *A (partial) type $p(x, b)$ over Ab does not divide over A if and only if for every infinite A-indiscernible sequence I containing b, there is c realising $p(x, b)$ such that I is Ac-indiscernible.*

Definition 2.2.4

1. *A formula $\phi(x, b)$* forks *over a set A if there is $n \in \omega$ and formulas $\phi_i(x, b^i)$ for $i < n$ such that $\phi(x, b) \vdash \bigvee_{i<n} \phi_i(x, b^i)$ and for each $i < n$, $\phi_i(x, b^i)$ divides over A.*

2. *A (partial) type $p(x)$* forks *over A if there is some formula $\phi(x, b)$ such that $p(x) \vdash \phi(x, b)$ and $\phi(x, b)$ forks over A.*

Thus a formula forks over a set A if it can be 'covered' by finitely many formulas each of which divide over A. Obviously if a formula (or partial type) divides over a set A, then it forks over A. Also, if $\phi(x, b) \vdash \bigvee_{i<n} \phi_i(x, b^i)$ and each $\phi_i(x, b^i)$ forks over A then $\phi(x, b)$ forks over A. The advantage of working with the concept of forking, rather than that of dividing, is that it is possible to prove the extension axiom. We demonstrate this below and also give some other properties of non-forking in an arbitrary theory.

Proposition 2.2.5 *Let T be any theory. Then non-forking satisfies the following:*

1. *(Extension) Suppose that p does not fork over A and that $B \supseteq dom(p)$. Then there exists $q \in S(B)$, extending p, such that q does not fork over A.*

2. *(Partial transitivity) Let $A \subseteq B \subseteq C$ and suppose $tp(d/C)$ does not fork over A, then $tp(d/C)$ does not fork over B and $tp(d/B)$ does not fork over A.*

3. *(Finite character) $tp(c/B)$ does not fork over A if and only if $tp(c/Ab)$ does not fork over A for each finite $b \subseteq B$.*

4. *(Automorphism invariance) p does not fork over A if and only if for every A-automorphism f, $f(p)$ does not fork over A.*

Proof. (of extension) Let $\Gamma = \{\neg\psi(x,b) : b \in B,\ \psi(x,b) \text{ forks over } A\}$. Then, if $p \cup \Gamma$ is inconsistent, there is a finite subset $\{\neg\psi_i(x,b_i) : i < n\} \subseteq \Gamma$ such that $p \cup \{\neg\psi_i(x,b_i) : i < n\}$ is inconsistent. Thus $p \vdash \bigvee_{i<n} \psi_i(x,b_i)$ and each $\psi_i(x,b_i)$ forks over A. Thus p forks over A contradicting the assumption. Therefore $p \cup \Gamma$ is consistent so we can choose $q \in S(B)$ extending $p \cup \Gamma$. If q forks over A, there is some formula $\psi(x,b) \in q$ such that $\psi(x,b)$ forks over A. But then $\neg\psi(x,b) \in \Gamma \subseteq q$ which is a contradiction. Hence q does not fork over A. ∎

Definition 2.2.6 *Let $p \in S(B)$ and let $A \subseteq B$. A sequence $I = \{c_i : i \in \omega\}$, of realisations of p, is called a* Morley sequence *of p over A if:*

1. *I is indiscernible over B;*
2. *for each $j \in \omega$, $tp(c_j/B \cup \{c_i : i < j\})$ does not fork over A.*

A Morley sequence of $p \in S(B)$ is a Morley sequence of p over B.

Lemma 2.2.7 *Let $A \subseteq B$. Suppose that $p \in S(B)$ and p does not fork over A. Then there exists a Morley sequence for p over A.*

Definition 2.2.8 *A theory T is* simple *if for every type $p \in S(A)$, there is some subset $A_0 \subseteq A$ such that p does not fork over A_0 and $|A_0| \leq |T|$. (Equivalently, there is some subset $A_0 \subseteq A$ with $|A_0| \leq |T|$ such that p does not divide over A_0.)*

There is an alternative definition of simple theories given by ruling out the possibility of the existence of certain trees of uniformly definable sets.

Definition 2.2.9 *A formula $\phi(x, y)$ has the* tree property *if there is $k \in \omega$ and tuples $(\bar{a}_\eta : \eta \in \omega^{<\omega})$ such that :*

1. *$\{\phi(x, \bar{a}_{\eta\hat{\ }r}) : r \in \omega\}$ is k-inconsistent for all $\eta \in \omega^{<\omega}$.*

2. *$\{\phi(x, \bar{a}_{\beta\restriction r}) : r \in \omega\}$ is consistent for all $\beta \in \omega^\omega$.*

Proposition 2.2.10 *[62] A theory T is simple if and only if T does not have the tree property (i.e. if and only if no formula has the tree property).*

Proposition 2.2.10 says (roughly) that a theory is simple if and only if it is impossible to define a tree of uniformly definable sets with infinite height and with each node sprouting infinitely many branches such that all paths up the tree are consistent but the set of branches at each node is k-inconsistent.

Definition 2.2.11 *Let p be a partial type, Δ a finite set of formulas, and k a positive integer. Define the ordinal valued rank $D(p, \Delta, k)$ by induction as follows:*

- *$D(p, \Delta, k) \geq 0$ iff p is consistent.*
- *$D(p, \Delta, k) \geq \delta$ for δ a limit ordinal iff $D(p, \Delta, k) \geq \alpha$ for all $\alpha < \delta$.*
- *$D(p, \Delta, k) \geq \alpha + 1$ if there exists a formula $\phi(x, y) \in \Delta$ and tuples $\{a_i : i \in \omega\}$ such that $D(p \cup \{\phi(x, a_i)\}, \Delta, k) \geq \alpha$ for each $i \in \omega$ and $\{\phi(x, a_i) : i \in \omega\}$ is k-inconsistent.*
- *$D(p, \Delta, k) = \alpha$ if $D(p, \Delta, k) \geq \alpha$ and $D(p, \Delta, k) \not\geq \alpha + 1$.*
- *$D(p, \Delta, k) = \infty$ if $D(p, \Delta, k) \geq \alpha$ for all ordinals α.*

It is, in fact, the case that $D(p, \Delta, k) = \infty$ if and only if $D(p, \Delta, k) \geq \omega$ and so the rank is either a natural number or ∞. It is easily seen that the rank has the following monotonicity property: if $p_1 \vdash p_2$, $\Delta_1 \subseteq \Delta_2$ and $k_1 \leq k_2$ then $D(p_1, \Delta_1, k_1) \leq D(p_2, \Delta_2, k_2)$. From this it follows (by a straightforward induction) that:

Proposition 2.2.12 *[62] A formula $\phi(x, y)$ has the tree property if and only if for some $k \in \omega$, $D(x = x, \phi(x, y), k) = \infty$. Hence, by Proposition 2.2.10, a theory T is simple if and only if $D(p, \Delta, k)$ is finite for all partial types p, finite sets of formulas Δ, and integers k.*

All stable theories are simple. (See Section 2.4.) Examples of simple unstable theories include the random graph (see Section 4.3 for a proof of this), algebraically closed fields with an automorphism ([14]) and more generally stable theories with a generic predicate or a generic automorphism ([13]). (See Chapter 4.) Smoothly approximable structures (as defined in Chapter 6) are known to be simple. This is due to Hrushovski ([33]). In particular, vector spaces over finite fields with non-degenerate bilinear forms are examples of simple unstable theories. (See Proposition 6.2.5 for a proof of this.) In Chapter 8 we give an example of a simple unstable theory arising from a vector space with a bilinear form over an infinite field.

2.3 Forking and Dividing in Simple Theories

A fundamental result about simple theories (due to Kim, see [38]) is that the concepts of forking and dividing are equivalent. This leads to a quick proof that the non-forking relation in simple theories is symmetric and transitive. We outline the main steps in the proofs of these facts. All the results of this section are due to Kim and Pillay and are contained in [38] and [40]. The proof given for Theorem 2.3.2 is due to Shelah (in [28]) and is a simplification of the original proof given by Kim.

We first note that Morley sequences exist for types in simple theories. Suppose $p \in S(A)$ is a type in a simple theory. There is some $A_0 \subseteq A$ such that p does not fork over A_0. Then, by the partial transitivity of non-forking, p does not fork over A and hence a Morley sequence for p (over A) exists.

Proposition 2.3.1 *Let T be simple. The following are equivalent:*

1. *$\phi(x, b)$ divides over A;*

2. *For every Morley sequence $\{b_i : i \in \omega\}$ for $tp(b/A)$, the set $\{\phi(x, b_i) : i \in \omega\}$ is inconsistent;*

3. *For some Morley sequence $\{b_i : i \in \omega\}$ for $tp(b/A)$, the set $\{\phi(x, b_i) : i \in \omega\}$ is inconsistent.*

Proof. (3)⇒(1) is by the definition of dividing and (2)⇒(3) is because Morley sequences exist.

(1)⇒(2) If $\phi(x, b)$ divides over A then, by definition, there is some sequence $\{a_i : i \in \omega\}$, indiscernible over A, with $a_0 = b$ and such that $\{\phi(x, a_i) : i \in \omega\}$ is k-inconsistent for some $k \in \omega$. Let $\{b_i : i \in \omega\}$ be some Morley sequence for $tp(b/A)$ and assume (for a contradiction) that $\{\phi(x, b_i) : i \in \omega\}$ is consistent. By the Concatenation Lemma (See [28] 1.13) we can find a sequence $\{c_i : i \in \omega\}$ such that for each $n \in \omega$:

$$tp(a_n, c_0, c_1, \ldots /A) = tp(c_0, c_1, c_2, \ldots /A) = tp(b_0, b_1, b_2, \ldots /A).$$

Let $p = \{\phi(x, c_n) : n \in \omega\}$. Then, by assumption, p is consistent. Let $\alpha = D(p, \phi(x, b), k)$. Then, for each $n \in \omega$, there is an A-automorphism f_n such that $f_n(a_n) = c_0$ and $f_n(c_{m+1}) = c_m$ for all $m \in \omega$. Therefore, since the rank is invariant under automorphisms, for every $n \in \omega$:

$$\begin{aligned} D(\phi(x, a_n) \cup p\,,\ \phi(x, y), k) &= D(f_n(\phi(x, a_n) \cup p)\,,\ \phi(x, y), k) \\ &= D(p\,,\ \phi(x, y), k) = \alpha. \end{aligned}$$

Thus, by definition of rank, (as $\{\phi(x, a_i) : i \in \omega\}$ is k-inconsistent) it follows that $D(p\,,\ \phi, k) \geq \alpha + 1$ which contradicts the fact that $D(p\,,\ \phi, k)$ is finite. ■

We can now prove the promised equivalence of forking and dividing.

Theorem 2.3.2 *Let T be a simple theory. The following are equivalent:*

1. *$\phi(x, b)$ divides over A;*

2. *$\phi(x, b)$ forks over A.*

Proof. (1)⇒(2) is clear.

(2)⇒(1): Since $\phi(x, b)$ forks over A there is $m \in \omega$ and formulas $\phi_j(x, a^j)$ for $j < m$ such that $\phi(x, b) \vdash \bigvee_{j<m} \phi_j(x, a^j)$ and each $\phi_j(x, a^j)$ divides over A. Let $\bar{a} = (a^0, \ldots, a^{m-1})$ and fix a Morley sequence $\{\bar{a}_i b_i : i \in \omega\}$ for $tp(\bar{a}b/A)$ with $\bar{a}_0 = \bar{a}$ and $b_0 = b$. Write $\bar{a}_i = (a_i^0, \ldots, a_i^{m-1})$. It is now sufficient to prove that $\{\phi(x, b_i) : i \in \omega\}$ is inconsistent. Suppose (for a contradiction) that it is consistent, realised by c. By applying suitable A-automorphisms we can assume that:

$$\phi(x, b_i) \vdash \bigvee_{j<m} \phi_j(x, a_i^j).$$

Therefore c satisfies: $\left\{\bigvee_{j<m} \phi_j(x, a_i^j) : i \in \omega\right\}$ and hence, by the pigeonhole principle, there is $j < m$ such that for some infinite $S \subseteq \omega$, c satisfies $\left\{\phi(x, a_i^j) : i \in S\right\}$. But, $\left\{a_i^j : i \in \omega\right\}$ is a Morley sequence for $tp(a_0^j/A)$ and hence by indiscernibility and compactness $\left\{\phi(x, a_i^j) : i \in \omega\right\}$ is consistent, thus, by Proposition 2.3.1, contradicting the fact that $\phi_j(x, a^j)$ divides over A. ■

Corollary 2.3.3 (Symmetry) *Let T be a simple theory. Then $tp(c/Ab)$ forks over A if and only if $tp(b/Ac)$ forks over A.*

Proof. Suppose that $tp(c/Ab)$ forks over A and choose $\phi(x, a, b) \in tp(c/Ab)$ such that $\phi(x, a, b)$ forks over A. Then, by Theorem 2.3.2, $\phi(x, a, b)$ divides over A. Suppose, for a contradiction, that $tp(b/Ac)$ does not fork over A. Then there is a Morley sequence $\{b_i : i \in \omega\}$ for $tp(b/Ac)$ over A with $b_0 = b$. Since c satisfies $\phi(x, a, b)$, by indiscernibility we have that $\{\phi(x, a, b_i) : i \in \omega\}$ is consistent (realised by c). Then, $\{ab_i : i \in \omega\}$ is a Morley sequence for

$tp(ab/A)$. Therefore by Proposition 2.3.1, $\{\phi(x, a, b_i) : i \in \omega\}$ is inconsistent and we have the required contradiction. ■

If c is a tuple of elements and c' is another tuple which contains the same elements as c but with a possibly different ordering, then it is clear that $tp(c/B)$ does not fork over A if and only if $tp(c'/B)$ does not fork over A. Thus if C is a possibly infinite set it makes sense to define non-forking and such 'infinitary' types by: $tp(C/B)$ does not fork over A if $tp(c/B)$ does not fork over A for each finite subset (tuple) $c \subseteq C$. We write $C \downarrow_A B$ (C and B are *independent* over A) to mean that $tp(C/B)$ does not fork over A.

Corollary 2.3.4 (Transitivity) *Let T be a simple theory and suppose $A \subseteq B \subseteq C$. Then $tp(d/C)$ does not fork over A if and only if $tp(d/C)$ does not fork over B and $tp(d/B)$ does not fork over A.*

Proof. We already have ($\Rightarrow$) by Proposition 2.2.5. To prove right to left it is sufficient (by finite character of non-forking) to prove that $tp(d/Ac)$ does not fork over A for any finite $c \subseteq C$. By symmetry, it suffices to prove that $tp(c/Ad)$ does not fork over A. Suppose, for a contradiction, that $tp(c/Ad)$ forks over A. Then by Theorem 2.3.2, there is $\phi(x, a, d) \in tp(c/Ad)$ such that $\phi(x, a, d)$ divides over A.

Since $tp(d/C)$ does not fork over B it follows that $tp(d/Ac)$ does not fork over B and hence by symmetry and 'forking equals dividing', we have that $tp(c/Ad)$ does not divide over B. However, $tp(d/B)$ does not fork over A and so there is a Morley sequence $\{d_i : i \in \omega\}$ for $tp(d/B)$ over A. Then $I = \{ad_i : i \in \omega\}$ is a Morley sequence for $tp(ad/B)$. Therefore, by definition of dividing, $\{\phi(x, a, d_i) : i \in \omega\}$ is consistent. Since I is a Morley sequence for $tp(ad/A)$ we have that $\phi(x, a, d)$ does not divide over A. This contradicts the choice of $\phi(x, a, d)$ and hence $tp(d/C)$ does not fork over A. ■

From the above it is possible to show that the ranks $D(-, \Delta, k)$ define forking in simple theories in the following sense.

Theorem 2.3.5 *[40] Let T be a simple theory. Let p be a type over B and let $A \subseteq B$. Then the following are equivalent:*

1. *p does not fork over A;*

2. *$D(p, \Delta, k) = D(p \restriction A, \Delta, k)$ for all $k \in \omega$ and all finite Δ.*

Definition 2.3.6 *Let T be any complete theory. We define a rank $SU(p)$ on complete types by:*

- *$SU(p) \geq 0$ for any complete type p;*

- *$SU(p) \geq \alpha + 1$ if there is $B \supseteq A$ ($A = dom(p)$) and an extension $q \in S(B)$ of p such that q forks over A and $SU(q) \geq \alpha$;*

- *$SU(p) \geq \delta$ for δ a limit ordinal, if $SU(p) \geq \alpha$ for all $\alpha < \delta$.*

Definition 2.3.7 *A theory T is* supersimple *if for every type $p \in S(A)$, there is a finite subset $A_0 \subseteq A$ such that p does not fork over A.*

Proposition 2.3.8 *1. If T is supersimple then T is simple.*

2. *T is supersimple if and only if for every type $p(x)$, $SU(p) < \infty$.*

3. *Suppose that T is simple, $p \subseteq q$ are complete types and at least one of p and q has ordinal valued SU-rank, then $SU(p) = SU(q)$ if and only if q is a non-forking extension of p.*

Proof. See [40]. ■

Note that if T is stable then SU-rank coincides with U-rank in stable theories. (See [51] or [53] for a definition of U-rank.) In particular a theory T is superstable if and only if it is stable and supersimple. (See Section 2.4 for definitions of stable and superstable theories.)

The following theorem was proved for simple theories by Kim and Pillay in [40]. It says that any two non-forking extensions, over independent sets, of a type over a model can be amalgamated into a common non-forking extension. This generalises the fact that types over models in a stable theory have unique non-forking extensions.

Theorem 2.3.9 (Independence over a model) *[40] Let T be a simple theory and $M \models T$. Let $p \in S(M)$. Let $A, B \supseteq M$ such that $A \downarrow_M B$ ($tp(A/B)$ does not fork over M). Suppose that $p_1 \in S(A)$ and $p_2 \in S(B)$ are non-forking extensions of p. Then there is $q \in S(A \cup B)$ extending both p_1 and p_2 such that q does not fork over M.*

It is possible to characterise simple theories as the class of theories admitting some notion of independence satisfying symmetry, transitivity, and certain other axioms, as well as the independence over a model theorem for that notion of independence.

Definition 2.3.10 *Let T be any theory and let Γ be a collection of triples (from some $\tilde{M} \models T$) of the form (c, B, A) with c a finite tuple and $A \subseteq B$ small subsets of $\tilde{M}$. Γ is called a* notion of independence *if the following axioms hold:*

1. ***(Invariance)*** *If $\sigma \in Aut(\tilde{M})$ then $(c, B, A) \in \Gamma$ if and only if $(\sigma(c), \sigma(B), \sigma(A)) \in \Gamma$;*

2. ***(Symmetry)*** *If $(c, A \cup \{b\}, A) \in \Gamma$, then $(b, A \cup \{c\}, A) \in \Gamma$;*

3. ***(Transitivity)*** *Suppose $A \subseteq B \subseteq C$. Then $(d, C, A) \in \Gamma$ if and only if $(d, C, B) \in \Gamma$ and $(d, B, A) \in \Gamma$;*

4. ***(Finite Character)*** *$(c, B, A) \in \Gamma$ if and only if for every finite $b \subseteq B$, $(c, A \cup \{b\}, A) \in \Gamma$;*

5. ***(Local Character)*** *For any c, B there is some $A \subseteq B$ such that $(c, B, A) \in \Gamma$ and $|A| \leq |T|$;*

6. ***(Extension)*** *For any c, A and $B \supseteq A$ there is c' such that $(c', B, A) \in \Gamma$ and $tp(c'/A) = tp(c/A)$;*

7. ***(Reflexivity)*** *If $c \notin acl(A)$, then $(c, A \cup \{c\}, A) \notin \Gamma$.*

If Γ is a notion of independence then we say that $tp(c/B)$ does not Γ-fork over A if $(c, B, A) \in \Gamma$. We say that C and B are Γ-independent over A if $(c, B, A) \in \Gamma$ for every finite $c \subseteq C$.

A notion of independence, Γ, satisfies the independence theorem over a model if Theorem 2.3.9 holds with Γ-non-forking and Γ-independence in place of usual non-forking and independence.

Theorem 2.3.11 *[40] Let T be any theory. Then T is simple if and only if T admits a notion of independence, Γ, which satisfies the independence theorem over a model. If Γ is such a notion of independence then $(c, B, A) \in \Gamma$ if and only if $tp(c/B)$ does not fork over A. (i.e. the notion of independence is exactly non-forking.)*

2.4 Stable Theories

Stable theories were introduced by Shelah as a tool in solving the classification problem of determining the spectrum function of a first order theory. (See [61].) If a theory T is unstable then it has the maximal possible number, 2^λ, of non-isomorphic models in every cardinal $\lambda > |T|$. Thus in order to count the models (of cardinality $\lambda > |T|$) of a theory T it can be assumed that T is stable. Shelah then defined the notion of non-forking (as in Definition 2.2.4) in stable theories in order to classify their models and thus solve the spectrum problem. However, in many stable (and also simple) theories, non-forking often turns out to have some kind of algebraic meaning. For instance, in vector spaces, non-forking is essentially linear independence and non-forking in algebraically closed fields turns out to be algebraic independence. In this section we define stability and give some important properties of the class of stable theories.

Definition 2.4.1 *Let T be a theory with a monster model $\tilde{M}$.*

1. *A formula $\phi(x, y)$ has the* order property *if there exist tuples $(a_i : i \in \omega)$ and $(b_i : i \in \omega)$ such that:*

$$\models \phi(a_i, b_j) \iff i < j.$$

2. *T is* stable *if no formula has the order property.*

Non-forking in a stable theory is a notion of independence and satisfies a stronger version of the independence over a model theorem:

T stable: If $M \models T$, $p \in S(M)$ and $A \supseteq M$ then,
if $p_1, p_2 \in S(A)$ are non-forking extensions of p then $p_1 = p_2$.

i.e. types over models are 'stationary'. In particular, any stable theory is simple. In fact, analogously to Theorem 2.3.11, a theory T is stable if and only if T admits a notion of independence, Γ, such that all types, p, over models have a unique Γ-non-forking extension to any set containing $dom(p)$. This is clear from the following theorem. The proof uses some results from later in the chapter.

Theorem 2.4.2 *For a theory T the following are equivalent:*

1. *T is stable;*

2. *T is a simple theory such that all types over models are stationary; (A type is* stationary *if it has a unique non-forking extension to any set containing its domain.)*

3. *there is some cardinal λ such that for all sets $A \subseteq \tilde{M}$, $|A| \leq \lambda \Rightarrow |S(A)| \leq \lambda$.*

Proof. (1)$\Leftrightarrow$(3) is by a counting types argument. (See [51] Chapter 2.) (1)$\Rightarrow$(2) follows from the fact that non-forking in a stable theory defines a notion of independence such that all types over models are stationary. (See almost any account of stability theory for this, e.g. [51] or [48].) To prove (2)$\Rightarrow$(1) note that if T is not stable then there is $M \models T$ and $p \in S(M)$ such that p is not definable. Then by [46] p has an unbounded number of heirs. But an heir of a type in a simple theory is a non-forking extension (Proposition 2.5.8) and hence p is not stationary. ■

Remark 2.4.3 *A theory T is called* λ-stable *if for all sets $A \subseteq \tilde{M}$, $|A| \leq \lambda \Rightarrow |S(A)| \leq \lambda$. So T is stable if and only if T is λ-stable for some cardinal λ. This allows us to define the class of* ω-stable theories*: the (countable) theories which are λ-stable for all infinite cardinals λ, and the* superstable *theories which are λ-stable for all $\lambda \geq 2^{|T|}$.*

Definition 2.4.4 1. *A formula $\phi(x, y)$ has the* strict order property *(SOP) if there exist tuples $(a_i : i \in \omega)$ such that:*

$$\models \exists x \, [\neg\phi(x, a_i) \leftrightarrow \phi(x, a_j)] \iff i < j.$$

i.e. $\phi(x, y)$ defines a pre-order on models of T with infinite chains.

2. *A formula $\phi(x, y)$ has the* independence property *if there exist tuples $(a_i : i \in \omega)$ and tuples $(b_I : I \subseteq \omega)$ such that:*

$$\models \phi(a_i, b_I) \iff i \in I.$$

Shelah ([61]) proved that a formula has the order property if and only if it has either the strict order property or the independence property. Thus a stable theory has neither the strict order property nor the independence property. Clearly the tree property implies the order property, providing another proof that any stable theory is simple. Also, the strict order property implies the tree property and hence no simple theory has the strict order property. In particular, a simple but unstable theory has the independence property.

According to the view of Pillay and others (see e.g. [41]) there are two aspects to non-forking in a stable theory: the first being the 'independence theory' saying that non-forking defines a well-behaved notion of independence, as described in Definition 2.3.10, all of which applies to simple theories as well as to stable theories; the second aspect is the theory of 'multiplicity' which concerns the definability of types and the related fact that any type has a bounded number of non-forking extensions. This multiplicity theory does not extend to simple theories although some parts of it can be recovered in different forms. This is investigated in Section 2.5. In this section we briefly describe some aspects of the multiplicity theory in the stable case.

A fundamental result about stable theories (which does not apply to simple theories) is the following:

Theorem 2.4.5 (The Finite Equivalence Relation Theorem) *Let T be a stable theory, let $p \in S(A)$ and let $B \supseteq A$. Suppose that p_1 and p_2 are distinct non-forking extensions of p to B. Then there is an equivalence relation $E(x, y)$ definable over A with finitely many equivalence classes such that*
$p_1(x) \cup p_2(y) \vdash \neg E(x, y)$.

From this it follows immediately that any type over a model in a stable theory is stationary. By working in T^{eq} a finite equivalence relation can be considered as an element in its own right and hence every equivalence class of a finite equivalence relation over A is an element of $acl^{eq}(A)$. Therefore Theorem 2.4.5 can be restated in the form: *in a stable theory, every type over an algebraically closed set is stationary.*

In most expositions of the theory of non-forking in stable theories, non-forking is not defined originally as in Definitions 2.2.1 and 2.2.4 in terms of dividing of formulas but is defined on types (or ϕ-types) via the fundamental order of Poizat (e.g. [46] and [51]), by definability of types (e.g. [53] and [3]) or by using an appropriate rank function ([6]), and then shown to be equivalent to Shelah's definition of non-forking (2.2.4).

We first explain how non-forking in a stable theory can be described in terms of the definability of types. It is possible to do this locally, in a possibly unstable theory, with respect to a boolean closed set of formulas without the order property. This approach is used in [34] to apply stability-theoretic methods to the study of groups definable in psuedofinite fields (which are unstable structures).

A formula $\phi(x, y)$ is *stable* if it does not have the order property. Note that stable formulas are closed under boolean combinations, see [52] or [53].

Definition 2.4.6 *Let $\phi(x, y)$ be a formula. A ϕ-*type *is a consistent set of formulas, each equivalent to a boolean combination of instances of $\phi(x, y)$ and $\neg\phi(x, y)$. A* complete ϕ-type *over A is a maximal consistent set of ϕ-formulas of L_A. The set of complete ϕ-types over A is denoted by $S_\phi(A)$.*

If $p \in S(A)$ or $S_\phi(A)$, then p is ϕ-definable *over $B \subseteq A$ if there is some formula $\psi(y) \in L_B$ such that for all $a \in A$: $\phi(x, a) \in p \iff \models \psi(a)$.*

Another fundamental result about stable theories is that types are definable.

Proposition 2.4.7 *(See [53]) Let T be any theory and let $\phi(x, y)$ be a stable formula. Let $M \models T$ and let $p \in S(M)$ or $S_\phi(M)$. Then p is ϕ-definable over M by a formula $\psi(y)$ which is a positive boolean combination of formulas $\phi(c, y)$ with $c \in M$.*

It is then possible to characterise non-forking in terms of definability. Assume, from now on, that $\phi(x, y)$ is a stable formula.

If $p \in S_\phi(M)$ is a ϕ-type over a model M, defined by the formula $\psi(y)$ then for $B \supseteq M$ consider the type $q \in S_\phi(B)$ given by $\phi(x, b) \in q \iff \models \psi(b)$. Then, since M is a model, q is complete (as a ϕ-type) and consistent and is clearly the unique such type. The type q will be the non-forking extension of p. Conversely, if we start with a type $q \in S_\phi(B)$ with $M \subseteq B$ then q is definable over B. It may also be the case that q is definable over M, i.e. the formula $\psi(y)$ defining q is over M. In this case we have that q does not fork over M.

In the general case things are not quite so straightforward. Suppose $A \subseteq B$ are any sets with $p \in S_\phi(A)$ and $p \subseteq q \in S_\phi(B)$. We want to know when q does not fork over A (i.e. when q is a non-forking extension of p). This is roughly when q is definable over A but there is a slight complication. In general, we must refine the concept of 'defined over' to that of 'defined almost over', where a formula ϕ is almost over A, if ϕ is a union of equivalence classes of a finite equivalence relation over A. Then $q \in S_\phi(B)$ does not fork over A if and only if q is definable by a formula almost over A. If we work in T^{eq} then we obtain the following characterisation of non-forking over algebraically closed sets: for $A = acl^{eq}(A)$ and $B = acl^{eq}(B)$, a type $q \in S_\phi(B)$ does not fork over A if and only if p is definable over A. The following Proposition is just a rephrasing of the Finite Equivalence Relation Theorem for ϕ-types, giving the additional information that all non-forking extensions of a type over a given set are conjugate.

Proposition 2.4.8 *[53] Let $\phi(x, y)$ be a stable formula and let $p(x) \in S_\phi(A)$. Let $B = acl^{eq}(A)$. Let $X = \{q(x) \in S_\phi(B) : q(x) \supseteq p(x)\}$. Let G be the group of elementary permutations of B which fix A pointwise. Then*

1. *G acts transitively on X.*

2. *There is an A-definable equivalence relation $E(x_1, x_2)$ with finitely many classes such that each E-class is defined by a ϕ-formula, and such that for $q_1, q_2 \in X$, $q_1 = q_2$ iff $q_1(x_1) \cup q_2(x_2) \models E(x_1, x_2)$. In particular X is finite.*

Note that any $q \in X$ as above is a non-forking extension of p (essentially because non-forking in a simple theory is automorphism invariant and p must have some non-forking extension in X, but since all such extensions are conjugate over A they must all be non-forking extensions.)

Definition 2.4.9 *Let $A \subseteq B$ and $q(x) \in S_\phi(B)$, then q is* definable almost over A *if for some (equivalently all) $q_1(x) \in S_\phi(acl(B))$ extending q, the ϕ-definition of q_1 is equivalent to the ϕ-definition of some $p_1 \in S_\phi(acl(A))$ extending $q \restriction A$.*

This is actually given in [53] as the definition of non-forking of a ϕ-type (for stable ϕ) and all the usual properties of non-forking in a stable theory are then derived from it. It is then shown, by the following theorem, which describes when a formula forks (or divides) over a set in a stable theory, that forking as given in Definition 2.2.4 coincides with forking given in terms of definability of types.

Theorem 2.4.10 *[53] Let A be a set of parameters and $\phi(x, b)$ an instance of a stable formula $\phi(x, y)$. The following are equivalent:*

1. *There is $q(x) \in S_\phi(Ab)$ such that $\phi(x, b) \in q$ and q is almost definable over A;*
2. *Some positive boolean combination of A-conjugates of $\phi(x, b)$ is A-definable and consistent;*
3. *Any set of $acl(A)$-conjugates of $\phi(x, b)$ is consistent;*
4. *Whenever $\{b_i : i \in \omega\}$ is an A-indiscernible sequence with $tp(b_0/A) = tp(b/A)$, then $\{\phi(x, b_i) : i \in \omega\}$ is consistent (i.e. $\phi(x, b)$ divides over A);*
5. *For any model $M_0 \supseteq A$, $\phi(x, b)$ is satisfied by an element of M_0.*

Thus $q \in S_\phi(B)$ does not fork over $A \subseteq B$ if and only if q is definable almost over A. If the theory T is stable then a complete type $q \in S(B)$ does not fork over a set $A \subseteq B$ if and only if for every $\phi(x, y) \in L$ the ϕ-type $q \restriction \phi$ does not fork over A. In this case there is a map (called a *defining schema*) $d : \phi(x, y) \to \psi_\phi(y)$, from L-formulas to L_B-formulas, such that for all $b \in B$, $\phi(x, b) \in q \iff \models \psi_\phi(b)$. In particular, if B is a model then $q(x)$ does not fork over A if and only if for each $\phi \in L$, the formula $\psi_\phi(y)$ is almost over A (i.e. is over $acl^{eq}(A)$).

Another useful fact about stable theories is that all stationary types possess 'canonical bases'. Recall that if T is any theory, $\tilde{M} \models T$ and X is a definable subset of $\tilde{M}$ then there is a 'code' for the set X: namely an element $a \in \tilde{M}^{eq}$ (unique up to interdefinability) such that for any $f \in Aut(\tilde{M})$, f fixes X setwise if and only if $f(a) = a$.

Suppose that $p \in S(A)$ is a stationary type in a stable theory and let p_1 be its unique non-forking extension to $acl^{eq}(A)$. Then for each $\phi(x, y) \in L$ let $\psi_\phi(y)$ be the ϕ-definition of a non-forking extension of p_1 to a model containing A. (Note that ψ_ϕ is well-defined and is over $acl^{eq}(A)$.) Let c_ϕ be the canonical parameter for the definable set ψ_ϕ. Define the *canonical base* of p to be $Cb(p) := dcl(\{c_\phi : \phi(x, y) \in L\})$. Then $Cb(p)$ is well defined for any stationary type in a stable theory and has the property that if M is a sufficiently saturated model of T and $f \in Aut(M)$, then f fixes the set of realisations of p in M setwise if and only if it fixes $Cb(p)$ pointwise. In fact we can define $Cb(p)$ for p a stationary type to be a definably closed set with this property and in stable theories canonical bases exist and are unique (as defined above).

We can then describe non-forking of stationary types in stable theories in terms of canonical bases:

Proposition 2.4.11 *[53] Let T be stable and let $p \in S(A)$ be stationary.*

1. *For any $B \subseteq A$, p does not fork over B if and only if $Cb(p) \subseteq acl(B)$.*

2. *For any $B \subseteq A$, p does not fork over B and $p \restriction B$ is stationary if and only if $Cb(p) \subseteq dcl(B)$.*

There is another way of defining non-forking in stable theories, described by the fundamental order of Poizat. See [46] or [51] for details.

Definition 2.4.12 *1. Let $p \in S(A)$. A formula $\phi(x, y)$ is* represented *in p if there is $a \in A$ such that $\phi(x, a) \in p$.*

2. *The* class *of the type p is the set: $cl(p) := \{\phi(x, y) \in L : \phi(x, y) \in L$ and there is $a \in A$ such that $\phi(x, a) \in p\}$. (i.e. $cl(p)$ is the set of L-formulas represented in p.)*

3. *The fundamental order (on n-types for a given n) is the ordered set $O_n(T) := (\{cl(p) : p \in S(M),\ M \models T\}, \subseteq)$.*

4. *For types p and q over models of T, write $p \leq q$ if $cl(p) \subseteq cl(q)$ and $p \sim q$ if $cl(p) = cl(q)$.*

5. *For $p \in S(B)$ and $A \subseteq B$ we can define $cl_A(p)$ to be the set of L_A-formulas represented in p and define the fundamental order relative to A as the set of A-classes of types over models containing A ordered by inclusion.*

Given a type $p \in S(A)$ we then define C_p to be the subset of the fundamental order given by $C_p := \{cl(q) : q \in S(M),\ p \subseteq q,\ M \models T,\ A \subseteq M\}$. Assuming that the theory T is stable it is then possible to prove that for any type p, the set C_p has a unique minimal element which we call $\beta(p)$, the *bound* of p. Non-forking can then be described in a stable theory in terms of the bounds of types:

Theorem 2.4.13 *[46] Let $A \subseteq B$ and let $p \in S(B)$. Then p does not fork over A if and only if $\beta(p) = \beta(p \restriction A)$.*

In the case of types over models we then have the familiar heir/coheir definition of non-forking in a stable theory.

Definition 2.4.14 *Let $M \models T$, let $M \subseteq B$ and let $p \in S(M)$, $q \in S(B)$ be such that $p \subseteq q$.*

1. *q is an* heir *of p if $cl_M(q) \subseteq cl_M(p)$.*

2. *q is a* coheir *of p if every L_B-formula $\phi(x,b) \in q$ is satisfied in M (q is said to be* finitely satisfiable *in M).*

Proposition 2.4.15 *Let T be stable and let $p \in S(M)$, $M \models T$. Let $B \supseteq M$ and let $q \in S(B)$ be an extension of p. The following are equivalent:*

1. *q is a non-forking extension of p;*

2. *q is an heir of p;*

3. *q is a coheir of p.*

Thus in particular a type over a model of a stable theory has a unique heir which is equal to its unique coheir.

2.5 Definability and Canonical Bases in Simple Theories

The results of the previous section on definability and boundedness of types, uniqueness of heirs and coheirs and existence of canonical bases are crucial results for stable theories. Unlike the fact that non-forking induces a well-behaved dependence relation these facts fail to generalise to the class of simple theories. In fact a theory is stable if and only if all types over models are definable, if and only if the theory is simple and all types over models (equivalently algebraically closed sets) are stationary.

However many of these properties of stable theories generalise in some weaker form to simple theories. Types are definable by defining schemas consisting of types rather than just formulas, canonical bases exist as sets of hyperimaginaries, a more general concept than that of imaginary, and forking can be described in terms of the fundamental order.

2.5.1 Definability in Simple Theories

We first deal with the definability of types in simple theories.

Remark 2.5.1 *Let $p \in S(A)$ be a type in a stable theory. Then not only is p definable over A, but p is strongly definable over A in the following sense: there is a defining schema $d : \phi(x, y) \to \psi_\phi(y)$ (mapping L-formulas to L_A-formulas) such that for any set $B \supseteq A$ and any $b \in B$, $\phi(x, b)$ is in some non-forking extension of p to B if and only if $\models \psi_\phi(b)$. (See [51] Chapter 4.)*

From now on we assume the underlying theory T to be simple but not necessarily stable.

Lemma 2.5.2 *[54] Let $p \in S(A)$. Then there is a type-definable defining schema $\Delta : \phi(x, y) \to \Delta(\phi)(y)$ (mapping L-formulas $\phi(x, y)$ to partial types $\Delta(\phi)(y)$ over A) with the following property: For all $\phi(x, y) \in L$ and all b, $\phi(x, b)$ is in some non-forking extension of p to $A \cup \{b\}$ if and only if $\models \Delta(\phi)(b)$.*

$$\textit{In particular}: \phi(x, a) \in p \iff \models \Delta(\phi)(a).$$

Proof. For each $\psi(x, z) \in L$ and $k \in \omega$, let $n_{\psi,k} := D(p(x)\,,\, \psi\,, k)$. Then $\phi(x, b)$ is in some non-forking extension of p to $A \cup \{b\}$ if and only if for all $\psi(x, z) \in L$ and all $k \in \omega$, $D(p(x) \cup \phi(x, b)\,,\, \psi\,, k) \geq n_{\psi,k}$. But, by [62], the condition $D(p(x) \cup \phi(x, b)\,,\, \psi\,, k) \geq n_{\psi,k}$ is definable by a partial type $\Delta(\phi, \psi, k)(y)$ over A. Then, simply take $\Delta(\phi)(y) = \bigcup_{\psi \in L} \bigcup_{k \in \omega} \Delta(\phi, \psi, k)(y)$. ■

Corollary 2.5.3 *Suppose that $p \in S(A)$ is stationary. Then p is well definable over A. There is a defining schema $d : \phi(x, y) \to \psi_\phi(y)$ (mapping L-formulas to L_A-formulas) such that for all b, $\phi(x, b)$ is in some (equivalently every) non-forking extension of p to $A \cup \{b\}$ if and only if $\models \psi_\phi(b)$.*

Proof. By Lemma 2.5.2 and compactness. ■

We now show that in the case of simple theories there is a dichotomy between the so called 'bounded' types which behave very much like types in a stable theory, and unbounded types which behave very differently.

Definition 2.5.4 *A type p is said to be* bounded *if there is a cardinal λ such that for all sets $B \supseteq dom(p)$, p has $< \lambda$ non-forking extensions to types over B. Otherwise p is* unbounded.

Note that all types in a stable theory are bounded. In fact a simple theory T is stable if and only if every type is bounded.

Proposition 2.5.5 *Suppose that T is simple and that $p \in S(A)$ is a bounded type. Then every extension of p to a complete type over $acl(A)$ is stationary. Moreover p is definable by a defining schema $d : \phi(x,y) \to \psi_\phi(y)$ (mapping L-formulas to L_A-formulas) such that for all b, $\phi(x,b)$ is in some non-forking extension of p to $A \cup \{b\}$ if and only if $\models \psi_\phi(b)$.*

Proof. The fact that every extension of p to $acl(A)$ is stationary is due to Pillay. ([54] 2.3, 2.4.) The proof is roughly as follows. It is first shown that if M is a model and $p \in S(M)$ is bounded then p is stationary. This is proved by using the independence theorem over a model to show that if p is not stationary then every non-forking extension of p is not stationary and then applying a standard argument to show that p is unbounded. Then if $p \in S(A)$ is bounded, let p' be a non-forking extension of p to $acl(A)$. The aim is to prove that p' is stationary. Suppose that $M \supseteq acl(A)$ and p_1, p_2 are non-forking extensions of p to M. By the comments above, both p_1 and p_2 are stationary and hence definable. In fact it can be shown, by the boundedness of p' that p_1 and p_2 are definable over $acl(A)$. From this, and the fact that p' is type definable over $acl(A)$, it follows that $p_1 = p_2$ and hence p' is stationary.

To prove the definability part suppose that $p \in S(A)$ is bounded. Let $\phi(x,y) \in L$. By a standard argument (see e.g. [3] p108 Th 2.1) $mult_\phi(p)$ is finite. That is, there is $r \in \omega$ such that for any $B \supseteq A$, p has at most r non-forking extensions with inequivalent restrictions to ϕ-types. (If B is a model p has exactly r such extensions.) Choose a very saturated model $M \supseteq A$ and let $p_1, \ldots, p_r$ be non-forking extensions of p to M with inequivalent restrictions to ϕ-types. By the first part of the proposition, each of the types p_i is stationary and hence is definable. Let $\psi_i(y) \in L_M$ be a formula such that for all $b \in \tilde{M}$, $\phi(x, b)$ is in some (every) non-forking extension of p_i to $M \cup \{b\}$ if and only if $\models \psi_i(b)$. Let $\psi(y)$ be the formula $\bigvee_{i \leq r} \psi_i(y)$. Let $\Delta(y)$ be the partial type over A, given by 2.5.2 such that $\phi(x, b)$ is in some non-forking extension of p if and only if $\models \Delta(b)$. Thus in the model M, $\Delta(y)$ is equivalent to the formula $\psi(y)$. Thus by an easy compactness argument (and saturation of M) $\psi(y)$ is definable over A and hence $\phi(x, b)$ is in some non-forking extension of p if and only if $\models \psi(b)$. ■

Proposition 2.5.5 says that bounded types behave very much like types in stable theories. For instance the finite equivalence relation holds for such types: if $p \in S(A)$ is bounded and q_1, q_2 are non-forking extensions of p to some set $B \supseteq A$ then $p_1 \neq p_2$ if and only if there is a finite equivalence relation $E(x, y)$ definable over A such that $p_1(x) \cup p_2(y) \Rightarrow \neg E(x, y)$. (This is really just a restatement of the fact that any extension of p to $acl(A)$ is stationary.) Also if $p \in S(A)$ is a bounded type and $q \in S(B)$ is a bounded extension of p, then q does not fork over A if and only if q is definable almost over A.

Corollary 2.5.6 *Let T be a simple theory and let $p \in S(M)$. Then if p is definable, p is stationary.*

Proof. Suppose that p is definable by a defining schema $d : \phi(x, y) \to \psi_\phi(y)$. By Proposition 2.5.2, there is a defining schema $\Delta : \phi(x, y) \to \Delta(\phi)(y)$ (with $\Delta(\phi)(y)$ a partial type over M) such that for all $b \in \tilde{M}$, $\phi(x, b)$ is in some non-forking extension of p if and only if $\models \Delta(\phi)(b)$. Then by compactness and the fact that M is a model, it is clear that for each $\phi(x, y) \in L$, $\Delta(\phi)(y)$

is equivalent to $\psi_\phi(y)$ and hence $\phi(x,b)$ is in some non-forking extension of p if and only if $\models \psi_\phi(b)$. Since M is a model it is clear that $\models \forall y(\psi_{(\neg\phi)}(y) \leftrightarrow \neg\psi_\phi(y))$. Now suppose that $p_1, p_2 \in S(B)$ are non-forking extensions of p and suppose that $\phi(x,b) \in p_1$ and $\phi(x,b) \notin p_2$. Then $\models \psi_\phi(b)$. As p_2 is a complete type, $\neg\phi(x,b) \in p_2$. Hence $\models \psi_{(\neg\phi)}(b)$ and therefore $\models \neg\psi_\phi(b)$ which is clearly a contradiction. Therefore p is stationary. ∎

Remark 2.5.7 *Suppose T is simple and let M be a model of T and $B \supseteq M$ with $p \in S(M)$ and $q \in S(B)$, a non-forking extension of p, then p is bounded (equivalently stationary) if and only if q is bounded.*

Proof. Left to right is obvious by transitivity of non-forking in simple theories. To prove right to left we must use the independence over a model theorem. Suppose that p is unbounded. Then given any cardinal λ we can choose A independent from B over M and $\{p_i : i < \lambda\}$ inequivalent non-forking extensions of p to A. Then by the independence theorem, 2.3.9, for each $i < \lambda$ there is a type $q_i \in S(A \cup B)$ which extends both q and p_i and does not fork over M. In particular each q_i is a non-forking extension of q. Clearly $p_i \neq p_j \Rightarrow q_i \neq q_j$ and hence q is unbounded. ∎

2.5.2 The Fundamental Order in Simple Theories

Let $p \in S(B)$. Recall that for $A \subseteq B$, $cl_A(p)$ is the set of A-formulas represented in p. If q is an extension of p to a model M, then q is an *A-bound* of p if $cl_A(q)$ is a minimal element (under inclusion) of the set $\{cl_A(r) : r \in S(N)$, N is a model extending A and $p \subseteq r\}$. Thus, in a stable theory, the A-bound of a type is unique and if $p \in S(A)$ and $q \supseteq p$, then q is a non-forking extension of p if and only if q and p have the same A-bound. In particular types over models have unique non-forking extensions to any given set and this non-forking extension is the necessarily unique heir which is equal to the unique coheir.

In the case of simple theories, the situation is much more complicated. By a result of Lascar and Poizat, [46], a type over a model, $p \in S(M)$, is definable if and only if for every $B \supseteq M$, p has a unique heir over B. Moreover, if $p \in S(M)$ does not have a unique heir over every $B \supseteq M$ (equivalently p is not definable), then p has an unbounded number of heirs (in the sense that for any cardinal λ, there is a model $N \succ M$ (with $|N| \leq \lambda$) such that p has at least λ^+ heirs over M). Thus, in a simple unstable theory, there will always be types over models with an unbounded number of heirs.

Lemma 2.5.8 *Let T be a simple theory. Let $M \models T$ and let $p \in S(M)$. Let $B \supseteq M$ and suppose that $q \in S(B)$ is an heir of p. Then q does not fork over M.*

Proof. Suppose, for a contradiction, that q forks over M. Choose $\phi(x,b) \in q$ such that $p \cup \{\phi(x,b)\}$ forks over M. By Lemma 2.5.2 there is a partial type $\Delta(\phi)(y)$ such that for all $a \in \tilde{M}$, $\phi(x,a)$ is in some non-forking extension of p iff $\models \Delta(\phi)(a)$. Thus there is there is an L_M-formula $\delta_\phi(y) \in \Delta(\phi)(y)$ such that $\models \neg\delta_\phi(b)$. Therefore $\phi(x,y) \wedge \neg\delta_\phi(y) \in cl_M(q)$. But clearly, since $\Delta(\phi)(y)$ defines $p \restriction \phi$, there is no $a \in M$ such that $\phi(x,a) \in p$ and $\models \neg\delta_\phi(a)$. Thus $\phi(x,y) \wedge \neg\delta_\phi(y) \notin cl_M(p)$ contradicting the assumption that q is an heir of p. ∎

Corollary 2.5.9 *Let T be simple. Let $M \models T$ and let $p \in S(M)$. Let $B \supseteq M$ and suppose that $q \in S(B)$ is a coheir of p. Then q does not fork over M.*

Proof. This is immediate form Lemma 2.5.8 and forking symmetry in simple theories. ∎

In fact, Lemma 2.5.9 (but not Lemma 2.5.8) is true in any complete theory using a topological argument. (See [45].)

Given a type $p \in S(A)$ in any complete theory such that p does not fork over A, there is a model $M \supseteq A$ and a non-forking extension $q \in S(M)$ of p such that for any non-forking extension r of p, extending q we have that r is an heir of q. Thus, by Lemma 2.5.8 and by transitivity and local character in simple theories, we have:

Proposition 2.5.10 *[45] Let T be simple and let $p \in S(A)$. Then there is a model $M \supseteq A$ and a non-forking extension q of p to M such that any non-forking extension of q is an heir of q.*

Since, in a simple theory, any heir is a non-forking extension the above proposition says that given any type p there is a non-forking extension of p to a model whose non-forking extensions are exactly its heirs.

The following proposition generalises, to types over arbitrary sets, the fact, that in simple theories, heirs are non-forking extensions.

Proposition 2.5.11 *[45] Let T be simple, let $p \in S(A)$, let $M \supseteq A$ and let $q \in S(M)$ be an A-bound of p. Then q does not fork over A.*

Note that if $q \in S(M)$ is a type as given by Proposition 2.5.10 such that the non-forking extensions of q are heirs of q then for any $B \supseteq M$ and any non-forking extension $r \in S(B)$ of q, the non-forking extensions of r are exactly the extensions of r which are heirs of q and hence we have the same criterion for non-forking extensions for r as we would have in a stable theory: for r' an extension of r, r' is a non-forking extension of r if and only if $cl_M(r) = cl_M(r')$.

Lascar and Pillay then give the following characterisation of non-forking in a simple theory in terms of the fundamental order:

Theorem 2.5.12 *[45] Let T be simple. The following are equivalent:*

1. *$tp(a/Ab)$ does not fork over A;*
2. *there is a model $M \supseteq A$ such that $tp(a/M)$ is an A-bound of $tp(a/A)$ and $tp(a/Mb)$ is a coheir of $tp(a/M)$;*
3. *if q is any A-bound of $tp(a/A)$ then there is a model M such that $cl_A(q) = cl_A(tp(a/M))$ and $tp(a/Mb)$ is a coheir of $tp(a/M)$.*

In particular, if we work over a model, then we have that a and b are independent over a model M if and only if there is a model $N \succeq M$ such that $tp(a/N)$ is an heir of $tp(a/M)$ and $tp(a/Nb)$ is a coheir of $tp(a/N)$.

In a stable theory a and b are independent over a model M if and only if $tp(a/Mb)$ is a coheir of $tp(a/M)$ (if and only if $tp(a/Mb)$ is a heir of $tp(a/M)$). In this way it can be seen how forking in a simple theory, given in terms of the fundamental order, generalises the corresponding characterisation of non-forking in a stable theory.

Note that in an arbitrary theory T, if $M \models T$ and $p \in S(M)$ then if for every $B \supseteq M$, it is the case that every heir of p over B is also a coheir, then p is definable. (See e.g. [51].) Thus in a simple unstable theory there are certainly heirs which are not coheirs.

We consider again the dichotomy between bounded and unbounded types in simple theories. Suppose $p \in S(A)$ is a bounded type. Then every extension of p to a model (or an algebraically closed set) is stationary. Thus every extension of p to a model has a unique heir (which equals the unique coheir) over any set containing its domain and this type is the unique non-forking extension. We also have the converse: if $p \in S(A)$ is a type with the property that every extension of p to a model has a unique heir, then every such extension of p is definable and therefore stationary and hence p is bounded. In particular we have the following characterisation of bounded types in simple theories:

Corollary 2.5.13 *Let T be simple. Let $M \models T$ and let $p \in S(M)$. The following are equivalent:*

1. *p is stationary;*

2. *p is bounded;*

3. *p is definable;*

4. *p has a unique heir;*

5. *every heir of p is also a coheir.*

Proof. The equivalence of (1),(2) and (3) was established by 2.5.5 and 2.5.6. (4)$\Leftrightarrow$(3) is due to Lascar and Poizat ([46]). (5)$\Rightarrow$(3) is from [51]. (1)$\Rightarrow$(5) is clear, since if p is stationary, then (as heirs and coheirs are always non-forking extensions) p has a unique heir and a unique coheir which must be equal. ■

2.5.3 Canonical Bases in Simple Theories

We now explain how a suitable notion of canonical bases can be obtained for types in simple theories. In a stable theory, every stationary type has a canonical base. In particular, all types over models (and algebraically closed sets) are stationary, and hence possess canonical bases. In a simple (unstable) theory we wish to find a notion of canonical base strong enough so that at least types over models possess canonical bases. We eventually show how to define canonical bases for types called amalgamation bases which include types over models and are exactly the stationary types in a stable theory. Almost everything in this section is due to Hart, Kim and Pillay and is described in [30].

In order to define these canonical bases we must consider elements more general than imaginaries in M^{eq}.

Definition 2.5.14 *A* hyperimaginary *is an equivalence class a/E where a is a tuple of possibly infinite length and E is a type definable equivalence relation on tuples of the same length as a.*

Note that essentially everything can be considered as a hyperimaginary. Every (small) set of elements is a hyperimaginary and even every (small) set of hyperimaginaries is again a hyperimaginary. It is possible to make sense of the notion of a hyperimaginary type as follows:

Definition 2.5.15 *Let $a = c/E$ and $b = d/F$ be hyperimaginaries. For a formula $\phi(x, y) \in L$, let $\phi(x/E\,,\, y/F)$ be the partial type expressing:*

$$\exists x' \exists y' \left(xEx' \wedge yFy' \wedge \phi(x', y')\right).$$

Let $\mathfrak{F}$ be the set (of types):

$$\mathfrak{F} = \{\phi(x, y) \in L : \textit{for some } c', d',\ cEc',\ dEd' \textit{ and } \models \phi(c', d')\}.$$

Let $tp(a/b) := \bigcup_{\phi \in \mathfrak{F}} \phi(x/E, b)$.

Hyperimaginary types have almost the same properties as normal types. For instance, if a, a', b are hyperimaginaries then a' realises $tp(a/b)$ if and only if there is an automorphism f (of the monster model), fixing b and taking a to a'. It is then possible to define indiscernible sequences, forking, dividing, Morley sequences, etc. for hyperimaginaries almost exactly as in the normal case. Then in simple theories, hyperimaginary types satisfy forking = dividing and symmetry, transitivity and independence over a model and so we can work with hyperimaginaries as we would with normal elements. See [30] for the details. From now on we assume that the underlying theory T is simple.

Definition 2.5.16 *A (possibly hyperimaginary) type $p \in S(A)$ is an* amalgamation base *if whenever $B, C \supseteq A$, B and C are independent over A and p_1, p_2 are non-forking extensions of p to B and C respectively, then $p_1 \cup p_2$ does not fork over A.*

Thus, by 2.3.9, all types over models in a simple theory are amalgamation bases. In a stable theory the amalgamation bases are exactly the stationary types.

Fix an amalgamation base $p(x, a)$. We aim to find a form of canonical base for $p(x, a)$. First, define a relation $\sim_1$ on realisations of $tp(a)$ by:

$a \sim_1 b$ iff $p(x, a)$ and $p(x, b)$ have a common non-forking extension.

It then follows that $a \sim_1 b$ iff $D(p(x,a) \cup p(x,b), \Delta, k) = D(p(x,a), \Delta, k)$ for all finite sets of formulas Δ and all $k \in \omega$. Hence (see e.g. proof of 2.5.2) $\sim_1$ is type-definable by a partial type R_1. Let $\sim$ be the transitive closure of $\sim_1$. Then $\sim$ is an equivalence relation on realisations of $tp(a)$. It is then possible to prove that $\sim$ is type-definable and to extend it to an equivalence relation on all tuples of the same length as a. Therefore the equivalence class $a/\sim$ is a hyperimaginary element. It also follows that $p(x,a)$ does not fork over $a/\sim$. We now explain how the hyperimaginary $a/\sim$ behaves like a canonical base for the type $p(x,a)$.

Define a relation E_1 on amalgamation bases in a simple theory by: q and r are E_1-related iff q and r have a common non-forking extension. Let E be the transitive closure of E_1. Then E is an equivalence relation on amalgamation bases. Given an amalgamation base p, let $\mathcal{P} = p/E$, the *parallelism class of* p. For $p(x,a)$ an amalgamation base in a simple theory, a (possibly hyperimaginary) element $\bar{a}$ is said to be a *canonical base* for p, if for all $f \in Aut(\tilde{M})$, f fixes $\mathcal{P}$ setwise if and only if $f(\bar{a}) = \bar{a}$.

Theorem 2.5.17 *Let $p(x,a)$ be an amalgamation base. Let $\mathcal{P} = p/E$ and let $\bar{a} = a/\sim$. Then:*

1. *For all $q \in \mathcal{P}$, if e realises q, then e realises $p \restriction \bar{a}$.*

2. *If $q \in \mathcal{P}$, then there is a' with $tp(a) = tp(a')$ such that $q \sim_1 p(x,a')$ (any $q \in p/E$ has a common non-forking extension with some conjugate of p).*

3. *$p \restriction \bar{a}$ is an amalgamation base.*

4. *$\bar{a} \in dcl(a)$.*

5. *The hyperimaginary $\bar{a}$, is a canonical base for $p(x,a)$.*

The hyperimaginary $a/\sim$ is the unique canonical base for $p(x,a)$ up to interdefinability. We can therefore write $Cb(p)$ for the canonical base of p.

Definition 2.5.18 *A theory T has canonical bases in $\tilde{M}^{eq}$ if for every amalgamation base p, there is a set $A \subseteq \tilde{M}^{eq}$ which is interdefinable with $Cb(p)$.*

If the theory T is stable, then the above definition corresponds to the usual definition of canonical bases in stable theories. In particular, T has canonical bases in $\tilde{M}^{eq}$. Small supersimple theories also have canonical bases in $\tilde{M}^{eq}$. This is because every type-definable equivalence relation in such a theory is equivalent to an intersection of definable equivalence relations. Thus any hyperimaginary is equivalent to a set of ordinary imaginaries and hence there are canonical bases in $\tilde{M}^{eq}$. In fact, the following result says that canonical bases are the most complicated kinds of hyperimaginary in simple theories.

Proposition 2.5.19 *Let T be simple and suppose that for every amalgamation base p, $Cb(p)$ is interdefinable with a set of elements of $\tilde{M}^{eq}$, then every hyperimaginary is equivalent to a set in M^{eq}.*

In particular, all hyperimaginaries in stable theories, or in small supersimple theories, are interdefinable with sets of normal imaginary elements. The fact that stable theories have 'elimination of hyperimaginaries' was proved originally in [56].

2.5.4 Lascar Strong Types

In a stable theory, not only are types over models stationary, but so are types over algebraically closed (in T^{eq}) sets (i.e. strong types). In a simple theory T, the analogous result to types over models being stationary is that the independence theorem over a model (Theorem 2.3.9) holds. It is then natural to ask if this independence theorem also holds over algebraically closed sets in a simple theory. Namely, if $B, C \supseteq A$ with $B \downarrow_A C$ and $tp(d/B)$ and $tp(e/C)$ are both types not forking over A with $stp(d/A) = stp(e/A)$ then is there a type extending $tp(d/B)$ and $tp(e/C)$ which does not fork over A and has the same strong type over A as d (and e)?

This turned out to be a rather intractable problem with, as yet, no known proof or counterexample. However there is a slightly different notion of a strong type more suited to simple theories: a Lascar strong type.

Definition 2.5.20 *Let $Laut_A(\tilde{M})$ be the group of automorphisms of $\tilde{M}$ generated by the set $\{Aut_M(\tilde{M}) : M \models T \text{ and } M \supseteq A\}$. Define $Lstp(a/A) = Lstp(b/A)$ (a and b have the same* Lascar strong type *over A) if and only if a and b are in the same orbit of the group $Laut_A(\tilde{M})$.*

Thus $Lstp(a/A) = Lstp(b/A)$ if and only if there is $n \in \omega$, a sequence $a = a_0, a_1, \ldots, a_n = b$ and a sequence $M_1, M_2, \ldots, M_n$ of models containing A such that for each $i = 1, \ldots, n$, $tp(a_{i-1}/M_i) = tp(a_i/M_i)$. In particular $Lstp(a/A) = Lstp(b/A) \Rightarrow stp(a/A) = stp(b/A)$. In a stable theory the converse holds: equality of strong types implies the equality of Lascar strong types. This was proved in [44].

The advantage of working with Lascar strong types is that it is possible to prove a version of the independence theorem.

Theorem 2.5.21 *[40] Let T be a simple theory. Let A, B, C be sets with $B, C \supseteq A$ and $B \downarrow_A C$. Let d and e be tuples such that $tp(d/B)$ does not fork over A, $tp(e/C)$ does not fork over A and $Lstp(d/A) = Lstp(e/A)$. Then there exists a tuple a such that $tp(a/BC)$ does not fork over A and extends both $tp(d/B)$ and $tp(e/C)$ such that $Lstp(a/A) = Lstp(d/A)$ $(= Lstp(e/A))$.*

Thus if T is a simple theory and Lascar strong type is equivalent to strong type in T, then the independence theorem holds over algebraically closed sets in T.

Lascar ([44]) defined a notion of G-compactness for an arbitrary theory T. This is equivalent to the requirement that $Laut_A(\tilde{M})$ is a closed subgroup of $Aut_A(\tilde{M})$ (for the permutation group topology) and for each $n \in \omega$, the equivalence relation on n-tuples given by $Lstp(a/A) = Lstp(b/A)$ is definable by some partial type. It is proved in [39] that every simple theory is G-compact.

(It is unknown whether there exists a theory which is not G-compact.) Thus if T is simple, for each set A and each $n \in \omega$, there is a partial type $\Sigma(x, y)$ such that for any n-tuple a, $Lstp(a/A) = Lstp(b/A)$ if and only if $\models \Sigma(a, b)$, i.e. $Lstp(a/A)$ can be considered as a partial type $\Sigma(x, a)$ over $A \cup \{a\}$.

We return to the question of whether or not Lascar strong type is equivalent to strong type in a simple theory. It is known (by an example of Poizat, see [39] for details) that there are (non-simple) theories where the notions of strong type and Lascar strong type do not coincide. In a G-compact theory it is clear that Lascar strong type is equivalent to strong type if and only if for each set A the partial type $\Sigma(x, y)$ defining Lascar strong type is equivalent to the partial type given by $\{E(x, y) : E$ is a formula over A which defines an equivalence relation with only finitely many classes$\}$.

By [56], if T is a small theory and $E(x, y)$ is a type definable equivalence relation such that $tp(a) = tp(b)$ implies $E(a, b)$, then there are *definable* equivalence relations $E_i(x, y)$ for $i \in \omega$ such that $E(x, y) \longleftrightarrow \bigwedge_{i \in \omega} E_i(x, y)$. (i.e. every type definable equivalence relation is an intersection of definable equivalence relations.) Together with the fact that Lascar strong type is a type definable equivalence relation, this eventually leads to a proof that if T is a small simple theory then for any set A, $Lstp(a/A) = Lstp(b/A) \iff stp(a/A) = stp(b/A)$. (See [39].)

In [7] Buechler succeeded in showing that another class of simple theory has the property that strong type and Lascar strong type are equivalent. Recall that if T is simple, Δ is a finite set of formulas and $k \in \omega$, then for any formula ϕ, the rank $D(\phi, \Delta, k)$ is finite. A (simple) theory T is called *low* if the rank $D(\phi, \Delta, k)$ is bounded as k varies. That is, given a formula ϕ and a finite set of formulas Δ, there is $n \in \omega$ such that for all $k \in \omega$, $D(\phi, \Delta, k) < n$. All the common examples of simple theories (psuedofinite fields, the random graph, smoothly approximated structures, etc.) are known to be low. However, there are examples of simple theories (e.g. the theory described in Section 5 of [11]) which are not low. Buechler proves in [7] that if T is a simple theory which is low then Lascar strong type is equivalent to strong type in T.

Note that the question of whether or not strong type and Lascar strong type are equivalent for a theory T is closely related to the question of the existence of canonical bases in $\tilde{M}^{eq}$. If Lascar strong type is equivalent to strong type then by 2.5.21, the independence theorem holds over algebraically closed sets and hence every type over an algebraically closed set is an amalgamation base (in the sense of Definition 2.5.16). Following the proof of the equivalence of strong types and Lascar strong types in a small simple theory, to prove $Lstp = stp$ in an arbitrary simple theory T it suffices to prove that every type definable equivalence relation is an intersection of definable equivalence relations. This is the situation if and only if every bounded hyperimaginary is equivalent to a set of normal imaginaries. (A hyperimaginary is *bounded* if it has small orbit under $Aut(\tilde{M})$.) In [8] this fact is proved for all supersimple theories. From this it follows that every finitary type in a supersimple theory has a canonical base in $\tilde{M}^{eq}$ and, in fact, that every hyperimaginary (bounded or not) is interdefinable with a set of elements from $\tilde{M}^{eq}$, i.e. T eliminates hyperimaginaries. In particular Lascar strong type and strong type are equivalent in any supersimple theory.

Chapter 3

Rings, Groups and Stability

3.1 Stable Groups

Definition 3.1.1 *A* stable group *is a group definable in a model of a stable theory. Thus G is a stable group if there is a stable theory T such that G is a definable subset of a model of T and there is a definable function, f, such that the group law on G is given by $f \upharpoonright G \times G$.*

Some authors aim for a greater level of generality by requiring only that the domain of G is type-definable in a model of a stable theory T and possibly also that f is only a type-definable function. (For example see [63] and [53].) However the stable groups we consider here will be definable in a stable theory as in the above definition. This allows us to consider the group G as a stable structure in its own right with functions and relations on G given by the (definable) restrictions of the corresponding definable functions on the model of the stable theory. Therefore there is no need to keep mentioning the ambient stable theory.

Definition 3.1.2 *Let G be a stable group. A family of subgroups of G, $\{H_i : i \in I\}$ is* uniformly definable *if there is some formula $\phi(x, y)$ such that each subgroup H_i is defined by an instance of ϕ.*

- *The* u.c.c. *is the* uniform chain condition *on families of uniformly definable subgroups: for any formula ϕ, there is an integer m_ϕ such that any chain of ϕ-definable subgroups has length $\leq m_\phi$.*

- *The* i.c.c. (intersection chain condition) *is the uniform chain condition on intersections of uniformly definable subgroups: for any formula ϕ, there is an integer n_ϕ such that any chain of intersections of ϕ-definable subgroups has length $\leq n_\phi$.*

Theorem 3.1.3 *1. Let G be a group without the strict order property. Then G satisfies the u.c.c.*

2. Let G be a stable group. Then G satisfies the i.c.c.

Proof. (1). Suppose that there is a formula $\phi(x, y)$ and parameters $(a_i : i \in \omega)$ such that $\phi(x, a_i)$ defines a group H_i. Suppose that for each $i \in \omega$, $H_{i+1} \lneqq H_i$. Then the formula $\forall x\,(\phi(x, y_1) \to \phi(x, y_2))$ defines a pre-order on models of T. The set $(a_i : i \in \omega)$ witnesses that this order has an infinite chain and therefore G has the strict order property.

(2). This is by exploiting the fact that a stable theory does not have the independence property. See [63] pp 57-58 for a proof. ∎

Note that a group definable in a simple theory (not called a simple group for obvious reasons) has the u.c.c. but does not necessarily have the i.c.c. For example let $M = (V, 0, +, [\ ,\ ])$ be an infinite dimensional vector space over a finite field with a non-degenerate bilinear form. Then M has simple theory (Proposition 6.2.5) and therefore $(V, +)$ is a group with simple theory definable in M, but V has an infinite chain of intersections of uniformly definable subgroups of the form $\{v \in V : [v, a] = 0\}$. However groups in simple theories have the i.c.c. *up to commensurability*. Two groups G and H are said to be *commensurable* if $G \cap H$ has finite index in both G and H. In [65] Frank Wagner proves the following:

Theorem 3.1.4 *Let G be a group definable in a simple theory and let $\mathcal{H}$ be a family of uniformly definable subgroups of G. Then there are integers k and k' such that there is no sequence $\{H_i : i \leq k\} \subseteq \mathcal{H}$ with $\left|\bigcap_{i<j} H_i : \bigcap_{i\leq j} H_i\right| > k'$ for all $j \leq k$. In particular any intersection N of elements of $\mathcal{H}$ is commensurable with a sub-intersection N_0, of size at most k and $|N_0 : N_0 \cap H| \leq k'$ for any $H \in \mathcal{H}$ with $H \geq N$.*

Returning to theories of stable groups we now briefly discuss the theory of generic types due to Poizat. See [57] for details.

Definition 3.1.5 *Let G be a stable group. A definable subset X of G is* generic *if there is $n \in \omega$ and $g_1, \ldots g_n \in G$ such that $G = g_1X \cup \ldots \cup g_nX$.*

Lemma 3.1.6 *Let X be a definable subset of a stable group G. Then either X or $G \backslash X$ is generic.*

Definition 3.1.7 *Let G be a stable group. A type $p(x)$ is* generic *if every formula $\phi(x) \in p$ is generic.*

Let $p(x) \in S(A)$ be a type in a stable group G and let $g \in G$. Then we can define the *translate* gp of p by g as follows: choose c realising $p(x)$ in some saturated elementary extension and define $gp = tp(gc/A)$. (Hence $gp = \{\phi(g^{-1}x) : \phi(x) \in p\}$.) It then follows that any stable group has generic types and a type $p(x)$ is generic if and only if gp does not fork over $\emptyset$ for all $g \in G$. It is then clear that if $p(x)$ is a generic type then gp is also generic. Similarly if $p(x)$ is generic then so is $p^{-1}(x) := \{\phi(x^{-1}) : \phi \in p\}$. Therefore G acts on the set of its generic types. If G is $|T|^+$-saturated (where $T = Th(G)$) then G acts transitively on the set of its generic types: if p and q are generic types of G then there is $a \in G$ such that $q = ap$.

Definition 3.1.8 *Let G be a stable group. The* connected component *G^0 of G is the intersection of all $\emptyset$-definable subgroups of G of finite index. It is a normal subgroup of G. A stable group G is* connected *if $G = G^0$.*

Theorem 3.1.9 *[57] Let G be a stable group. Then the index of G^0 in G is finite if and only if G has only finitely many generic types, and if this is the case then the index $[G : G^0]$ is equal to the number of generic types of G. In particular, G is connected if and only if G has a unique generic type.*

3.2 Stable Rings

Before going on to consider rings with stable theories, we give some definitions and properties of various classes of rings. We refer to [59] for proofs of the following results and for further details.

Definition 3.2.1 *Let R be a ring. The* Jacobson radical, *$J(R)$, is the ideal $J(R) = \bigcap\{maximal\ left\ ideals\ of\ R\}$.*

Equivalently, $J(R)$ is the intersection of the maximal right ideals of R.

Definition 3.2.2 *Let R be a ring:*

1. *R is* local *if $R/J(R)$ is a division ring.*
2. *R is* semilocal *if $R/J(R)$ is semisimple artinian.*
3. *R is* semiperfect *if $R/J(R)$ is semisimple artinian and $J(R)$ is idempotent lifting (i.e. every idempotent of $R/J(R)$ is the image of an idempotent in R).*
4. *R is* semiprimary *if $R/J(R)$ is semisimple artinian and $J(R)$ is nilpotent.*

Note that: Artinian $\Rightarrow$ semiprimary $\Rightarrow$ semiperfect $\Rightarrow$ semilocal.

Theorem 3.2.3 *A ring R is semiprimary if and only if there is an integer $m \in \omega$ such that every chain of principal left ideals has length $< m$. (if and only if there is $n \in \omega$ such that every chain of principal right ideals has length $< n$.)*

Theorem 3.2.4 *Let R be a semiprimary ring and let $a \in R$.*

1. *a is left invertible if and only if a is right invertible if and only if a is a unit.*

2. *For any $a \in R$, a is a unit if and only if it is not a zero divisor.*

3. *$r.Ann_R(a) = 0 \Leftrightarrow l.Ann_R(a) = 0$.*

Theorem 3.2.5 *A semiperfect ring has a complete set of orthogonal idempotents and hence is a finite direct product of indecomposable rings. (A ring R is indecomposable if the only central idempotents of R are 0 and 1.)*

The following results on stable rings are due to Cherlin, Reineke and Baldwin. (See [18] and [2].)

Theorem 3.2.6

1. *If R is a stable ring then $J(R)$ is nilpotent and $R/J(R)$ is semisimple artinian (i.e. R is semiprimary).*

2. *A ring R is λ-stable if and only if R is the direct product $R = R_1 \times \cdots \times R_k$ of finitely many λ-stable indecomposable rings R_i.*

3. *If R is a commutative λ-stable ring then each of the rings R_i as given in (2) is a local ring and hence each R_i has a unique maximal ideal M_i (which is nilpotent) and R_i/M_i is a λ-stable field.*

In the papers of Cherlin, Reineke and Baldwin the properties of stable rings, such as nilpotence of $J(R)$, $R/J(R)$ being semisimple artinian, every element of R being a unit or zero divisor, etc. are proved separately. Instead of taking this approach, we first prove:

Theorem 3.2.7 *Let R be any ring without the strict order property. Then R is semiprimary.*

Proof. Let $\phi(x, y)$ be the formula: $\exists z(x = yz)$. Then given any $b \in R$, the formula $\phi(x, b)$ defines a subgroup of the group $(R, +)$. But $(R, +)$ is a group without the strict order property (as the ring R does not have the SOP) and thus by Theorem 3.1.3 there is $m \in \omega$ such that every chain of instances of $\phi(x, y)$ has length $< m$. Hence there is no sequence $\{a_i : i \leq m\}$ such that $a_i \in a_j R \Leftrightarrow i \leq j$ and therefore every chain of principal left ideals of R has length $< m$. Thus R is semiprimary. ■

From this, most of the properties of stable rings given in [18] and [2] can be deduced directly from the structure theory of semiprimary rings. Rings with simple theories do not have the strict order property and hence any such ring is semiprimary, and every element of such a ring is either a unit or a zero divisor (on both sides).

Proposition 3.2.8 *Let R be a ring without the strict order property, then there exists an integer $k \in \omega$ such that all chains of annihilators (of single elements) have length $\leq k$. In particular, if $a \in R$ is nilpotent then a is nilpotent of index $\leq k$ and therefore the set of all nilpotent elements in R is first-order definable.*

Proof. Let $\phi(x, y)$ be the formula $xy = 0$. Then for any $b \in R$, $\phi(x, b)$ defines a subgroup of $(R, +)$ and hence by the u.c.c. there is $k \in \omega$ such that any chain of instances of $\phi(x, y)$ has length $\leq k$.

Suppose that $a \in R$ and $m \in \omega$ are such that $Ann_R(a^m) = Ann_R(a^{m+1})$. Then by an easy induction it follows that $Ann_R(a^m) = Ann_R(a^n)$ for every $n \geq m$. Suppose that $a \in R$ is nilpotent of index m (i.e. $Ann_R(a^m) = R$). Therefore $Ann_R(a) \subsetneq Ann_R(a^2) \subsetneq \ldots \subsetneq Ann_R(a^m)$ is a properly ascending chain of subgroups. Then, by the u.c.c. such a chain has length $\leq k$ and thus $Ann_R(a^k) = Ann_R(a^m) = R$. Therefore a is nilpotent of index $\leq k$. The set of nilpotent elements in R is given by the definable set $\{x \in R : x^k = 0\}$. ■

Remark 3.2.9 *Let R be a ring without the strict order property. Let A and B be definable subsets of R and let S be a definable multiplicatively closed subset of R. Then the formula $\phi(x, y) := x \in A \wedge y \in B \wedge x \in yS$ is stable. This is simply because $\phi(x, y)$ defines a partial order and hence if $\phi(x, y)$ were unstable there would be an infinite chain of uniformly definable sets, contradicting the u.c.c.*

Since any stable/semiprimary ring is a finite direct product of indecomposable semiprimary rings and these indecomposable factors are definable (as they are given by the complete set of orthogonal idempotents, which are known to exist by Theorem 3.2.5) we can restrict attention to such indecomposable rings.

Definition 3.2.10 *Let R be a ring.*

1. *R is of* unmixed characteristic *if the Abelian group $(R, +)$ is either p-primary (for some prime p) or is torsion free.*

2. *R is of* uniform characteristic *if every non-zero element of R has the same additive order.*

Theorem 3.2.11

1. *[23] If R is an indecomposable semiprimary ring then R has unmixed characteristic.*

2. *If R is indecomposable, semiprimary and non-singular then R has uniform characteristic.*

Note that if R is p-primary for some prime p, then the multiplicative identity 1_R has finite additive order p^k say. Then for any $r \in R$, $r = (1_R)r$ and therefore $p^k r = (p^k 1_R)r = 0$ and hence $(R, +)$ has bounded exponent p^k.

Proposition 3.2.12 *Let R be a ring. Define an operation $\circ$ on $J(R)$ by $r \circ s = r + s - rs$. Then $(J(R), \circ)$ is a group with identity 0_R.*

Proof. First note that $\circ$ defines a binary relation on $J(R)$ because $J(R)$ is an ideal and therefore $r, s \in J(R) \Rightarrow r + s - rs$. For associativity: $r \circ (s \circ t) = r \circ (s + t - st) = r + s + t - st - rs - rt + rst = (r + s - rs) \circ t = (r \circ s) \circ t$. Clearly $0_R \circ r = r = r \circ 0_R$. For inverses, note that any element of $J(R)$ is *quasi-invertible,* that is for any $r \in J(R)$ the element $1 - r$ is a unit. Let $r \in J(R)$ and put $s = -r(1-r)^{-1}$. Then $s \in J(R)$ and $r \circ s = r - r(1-r)^{-1} + r^2(1-r)^{-1} = r - (r - r^2)(1-r)^{-1} = r - r(1-r)(1-r^{-1}) = r - r = 0$. Therefore for every $r \in J(R)$ there is $s \in J(R)$ such that $r \circ s = 0_R$ and hence $(J(R), \circ)$ is a group. ■

Corollary 3.2.13 *The group $(J(R), \circ)$ is a nilpotent group. It is isomorphic to the multiplicative group $J' = (1 - J(R), \cdot)$ via the bijection $r \mapsto 1 - r$ for $r \in J(R)$. Moreover J' is a normal subgroup of the group $U(R)$ of units of R and $U(R)/J' \cong U(R/J(R))$.*

Proof. Let $J = J(R)$. Define $\phi : (J, \circ) \to (J', \cdot)$ by $\phi(r) = 1 - r$. Then ϕ is clearly a bijection. it is an isomorphism since: $\phi(r \circ s) = 1 - (r + s - rs) = (1-r)(1-s) = \phi(r)\phi(s)$. To show that $J \trianglelefteq U(R)$, choose $u \in U(R)$ and $r \in J'$. Then $r = 1 - r'$ for some $r' \in J$ and $u^{-1}ru = u^{-1}(1-r')u = 1 - u^{-1}r'u$. But J is a two-sided ideal and hence $u^{-1}r'u \in J$. Therefore $u^{-1}ru = 1 - u^{-1}r'u \in J'$ and hence J' is a normal subgroup of $U(R)$. It is then clear that the map $\psi : U(R)/J' \to U(R/J)$ given by $\psi(uJ') = u + J$ is an isomorphism since u is a unit of R if and only if $u + J$ is a unit of $R/J(R)$. For the nilpotence of the group $(J, \circ)$ see [36]. ■

Theorem 3.2.14 *Let R be an indecomposable semiprimary ring. Suppose that the Abelian group $(R, +)$ is not torsion free. Then:*

1. *$(J(R), \circ)$ is a group of bounded exponent.*
2. *$(J(R), \circ)$ is a locally finite group.*

Proof. **(1)** Let $r \in J(R)$ and define $r\circ^n$ inductively by $r\circ^1 = r$ and $r\circ^{n+1} = r \circ (r\circ^n)$. We must find some $M \in \omega$ such that for all $r \in J(R)$, $r\circ^M = 0_R$. The ideal $J(R)$ is nilpotent and hence there is $N \in \omega$ such that $r^N = 0$ for all $r \in J(R)$. As $(R, +)$ has unmixed characteristic there is a prime p and $k \in \omega$ such that $(R, +)$ has bounded exponent p^k. Let $m = k + N$. We claim that $(J(R), \circ)$ has bounded exponent $\leq M = p^{N+k}$.

By definition of $r\circ^n$ we have $r\circ^n = \sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}r^i$. Then $p^k \mid \binom{p^m}{i}$ for all $i < N$. Therefore $r\circ^{p^m} = \sum_{i=1}^{p^m}(-1)^{i+1}\binom{n}{i}r^i = \sum_{i=1}^{N-1}(-1)^{i+1}\binom{n}{i}r^i$. But every element of $(R, +)$ has additive order dividing p^k and there since $p^k \mid \binom{p^m}{i}$ for all $i < N$ it follows that $r\circ^{p^m} = 0_R$. Therefore $(J(R), \circ)$ has bounded exponent $\leq p^m$.

(2) Let $n \in \omega$ and let $a_1, \ldots, a_n \in J(R)$ and let H be the subgroup of $(J(R), \circ)$ generated by $a_1, \ldots, a_n$. We must show that H is finite. By (1) we know that each a_i has finite order $\leq p^m$. Let $h \in H$. Then there is $r \in \omega$ such that $h = h_1 \circ h_2 \circ \cdots \circ h_r$, where for each $j \leq r$, $h_j = a_i$ for some $i \leq n$. Note that $h_1 \circ h_2 \circ h_3 = h_1 + h_2 + h_3 - h_1h_2 - h_2h_3 - h_1h_3 + h_1h_2h_3$ and in general:

$$\begin{aligned} h_1 \circ h_2 \circ \cdots \circ h_r &= \sum_{i=1}^{r} h_i - \left(\sum_{i_1=1}^{r}\sum_{i_2=i_1+1}^{r} h_{i_1}h_{i_2}\right) + \cdots \\ &\quad \ldots + (-1)^{k+1}\left(\sum_{i_1=1}^{r}\sum_{i_2=i_1+1}^{r}\cdots\sum_{i_k=i_{k-1}+1}^{r} h_{i_1}h_{i_2}\cdots h_{i_k}\right). \end{aligned}$$

Since $J(R)$ is nilpotent of index N (as an ideal) we need only consider the first $N-1$ summands in the above sum. (Because all other summands are (sums of) products of $\geq N$ elements form $J(R)$ and are therefore zero.) Consider the first summand $\sum_{i=1}^{r} h_i$. There are only n different elements in this sum (namely $a_1, \ldots a_n$) and since $(R, +)$ is a group of bounded exponent q say, each a_i can appear in the summand a maximum of q times. Therefore there are at most q^n choices for this summand. Similarly there are only q^{n^k} choices for the k^{th} summand, $\left(\sum_{i_1=1}^{r}\sum_{i_2=i_1+1}^{r}\cdots\sum_{i_k=i_{k-1}+1}^{r} h_{i_1}h_{i_2}\cdots h_{i_k}\right)$, and hence there are only q^{n^N} possible choices for h. Therefore H is finite of order $\leq q^{n^N}$ and hence $(J(R), \circ)$ is locally finite.

In fact, the above proof shows that $(J(R), \circ)$ is *uniformly locally finite*: that is for each $n \in \omega$ there is an integer k_n such that any n-generator subgroup of $(J(R), \circ)$ has order $\leq k_n$. This is stronger than ordinary local finiteness in that the order of a finitely generated subgroup depends only on the number of generators and not on the particular generators chosen. ■

Proposition 3.2.15 (Local rings) *For a ring R the following are equivalent:*

1. *R has a unique maximal left (equivalently right) ideal;*

2. *$R/J(R)$ is a division ring;*

3. *the set $R \backslash U(R)$ is an ideal of R;*

4. *$R \backslash U(R)$ is a group under $+$;*

5. *$a + b \in U(R) \Rightarrow a \in U(R)$ or $b \in U(R)$.*

Proof. See [59]. ■

From the above it follows immediately that if R is local then the unique maximal ideal is $J(R)$.

Theorem 3.2.16 *Let R be a stable local ring. Then $R/J(R)$ is infinite if and only if the groups $(R, +)$ and $U(R)$ are both connected.*

Proof. Right to left is obvious: if $R/J(R)$ is finite then since (in a local ring) $J(R)$ is definable the Abelian group $(R, +)$ is not connected.

To prove left to right, suppose that H is a definable subgroup of $(R, +)$ of finite index. Let $G = U(R)$ and let $I = \bigcap\{gH : g \in G\}$.

Claim. I is a left ideal of R.

Proof of claim. Each gH is an additive subgroup of R and hence the intersection I is also an additive subgroup. Let $r \in R$. We aim to prove that $rI = I$. Since R is local and semiprimary, either $r \in U(R)$ or $r \in J(R)$

(in which case r is nilpotent). If r is a unit and $a \in I$, then for all $g \in G$, $a \in r^{-1}gH$ and therefore $ra \in gH$ for all $g \in G \Rightarrow ra \in I$. If $r \in J(R)$, then $1 - r$ is a unit and hence for any $a \in I$, $(1-r)a = a - ra \in I$. But, by assumption, $a \in I$ and I is an additive subgroup of R, therefore $ra \in I$ as required. Hence I is a left ideal of R and the claim is proved.

Now I is a intersection of uniformly definable sets in a stable theory. Thus, by the d.c.c. on intersections of uniformly definable sets in a stable theory, there is $n \in \omega$ and $g_1, \ldots, g_n \in G$ such that $I = \bigcap_{i=1}^n g_i H$. Each $g_i H$ is isomorphic to H (as an Abelian group). Hence (as H has finite index) each $R/g_i H$ is a finite Abelian group. Consider the Abelian group homomorphism

$$\phi : R \to R/g_1 H \times \cdots \times R/g_n H.$$

Then, $\mathrm{Ker}(\phi) = \{r \in R : r \in g_i H \text{ for all } i = 1 \ldots n\} = \bigcap_{i=1}^n g_i H = I$. Therefore R/I is isomorphic to a subgroup of $R/g_1 H \times \cdots \times R/g_n H$ which is finite. Thus I is a left ideal of R of finite index. However R has a unique maximal ideal $J(R)$. Therefore $I \subseteq J(R)$ and in particular $J(R)$ has finite index. Hence if $R/J(R)$ is infinite then the Abelian group $(R, +)$ is connected.

We now prove that the group of units $U(R)$ of R is also connected. Since $(R, +)$ is connected R has a unique additive generic type $p(x)$. Suppose that $p(x) \vdash x \in J(R)$. Then, as $p(x)$ is generic, there is $n \in \omega$ and $a_1, \ldots, a_n \in R$ such that $R \vDash \forall x \bigvee_{i=1}^n x \in a_i + J(R)$ which implies that $J(R)$ has finite index which contradicts the assumption. Therefore, as R is local, we have that $p(x) \vdash x \in U(R)$.

Claim. $p(x)$ is the unique generic type of $U(R)$.

Proof of claim. Let $g \in U(R)$. Then the map $\phi_g : a \mapsto ga$ defines an isomorphism of $(R, +)$. Therefore $\phi_g(p)$ is also a generic type of $(R, +)$. But p is the unique (additive) generic type of $(R, +)$ and therefore $gp = p$. Hence (by definition) p is a (multiplicative) generic type for $U(R)$. But $U(R)$ acts transitively on the set of its generic types (as it is a stable group). Thus p is the unique generic type for $U(R)$ which implies that $U(R)$ is connected. ■

Suppose that R is local, $R/J(R)$ is infinite and either R is 1-based or the (unique) generic type $p(x)$ given in the above proof is regular, then the (connected) group $U(R)$ is Abelian. (This is standard theory on stable groups and generic types: a connected stable group in a 1-based theory is Abelian.). Since $U(R/J(R)) \cong U(R)/J'$ it follows that the division ring $R/J(R)$ is a field.

A ring R is a *central idempotent lift* (CIL) ring if every central idempotent of $R/J(R)$ is the image of a central idempotent of R. Therefore if R is an indecomposable CIL-ring then $R/J(R)$ is also indecomposable. Note that if R is a ring with stable (or simple) theory then R is semiprimary and therefore R is an idempotent lifting ring. Therefore any commutative stable ring is a CIL-ring. However, a non-commutative (indecomposable) stable ring may or may not be CIL. (In [18] Cherlin and Reineke give an example of an $\aleph_1$-categorical ring which is not CIL.)

Suppose that R is a stable CIL-ring, then R is a finite direct product of indecomposable stable CIL-rings R_i. Each $R_i/J(R_i)$ is therefore indecomposable and hence is isomorphic to $M_{n_i}(D_i)$ for some (stable) division ring D_i and thus each ring R_i is isomorphic to $M_{n_i}(T_i)$ for some stable local ring T_i. Therefore any stable CIL-ring (in particular any stable commutative ring) is a finite direct product of matrix rings over stable local rings. The theorem above shows if the radical has infinite index then both the additive group and the unit group of a stable local ring are connected groups.

Any further attempts at classifying rings with stable theories break down because of the impossibility of classifying nilpotent groups in a given stability class. This is due to work of Mekler ([49]) in which it is proved that the problem of classifying the nilpotent groups of class 2 and exponent p in a given stability class is equivalent to the problem of classifying all structures in that stability class. It is shown that given any stable structure M, it is possible to associate a nilpotent group of class 2 and exponent p with the same stability spectrum function as M. Since the Mal'cev correspondence associates nilpotent rings with nilpotent groups the problem of classifying stable nilpotent rings is equally intractable.

It is interesting to note that under certain conditions, a nilpotent group of class 2 and exponent p has a simple theory. Suppose that G is a nilpotent group of class 2 and exponent p, then $G/Z(G)$ is Abelian and hence the derived group G' is a subgroup of $Z(G)$. Suppose that we also have $Z(G) \leq G'$ and so the centre and the derived group are equal. Suppose also that $Z(G)$ is the finite cyclic group $\mathbb{Z}_p$ (which can be given the structure of the finite field $\mathbb{F}_p$). Then G is essentially a vector space over $\mathbb{F}_p$ with non-degenerate bilinear form. Let $\bar{G} = G/Z(G)$, define scalar multiplication of $\mathbb{F}_p$ on $\bar{G}$ by $\alpha\bar{x} = \bar{x}^\alpha$ and define $\Phi : \bar{G} \times \bar{G} \to Z(G)$ by $\Phi(\bar{x}, \bar{y}) = x^{-1}y^{-1}xy$. Then the scalar multiplication and the function Φ are well defined. In fact $\bar{G}$ is an $\mathbb{F}_p$-vector space with the given scalar multiplication function and Φ is a non-degenerate bilinear (alternating) form on $\bar{G}$. By the results of Chapter 6 the structure $(\bar{G}, \mathbb{F}_p, \Phi)$ has simple theory. Also the group G is determined up to isomorphism by the data $(\bar{G}, \mathbb{F}_p, \Phi)$, that is if H is another nil-2 exponent p with $Z(H) = H' = \mathbb{Z}_p$ such that the bilinear spaces $(\bar{G}, \mathbb{F}_p, \Phi_G)$ and $(\bar{H}, \mathbb{F}_p, \Phi_H)$ are isomorphic then the groups G and H are isomorphic (see e.g. [4]). Hence if G is a nil-2 group of exponent p with centre cyclic and equal to its derived subgroup, then G has a simple theory.

Chapter 4

Generic Predicates and Automorphisms

4.1 Model Completeness and Model Companions

This section summarises some basic results about model completeness and model companions, which will be used later in this chapter and in subsequent chapters. Throughout the section T denotes a not necessarily complete theory. We refer to [31] for the relevant proofs.

Definition 4.1.1 *Let L be a language and $\mathcal{K}$ a class of L-structures. A structure $M \in \mathcal{K}$ is* existentially closed *in $\mathcal{K}$ if given any $\exists_1$-formula $\phi(x) \in L$, any $a \in M$, and any $N \supseteq M$ with $N \in \mathcal{K}$, then if $N \models \phi(a)$, we also have $M \models \phi(a)$. (Write $M \preccurlyeq_1 N$.)*

Note, that in a language L without relation symbols, (e.g. groups, fields or modules) we have $M \preccurlyeq_1 N$ if and only if every finite set of equations and inequations with parameters from M and with a solution in some extension $N \supseteq M$ already has a solution in M.

Definition 4.1.2 *A theory T is* model complete *if all models of T are existentially closed models.*

Theorem 4.1.3 *Let T be an L-theory. The following are equivalent:*

1. *T is model complete (i.e. if $M, N \models T,\ M \subseteq N \Rightarrow M \preccurlyeq_1 N$);*

2. *Any embedding between models of T is elementary.*
   (i.e. if $M, N \models T,\ M \subseteq N \Rightarrow M \preccurlyeq N$);

3. *For every $M \models T$, $T \cup Diag(M)$ is a complete L_M-theory;*

4. *Every L-formula is equivalent (modulo T) to an $\exists_1$-formula;*

5. *Every L-formula is equivalent (modulo T) to an $\forall_1$-formula.*

Remark 4.1.4 *Let T be a model complete theory, then T is equivalent to an $\forall_2$-axiomatizable theory.*

Note that the property of model completeness, like that of quantifier elimination depends on the particular choice of language L. In fact, by (4) of the above theorem, if T has quantifier elimination in L then T is model complete in L.

Definition 4.1.5 *Let T be an L-theory. An L-theory T^* is a* model companion *for T if the following conditions hold:*

(a) Every model of T^* embeds in a model of T.

(b) Every model of T embeds in a model of T^*.

(c) T^* is model complete.

If the theory T has a model companion it is called companionable.

Theorem 4.1.6 *Let T be an $\forall_2$-theory. Then:*

1. *T is companionable if and only if the class of existentially closed models of T is axiomatizable.*

2. *If T is companionable then the model companion of T is unique and equals the class of existentially closed models of T.*

4.2 Structures with Generic Predicates

In this section we outline a method due to Chatzidakis and Pillay ([13]) which, given a stable (or simple) theory T, produces a simple, usually unstable, theory T_A with many properties in common with the original theory T. The theory T_A is constructed as the model companion of an expansion $T_{0,S}$ of T.

Definition 4.2.1 *A theory, T,* eliminates the quantifier $\exists^\infty$ *if for every formula $\phi(x, y) \in L$ the set $\{b \in M : \phi(M, b)$ is infinite$\}$ is definable.*

Note that whether or not T eliminates $\exists^\infty$ does not depend on the particular choice of language L but only on the definable sets in models of T. It is necessary for the following theorem that T has elimination of quantifiers in the language L. This can be achieved by expanding L, if necessary, by adding Skolem functions or some other elimination set for T.

Theorem 4.2.2 *[13] Let T be a complete theory in a language L, with quantifier elimination and assume that L contains a unary predicate S. Let P be a new unary predicate, and let L_P be the language $L_P = L \cup \{P\}$. Let $T_{0,S}$ be the L_P-theory $T \cup \{\forall x P(x) \to S(x)\}$. Then:*

1. *$T_{0,S}$ has a model companion $T_{P,S}$ if and only if T eliminates the quantifier $\exists^\infty$.*

2. *If $T_{P,S}$ exists then model-theoretic algebraic closure in T and $T_{P,S}$ coincide.*

3. *If T is simple (supersimple) then so is every completion of $T_{P,S}$.*

4. *If algebraic closure is non-trivial on the set S then $T_{P.S}$ has the independence property and hence is unstable.*

The remainder of this section outlines the main points of the proof of Theorem 4.2.2. (The details are contained in [13].)

Firstly, assuming that T eliminates the quantifier $\exists^\infty$, it is possible to define a dimension on all L-formulas $\phi(\bar{x})$, to be the maximal 'algebraic dimension' of a tuple $\bar{a}$ realising $\phi(\bar{x})$. This dimension is definable. From this it can be shown, by using compactness (and working in a sufficiently saturated model M), that for each formula $\phi(\bar{x}, \bar{y})$ the set $S(\phi) := \left\{\bar{b} \in M : \text{there is } \bar{a} \in M \text{ such that } M \vDash \phi(\bar{a}, \bar{b}) \text{ and } \bar{a} \cap acl(\bar{b}) = \emptyset\right\}$ is definable.

We then define $T_{P,S}$ to be the L_P-theory given by expressing that $(M, P) \vDash T_{P,S}$ if and only if:

1. $M \vDash T$; and

2. For every L-formula $\phi(\bar{x}, \bar{y})$, $\bar{x} = (x_1, \ldots, x_n)$ and for every subset $I \subseteq \{1, \ldots, n\}$

$$\forall \bar{z}([\exists \bar{x}\, \phi\,(\bar{x}, \bar{z}) \wedge (\bar{x} \cap acl_T(\bar{z}) = \emptyset) \wedge \textstyle\bigwedge_{i=1}^{n} S(x_i) \wedge \bigwedge_{i \neq j} (x_i \neq x_j)]$$

$$\rightarrow [\exists \bar{x}\, \phi\,(\bar{x}, \bar{z}) \wedge \textstyle\bigwedge_{i \in I} (x_i \in P) \wedge \bigwedge_{i \notin I} (x_i \notin P)]\,).$$

This says that for any L-formula $\phi(\bar{x}, \bar{z})$ and any $I \subseteq \{1, \ldots, n\}$, given any tuple $\bar{b}$, if there is a tuple $\bar{a}$ satisfying $\phi(\bar{x}, \bar{b})$ such that each $a_i \notin acl_T(\bar{b})$, each a_i satisfies S, and the a_i are distinct, then there is a tuple $\bar{c} = (c_1, \ldots c_n)$ satisfying $\phi(\bar{x}, \bar{b})$ such that $P(c_i)$ holds iff $i \in I$. Note, $acl_T(-)$ denotes the model-theoretic algebraic closure in the sense of the theory T.

By the above (assuming that T eliminates $\exists^\infty$) this is axiomatizable in the language L_P, and the next proposition shows that $T_{P,S}$ is the model companion of $T_{0,S}$.

Proposition 4.2.3 *Suppose that T eliminates $\exists^\infty$ then $T_{P,S}$ is the model companion of $T_{0,S}$.*

Proof. Clearly every model of $T_{P,S}$ embeds in (is) a model of $T_{0,S}$. Conversely, it can be shown, by the above axiomatisation, that every model of $T_{0,S}$ can be embedded in a model of $T_{P,S}$.

To show model completeness, let $(M,P) \models T_{P,S}$ and let $\psi(\bar{x},\bar{c})$ be a quantifier free L_P-formula which has a solution in some extension (N,P) of (M,P). We aim to prove that $\psi(\bar{x},\bar{c})$ already has a solution in (M,P).

The formula $\psi(\bar{x},\bar{z})$ is equivalent to (a disjunction of) formulas of the form:

$$\phi(\bar{x},\bar{z}) \wedge \bigwedge_{i\in I} (t_i(\bar{x},\bar{z}) \in P) \wedge \bigwedge_{j\in J} (u_j(\bar{x},\bar{z}) \notin P)$$

with $\phi(\bar{x},\bar{z}) \in L$ and $t_i(\bar{x},\bar{z})$ and $u_j(\bar{x},\bar{z})$ L-terms.

Let $\bar{y}$ be a new tuple of variables indexed by the disjoint union $I \cup J$ and let $\theta(\bar{x},\bar{y},\bar{z})$ be the following formula:

$$\phi(\bar{x},\bar{z}) \wedge \bigwedge_{i\in I} y_i = t_i(\bar{x},\bar{z}) \wedge \bigwedge_{j\in J} y_j = u_j(\bar{x},\bar{z}).$$

Then $\models \exists \bar{y}\, \theta(\bar{x},\bar{y},\bar{z}) \Leftrightarrow \phi(\bar{x},\bar{z})$.
Let $(\bar{a},\bar{b})$ be a tuple in (N,P) satisfying $\theta(\bar{x},\bar{y},\bar{c}) \wedge \bigwedge_{i\in I} (y_i \in P) \wedge \bigwedge_{j\in J} (y_j \notin P)$. By modifying the formula we can assume that the elements of the tuple $\bar{b}$ are distinct and contained in $(N\backslash M) \cap S$ and also that $I = \{i : b_i \in P\}$, $J = \{i : b_i \notin P)$. Therefore $\bar{b} \cap acl_T(\bar{c}) \subseteq \bar{b} \cap M = \emptyset$. Hence we can apply part (2) of the axiomatization of $T_{P,S}$ to the L-formula $\exists \bar{x}\, \theta(\bar{x},\bar{y},\bar{z})$ and the sets I and J to find a tuple $(\bar{d},\bar{e})$ such that:

$$M \models \theta(\bar{d},\bar{e},\bar{c}) \wedge \bigwedge_{i\in I} (e_i \in P) \wedge \bigwedge_{j\in J} (e_j \notin P).$$

Therefore $M \models \exists \bar{y} \left[\theta(\bar{d}, \bar{y}, \bar{c}) \wedge \bigwedge_{i \in I} (y_i \in P) \wedge \bigwedge_{j \in J} (y_j \notin P) \right]$ and hence $M \models \psi(\bar{d}, \bar{c})$. Therefore $\psi(\bar{x}, \bar{c})$ is satisfied in M and thus $T_{P,S}$ is model complete. ■

Proposition 4.2.4 *Let (M, P) and (N, Q) be models of $T_{P,S}$. Let A be a common subset of M and N. Then:*

$$(M, P) \equiv_A (N, Q) \Longleftrightarrow (acl_T(A), P \cap acl_T(A)) \simeq_A (acl_T(A), Q \cap acl_T(A))$$

i.e. (M, P) and (N, Q) satisfy the same $L_P(A)$-formulas iff there is an L_P-isomorphism from $(acl_T(A), P \cap acl_T(A))$ to $(acl_T(A), Q \cap acl_T(A))$ fixing the common subset A.

This gives the following corollaries:

Corollary 4.2.5

1. *The completions of $T_{P,S}$ are given by describing $P \cap acl_T(\emptyset)$.*
2. *If $\bar{a}, \bar{b} \in (M, P) \models T_{P,S}$ and $A \subseteq M$ then $tp(\bar{a}/A) = tp(\bar{b}/A)$ if and only if there is an A-isomorphism of L_P-structures from $acl_T(A, \bar{a})$ to $acl_T(A, \bar{b})$ taking $\bar{a}$ to $\bar{b}$.*
3. *Let $\bar{a} \in M$, $A \subseteq M$. Then $\bar{a}$ is algebraic over A in the sense of the theory $T_{P,S}$ if and only if $\bar{a} \in acl_T(A)$ i.e. algebraic closure in the sense of $T_{P,S}$ and T_P coincide.*
4. *Modulo $T_{P,S}$ every formula $\phi(\bar{x})$ is equivalent to a disjunction of formulas of the form $\exists \bar{y}\, \psi(\bar{x}, \bar{y})$ where $\psi \in L_P$ is quantifier free and for every tuple $(\bar{a}, \bar{b})$ satisfying ψ, $\bar{b} \in acl_T(\bar{a})$.*

The next step is to show that if T is simple then the theory $T_{P,S}$ is also simple. (Recall that Theorem 2.3.11 says that if T has a notion of independence satisfying the independence theorem over a model then T is simple and the notion of independence is non-forking on T.) The idea is to take the notion of independence for $T_{P,S}$ to be non-forking on the reducts of models of $T_{P,S}$ to models of T. This is clearly a notion of independence on $T_{P,S}$ and, by using the characterisation of algebraically closed sets in models of $T_{P,S}$ and the partial quantifier elimination result in (4) of Corollary 1, we can prove the independence theorem over a model for $T_{P,S}$. Hence:

Proposition 4.2.6 *If T is simple (supersimple) then so is every completion of $T_{P,S}$. Moreover non-forking is described in (any completion of) $T_{P,S}$ by $tp(\bar{c}/B)$ does not fork over A in $T_{P,S}$ if and only if $tp_T(\bar{c}/B)$ does not fork over A in the sense of the theory T.*

The interest in this construction arises from the fact that if T is a stable theory with non-trivial algebraic closure on the set defined by S then the resulting theory $T_{P,S}$ has the independence property and therefore is simple unstable.

This completes the outline of the proof of Theorem 4.2.2. In the next section we show how to use this method to construct the theory of the random graph as a model companion of the theory of equality and show that this theory is simple unstable.

4.3 The Random Graph

Let L be a language with (equality and) a single binary relation R. Let T be the theory expressing that R is symmetric and anti-reflexive together with the axiom expressing that:

> if U and V are disjoint finite sets of elements then there is an element z such that $R(z,u)$ holds for all $u \in U$ and $\neg R(z,v)$ holds for all $v \in V$.

Then T is complete and $\aleph_0$-categorical. It is the theory of the *random graph*. The random graph is the (unique) countable graph chosen at random with any 2-element subset of the vertex set having probability $1/2$ of being an edge. (See [9] for details.)

Alternatively, we can construct the random graph as follows. Let L be the empty language, (by the empty language we mean the language with only a symbol for equality), and let T be the theory of pure infinite sets in L. By working in L^{eq}, we aim to construct a generic predicate on the sort consisting of unordered pairs as in Theorem 4.2.2, and then prove that this predicate P has the property that given any disjoint finite sets of elements $u_1, \dots, u_m$ and $v_1, \dots, v_m$ there is an element z such that $\{z, u_i\} \in P$ for $i = 1, \dots, m$ and $\{z, v_i\} \notin P$ for $i = 1, \dots, n$.

Clearly T has quantifier elimination in L, is stable, and eliminates $\exists^\infty$. Let S be the sort defined by the equivalence relation E on pairs given by:

$$(x, y)\, E\, (x', y') \iff x = x' \text{ and } y = y' \text{ or } x = y' \text{ and } y = x'$$

i.e. S is the sort consisting of unordered pairs. Let $T_{0,S}$ be the theory $T \cup \{\forall x P(x) \to S(x)\}$ in the language $L \cup \{P\}$. Then, by the results of Section 4.2, the model companion $T_{P,S}$ exists and has axiomatization as given in Proposition 4.2.3. Let f_E be the surjective map from ordered pairs of home sort elements to sort S, taking an ordered pair (x, y) to the unordered pair $\{x, y\}$. Let $\bar{x}$ be a tuple of S-sort variables indexed by $m+n$, and $\bar{z}$ a tuple of home sort variables indexed by $m+n$. Let $\phi(\bar{x}, \bar{z})$ be the formula $\exists y \bigwedge_{i=1}^{n+m} f_E(y, z_i) = x_i$. Then the axiomatization for $T_{P,S}$ says that if $b_1, \dots, b_m, d_1, \dots, d_n$ are distinct home sort elements (then as there is always some tuple $\bar{a}$ of S-sort elements satisfying $\phi(\bar{x}, \bar{b}\bar{d})$) there is a tuple $\bar{c} = (c_1, \dots, c_{m+n})$ satisfying $\phi(\bar{x}, \bar{b}\bar{d})$ such that $P(c_i)$ holds for $i = 1, \dots, m$ and $\neg P(c_i)$ holds for $i = m+1, \dots, m+n$. But if $\bar{c}$ is such a tuple then there is a home sort element e such that $c_i = \{e, b_i\}$ for $i = 1, \dots, m$ and $c_i = \{e, d_{i-m}\}$ for $i = m+1, \dots, n$. Let R be the relation on home sort elements given by $R(a_1, a_2) \iff P(f_E(a_1, a_2))$ holds. Then all this says exactly that if $b_1, \dots, b_m, d_1, \dots, d_n$ are distinct elements then there is e such that $R(e, b_i)$ for $i = 1, \dots, m$ and $\neg R(e, d_i)$ for $i = 1, \dots, n$ and hence

$T_{P,S}$ is the theory of the random graph. In particular theorem 4.2.2 shows that the theory of the random graph is simple (unstable) and that non-forking on models of $T_{P,S}$ is just non-forking on theories of equality and hence $T_{P,S}$ is supersimple of SU-rank 1.

4.4 Structures with Generic Automorphisms

In this section the construction of Section 4.2 is modified in order to consider structures with generic automorphisms rather than generic predicates. The results and proofs are broadly similar to those of Section 4.2 the main difficulty being that there are no good necessary and sufficient conditions for the existence of the model companion. However, if the model companion does exist, then its theory is simple whenever the original theory is stable. As in Section 4.2 the results are due to Chatzidakis and Pillay in [13].

Theorem 4.4.1 *[13] Let T be a complete theory with PAPA and quantifier elimination in a language L. Let σ be a new unary function symbol and $L_\sigma = L \cup \{\sigma\}$. Let T_0 be the L_σ-theory with models (M, σ) where $M \vDash T$ and $\sigma \in Aut(M)$. Then if a model companion T_A of T_0 exists we have:*

1. *$T_A = Th(\{(M, \sigma) : M \vDash T$ and σ is $\aleph_1$-generic$\})$.*

2. *$acl_{T_A}(A) = acl_\sigma(A)$ where $acl_\sigma(-)$ denotes the algebraic closure in the sense of T of the closure of the set A under σ and σ^{-1}.*

3. *If T is stable (respectively superstable) and T_A exists, then every completion of T_A is simple (resp. supersimple).*

4. *If there is $M \vDash T$ and tuples $\bar{a}$, $\bar{b}$ such that $acl_T(M, \bar{a}, \bar{b}) \neq dcl_T(acl_T(M, \bar{a}), acl_T(M, \bar{b}))$ then T_A has the independence property and is therefore unstable.*

We first need to define the terms $PAPA$ and $\aleph_1$-generic.

Definition 4.4.2 *(Fix a theory T in a language L) A* local automorphism *is a pair $\mathcal{A} = (A, \alpha)$ where A is an algebraically closed (in T^{eq}) substructure of $\tilde{M}$ (a monster model) and $\alpha \in Aut(A)$. An* embedding *of local automorphisms $(A, \alpha) \to (B, \beta)$ is a map $f : A \to B$ which is an embedding of L-structures such that $f(\alpha(a)) = \beta(f(a))$ for all $a \in A$.*

T has $PAPA$ (property of amalgamation of pairs of automorphisms) if whenever $f_1 : (A, \alpha) \to (B_1, \beta_1)$ and $f_2 : (A, \alpha) \to (B_2, \beta_2)$ are embeddings of local automorphisms then there is a local automorphism (C, γ) and embeddings $g_1 : (B_1, \beta_1) \to (C, \gamma)$ and $g_2 : (B_2, \beta_2) \to (C, \gamma)$ such that $g_2 \circ f_2 = g_1 \circ f_1$.

Stable theories and $Th(\mathbb{Q}, <)$ have PAPA.

Definition 4.4.3 *(Fix a countable theory T in a language L) Let $M \vDash T$ and $\sigma \in Aut(M)$. Then σ is* $\aleph_1$-generic *if for all countable local automorphisms (A, α) and (B, β), if $(A, \alpha) \subseteq (M, \sigma)$ and $f : (A, \alpha) \to (B, \beta)$ is an embedding of local automorphisms then there is an embedding of local automorphisms $g : (B, \beta) \to (M, \sigma)$ such that $g \circ f$ is the identity on A.*

It turns out that if $T' = Th\left(\{(M, \sigma) : M \vDash T \text{ and } \sigma \text{ is } \aleph_1\text{-generic}\}\right)$ then T_0 has a model companion T_A if and only if T' is model complete, and if T' is model complete (i.e. if T_0 is companionable) then $T' = T_A$. However, characterising T' and determining whether or not it is model complete is not easy.

Assuming that the model companion T_A exists we can then prove Theorem 4.4.1 in much the same way as Theorem 4.2.2. In particular, if T is stable, a notion of independence can be defined on models of T_A to be non-forking on the reducts to models of T, and this satisfies the independence theorem over a model in T_A thus proving (any completion of) T_A to be simple.

In general, proving the existence of a model companion T_A is a very difficult problem. There are some results in specific cases, for example when T is the theory of algebraically closed fields, differentially closed fields, or complete ω-stable theories of modules, when T_A is known to exist. In the case of algebraically closed fields, the model companion is called $ACFA$ and is an important example of a simple unstable theory, studied extensively in [14]. In this case an axiomatization of T_A can be given by saying that if U and V are varieties, with $V \subseteq U \times \sigma(U)$ such that the projections from V to U and $\sigma(U)$ are dense, then there is a tuple a with $(a, \sigma(a)) \in V$.

4.5 Modules with Generic Automorphisms

As in Section 4.4, let T be a complete theory in a language L with $PAPA$ (e.g. T is stable). Let σ be a new unary function symbol, $L_\sigma = L \cup \{\sigma\}$ and $T_0 = T \cup \{\sigma \text{ is an automorphism}\}$. If T_0 admits a model companion we call it T_A. In [13] it is shown that if T is a complete ω-stable theory of modules then the model companion T_A exists (in general T_A is stable but not ω-stable.) The following question is then asked:

Problem 4.5.1 *Is there an example of a complete non-ω-stable theory of modules, T, such that T_A exists? Even better, can we give necessary and sufficient conditions for a model companion T_A to exist for an arbitrary theory of modules, T ?*

Here we give a sufficient condition for T_A to exist and find an example of a non-ω-stable theory satisfying this condition, hence giving a positive answer to the first part of the above problem. We also show that if it were possible to give necessary and sufficient conditions for the existence of a model companion then this would answer various difficult questions in ring theory concerning the coherence of polynomial rings over regular rings.

Note that since the class of modules over a given ring has the amalgamation property then a model companion, if it exists, is in fact a model completion.

In order to characterise model companions of modules we need the following definition:

Definition 4.5.2 *A ring R is* left coherent *if every element of R has finitely generated left annihilator and the intersection of any two finitely generated left ideals is again a finitely generated left ideal. (Similarly right coherent)*

The following theorem was proved by Eklof and Sabbagh in [21].

Theorem 4.5.3 *Let R be a ring, and let $T = T_R$ be the theory of all (non-zero) left R-modules. (So in general T is not a complete theory.) Then T has a model companion if and only if R is left coherent.*

In fact, by [21], if the model companion of T exists, it is possible to characterise it as follows:

- For I a left ideal of R let $E\,(R/I)$ be the injective envelope of the cyclic left R-module R/I.
- Let $\mathcal{E} = \{E\,(R/I) : I \leq {}_R R\}$, i.e. $\mathcal{E}$ is the set of injective envelopes of cyclic left R-modules.
- Enumerate $\mathcal{E}$ as $\mathcal{E} = \{E_j : j \in J\}$.
- Let $M_0 = \bigoplus_{j \in J} E_j^{(\aleph_0)}$.
- Let $T^* = Th\,(M_0)$.
- Then if R is left coherent, T^* is the model companion of T.

Suppose T is a complete theory of R-modules. Then $(M, \sigma) \models T_0 \iff M \models T$ and $\sigma \in Aut(M)$. We can then think of (M, σ) as an $R[\sigma, \sigma^{-1}]$-module, where $R[\sigma, \sigma^{-1}]$ is the Laurent polynomial ring over R. In particular T_0 is stable, since all theories of modules are stable. (Hence T_A, if it exists, is also stable, as it will be a theory of modules.) Therefore if T is the theory of all non-zero R-modules then T_0 is the theory of all non-zero $R[\sigma, \sigma^{-1}]$-modules and hence, by the result of Eklof and Sabbagh, T_0 has a model companion T_A if and only if the ring $R[\sigma, \sigma^{-1}]$ is (left) coherent. Thus, in order to find an example giving a positive solution to the problem, it suffices to find a theory T, of R-modules (for some ring R) such that:

1. T is a complete theory;
2. T is the theory of all (non-zero) R-modules;
3. T is non-ω-stable;
4. $R[\sigma, \sigma^{-1}]$ is (left) coherent.

Definition 4.5.4 *A ring R is* (von-Neumann) regular *if for every $r \in R$ there exists $x \in R$ such that $r = rxr$.*

There are a number of characterisations of regular rings (see [24]), one of them being the condition that every finitely generated left (equivalently right) ideal of R is generated by an idempotent. From this it follows that a regular ring is both right and left coherent.

Theorem 4.5.5 *[58] Consider the following three statements about a ring R:*

1. *R is an infinite, simple, regular ring.*
2. *The theory of all left (equivalently, right) R-modules is model complete.*
3. *The theory of all (non-zero) left (equivalently, right) R-modules is complete.*

Then (1) and (2) are equivalent and (1)$\Rightarrow$(3). If R is coherent then (3)$\Rightarrow$(1). Thus if R is coherent all three statements are equivalent.

Note that if we require $R[\sigma, \sigma^{-1}]$ to be coherent, we certainly need R to be coherent.

Theories of modules over regular rings have elimination of quantifiers. In fact this property characterises regular rings. (See [58] for details.) This means that the construction of a model companion (as described in Theorem 4.4.1) of a theory of modules over a regular ring R takes place in the natural language of R-modules.

We now wish to find a ring R satisfying the following conditions:

1. R is infinite;

2. R is simple (i.e. R has no proper, non-trivial, two sided ideals);

3. R is regular;

4. T_R the theory of all (non-zero) R-modules is non-ω-stable;

5. $R[\sigma, \sigma^{-1}]$ is (left) coherent.

Note that if R is infinite, simple and regular then $Th({}_R R)$, the theory of R as left module over itself, is equal to the theory T_R of all non-zero left R-modules. So for (4) above, it is enough to prove that the module ${}_R R$ is non-ω-stable.

These conditions can be slightly simplified by the following results:

Lemma 4.5.6 *(See [58]) If R is a regular ring then R is ω-stable (i.e. $Th({}_R R)$ is ω-stable) if and only if R is (left) Artinian if and only if R is (left) Noetherian.*

Lemma 4.5.7 *[20] If R is a regular ring then $R[X]$ is left (right) coherent if and only if $R[X, X^{-1}]$ is left (right) coherent.*

We are now interested in knowing when a polynomial ring over a regular ring is coherent. In general (despite much work on specific cases, e.g. Carson [10], Dicks and Schofield [20], and Goodearl and Moncasi [26]) there are no known necessary and sufficient conditions for the coherence of polynomial rings over regular rings. It is known, however, that a polynomial ring over a *commutative* regular ring is coherent. The following result from [10] will prove useful:

Proposition 4.5.8 *Let R be a regular ring of* finite index, *(i.e. there is $n \in \omega$ such that $\forall m \geq n\ \forall r \in R\ (r^m = 0 \rightarrow r^n = 0)$). Then the polynomial ring $R[X]$ is coherent.*

If we could give a necessary and sufficient condition for the existence of a model companion for T_0, when T is a complete theory of modules and T_0 is $T \cup \{\sigma$ is an automorphism$\}$, then, by restricting to the case where T is a theory of modules over a simple regular ring, we would have a complete characterisation of the simple regular rings which have coherent polynomial rings.

Definition 4.5.9 *A ring R will be called* suitable *if R is infinite, simple, regular and non-Artinian.*

We now give an example of a suitable ring R such that $R[X]$ is coherent, thus, by the comments and results above, providing a positive answer to the first part of Problem 4.5.1.

- Let K be any field.
- For $n \in \omega$ let $R_n := M_{2^n}(K)$, the set of $2^n \times 2^n$- matrices over K.
- Let $\varphi_{n,n+1} : R_n \rightarrow R_{n+1}$ be the diagonal map, $\varphi_{n,n+1}(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$.
- For $m \geq n$ define $\varphi_{n,m} : R_n \rightarrow R_m$ by:

  $$\varphi_{n,m} = \varphi_{m-1,m} \circ \varphi_{m-2,m-1} \circ \cdots \circ \varphi_{n,n+1}.$$

  For example, if $a \in R_0$ (i.e. $a \in K$) then $\varphi_{0,2}(a) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$.
- Let $R = \lim_{n \rightarrow \infty} R_n$ be the direct limit of the rings R_n with respect to the maps $\varphi_{n,m} : R_n \rightarrow R_m$.

Formally, to construct the direct limit, put $S = \bigcup_{n\in\omega} R_n$, a disjoint union. Define an equivalence relation $\sim$ on S by: for $A, B \in S$, $A \in R_m$, $B \in R_n$,

$$A \sim B \Leftrightarrow \text{there is } t \geq \max\{m, n\} \text{ such that } \varphi_{m,t}(A) = \varphi_{n,t}(B).$$

Then $\sim$ is an equivalence relation on S. Let $R = S/\!\sim$. Then, for each n, there are natural injections $\varphi_n : R_n \to R$ and addition and multiplication are well defined on R, giving R a well defined ring structure. It is convenient to view R as an ascending union of the rings R_n. This can be done by identifying each ring R_n with its image under the embedding $\varphi_n : R_n \to R$.

Proposition 4.5.10 *Let R be the ring described above, then*

1. *R is a suitable ring, i.e.:*

   (a) R is infinite;

   (b) R is simple;

   (c) R is non-Artinian;

   (d) R is regular;

2. *The polynomial ring $R[X]$ is coherent.*

Proof: **(1)(a)** Clearly R is infinite.

(b) Suppose $0 \neq I \trianglelefteq R$. We aim to prove $I = R$. It is sufficient to prove $1 \in I$. Choose some non-zero $b \in I$. So $b \in R_n$ for some n. Then $I \cap R_n \trianglelefteq R_n$ and $I \cap R_n \neq 0$ as it contains b. But $R_n = M_{2^n}(K)$ is a simple ring, hence $I \cap R_n = R_n \Rightarrow 1 \in I \cap R_n$. Thus $1 \in I$ and therefore $I = R$ and the ring R is simple.

(c) To prove R is non-Artinian we construct an infinite, properly descending chain of left ideals of R. Let $e_n \in R_n$ $(= M_{2^n}(K))$ be the matrix with entry 1 in the top left corner and zeros everywhere else.

E.g. $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_1 \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and $e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Consider each element e_n as an element of R by identifying elements with their $\sim$ equivalence classes via the map $\varphi_n : R_n \to R$, and let $I_n = Re_n$. Then for $m \geq n$: $e_m e_n = e_m$ and so $e_m \in I_n$. Clearly $e_n \notin I_m$ for $m \geq n$. Thus $I_1 \supset I_2 \supset I_3 \supset \cdots$ is an infinite properly descending chain of left ideals, and hence R is not (left) Artinian.

(d) To show that R is regular, first note that any simple artinian ring is regular (see e.g. [24]). Hence each ring R_n is regular. But a union of a chain of regular rings is regular. (E.g. because the property of regularity is $\forall_2$-axiomatizable and $\forall_2$-formulas are preserved in unions of chains.) Thus the ring R is regular.

Therefore, part 1 of the proposition is proved and R is a suitable ring.

(2) We aim to prove that $R[X]$ is coherent.

The ring $R[X]$ is the polynomial ring over the direct limit of the rings R_n. This is clearly isomorphic to the ring $lim_{n\to\infty}(R_n[X])$, the direct limit of the polynomial rings $R_n[X]$ under the induced mappings, $\psi_{n,m} : R_n[X] \to R_m[X]$ given by $\psi_{n,m}(f(X)) = \psi_{n,m}(A_0 + A_1 X + \cdots + A_t X^t) = \varphi_{n,m}(A_o) + \varphi_{n,m}(A_1)X + \cdots + \varphi_{n,m}(A_t)X^t$. So from now on we consider $R[X]$ as the direct limit of the polynomial rings $R_n[X]$, and prove that this ring is coherent. We first note that each of the rings $R_n[X]$ is coherent. This follows immediately from Lemma 4.5.8, as each of the rings $R_n[X]$ is regular and has finite index 2^n. We then prove the coherence of $R[X]$ by showing:

(a) The (left) annihilator of every non-zero element of $R[X]$ is a finitely generated (left) ideal, and

(b) The intersection of two finitely generated (left) ideals of $R[X]$ is itself a finitely generated (left) ideal.

(a) Let $f(X) \in R[X]$. We must prove that $l.Ann_{R[X]}(f(X))$ is finitely generated (f.g.). Choose n such that $f(X) \in R_n[X]$. Then, as $R_n[X]$ is coherent, $l.Ann_{R_n[X]}(f(X))$ is f.g. Let $l.Ann_{R_n[X]}(f(X)) = R_n[X](f_1, \dots, f_r)$, where $f_i(X) \in R_n[X]$ for $i = 1 \dots, r$.

Claim. $l.Ann_{R[X]}(f(X)) = R[X](f_1, \dots f_r)$.

Proof of claim: Suppose $g(X)f(X) = 0$ for some $g(X) \in R[X]$. For some $m \in \omega$, $g(X) \in R_m[X]$ (without loss of generality $m \geq n$). We prove the case $m = n+1$, the general case being virtually identical. Let $f(X) = A_0 + A_1X + \cdots + A_tX^t$ with $A_l \in R_n$ for each $l = 0, \dots, t$. Let $g(X) = B_0 + B_1X + \cdots + B_sX^s$ with $B_l \in R_{n+1}$ for $l = 0, \dots, s$.

Write $B_l = \begin{pmatrix} B_l^{11} & B_l^{12} \\ B_l^{21} & B_l^{22} \end{pmatrix}$ $B_l^{jk} \in R_n$ $l = 0, \dots, s$ $j, k \in \{1, 2\}$.

Then $g(X)f(X) = 0 \iff$

$$\left(\begin{pmatrix} B_0^{11} & B_0^{12} \\ B_0^{21} & B_0^{22} \end{pmatrix} + \begin{pmatrix} B_1^{11} & B_1^{12} \\ B_1^{21} & B_1^{22} \end{pmatrix} X + \cdots + \begin{pmatrix} B_s^{11} & B_s^{12} \\ B_s^{21} & B_s^{22} \end{pmatrix} X^s\right) \times$$
$$\left(\begin{pmatrix} A_0 & 0 \\ 0 & A_0 \end{pmatrix} + \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix} X + \cdots + \begin{pmatrix} A_t & 0 \\ 0 & A_t \end{pmatrix} X^t\right) = 0$$

$$\Leftrightarrow \left\{ \begin{array}{l} (B_0^{11} + B_1^{11}X + \cdots + B_s^{11}X^s)(A_0 + A_1X + \cdots + A_tX^t) = 0 \ and \\ (B_0^{12} + B_1^{12}X + \cdots + B_s^{12}X^s)(A_0 + A_1X + \cdots + A_tX^t) = 0 \ and \\ (B_0^{21} + B_1^{21}X + \cdots + B_s^{21}X^s)(A_0 + A_1X + \cdots + A_tX^t) = 0 \ and \\ (B_0^{22} + B_1^{22}X + \cdots + B_s^{22}X^s)(A_0 + A_1X + \cdots + A_tX^t) = 0 \end{array} \right\}.$$

Now let $g^{jk}(X) = B_0^{jk} + B_1^{jk}X + \cdots + B_s^{jk}X^s, \in R_n[X]$,

$$\begin{aligned} \text{Then } g(X)f(X) &= 0 \iff g^{jk}(X)f(X) = 0 \ \ for\ all\ \ j = 1, 2 \ \ k = 1, 2 \\ \text{i.e. } g(X)f(X) &= 0 \iff g^{jk}(X) \in l.Ann_{R_n[X]}(f(X)) \ \ j = 1, 2 \ \ k = 1, 2. \end{aligned}$$

But $g^{jk}(X) \in l.Ann_{R_n[X]}(f(X))$ which is f.g. by $f_1(X), \dots, f_r(X)$. Therefore, for each pair $\{j, k\}$, there are $p_i^{jk}(X) \in R_n[X]$ such that

$g^{jk}(X) = p_1^{jk}(X)f_1(X) + p_2^{jk}(X)f_2(X) + \cdots + p_r^{jk}(X)f_r(X)$. Suppose $p_i^{jk}(X) = C_{i,o}^{j,k} + C_{i,1}^{j,k}X + \cdots + C_{i,v}^{j,k}X^v$ with $C_{i,d}^{j,k} \in R_{\dot{n}}$. Let $q_i(X) \in R_{n+1}(X)$ be the polynomial,

$$q_i(X) = \begin{pmatrix} C_{i,0}^{11} & C_{i,0}^{12} \\ C_{i,0}^{21} & C_{i,0}^{22} \end{pmatrix} + \begin{pmatrix} C_{i,1}^{11} & C_{i,1}^{12} \\ C_{i,1}^{21} & C_{i,1}^{22} \end{pmatrix} X + \cdots + \begin{pmatrix} C_{i,v}^{11} & C_{i,v}^{12} \\ C_{i,v}^{21} & C_{i,v}^{22} \end{pmatrix} X^v.$$

Then $g(X) = q_1(X)f_1(X) + \cdots + q_r(X)f_r(X)$.

Exactly the same argument works in the general case $m \geq n$ (using $2^{m-n} \times 2^{m-n}$ matrices instead of 2×2 matrices), showing that if $g(X)f(X) = 0$ for some $g(X) \in R[X]$ then $g(X) \in R[X](f_1, \ldots, f_r)$ and hence $l.Ann_{R[X]}(f(X))$ is finitely generated.

(b) The intersection of any two f.g. left ideals is itself a f.g. left ideal.

Let $I_1 = R[X](f_1(X), \ldots, f_r(X))$ and $I_2 = R[X](f_1'(X), \ldots, f_s'(X))$. We must prove that $I_1 \cap I_2$ is finitely generated.

Choose $n \in \omega$ such that $f_1(X), \ldots, f_r(X), f_1'(X), \ldots, f_s'(X) \in R_n[X]$. Let $I_1^* = R_n[X](f_1(X), \ldots, f_r(X))$, and $I_2^* = R_n[X](f_1'(X), \ldots, f_s'(X))$. Then, as $R_n[X]$ is coherent, $I_1^* \cap I_2^*$ is a f.g. left ideal of $R_n[X]$. Let $I_1^* \cap I_2^* = R_n[X](h_1(X), \ldots, h_t(X))$ with $h_i(X) \in R_n[X]$. For $m \in \omega$ let $J_m = (I_1 \cap I_2) \cap R_m[X]$. Then:

1. $J_n = (I_1 \cap I_2) \cap R_n[X] = I_1^* \cap I_2^*$
2. $\bigcup_{m \in \omega} J_m = I_1 \cap I_2$

Claim: For each $m \geq n$, $J_m = R_m[X]J_n$ (and hence $I_1 \cap I_2 = \bigcup_{m \in \omega} J_m = R[X]J_n$ implying that $I_1 \cap I_2$ is f.g.)

Proof of claim: Clearly $R_m[X]J_n \subseteq J_m$ for $m \geq n$.

Required to prove: $k(X) \in J_m \Rightarrow k(X) \in R_m[X]J_n$

By induction, it is enough to consider the case $m = n+1$.

Let $k(X) \in (I_1 \cap I_2) \cap R_{n+1}[X]$. Thus:

$$\begin{aligned} k(X) &= g_1(X)f_1(X) + \cdots + g_r(X)f_r(X) \quad g_i(X) \in R_{n+1}[X] \ i = 1, \ldots, r \text{ and} \\ k(X) &= g_1^{'}(X)f_1^{'}(X) + \cdots + g_s^{'}(X)f_s^{'}(X) \quad g_i^{'}(X) \in R_{n+1}[X] \ i = 1, \ldots, s. \end{aligned}$$

$$\begin{aligned} \text{Suppose } f_i(X) &= A_{i,0} + A_{i,1}X + \cdots + A_{i,v}X^v \text{ and} \\ f_i^{'}(X) &= A_{i,0}^{'} + A_{i,1}^{'}X + \cdots + A_{i,v}^{'}X^v \text{ with } A_{i,l}^{(')} \in R_n. \end{aligned}$$

Let

$$\begin{aligned} g_i(X) &= \begin{pmatrix} B_{i,0}^{11} & B_{i,0}^{12} \\ B_{i,0}^{21} & B_{i,0}^{22} \end{pmatrix} + \begin{pmatrix} B_{i,1}^{11} & B_{i,1}^{12} \\ B_{i,1}^{21} & B_{i,1}^{22} \end{pmatrix} X + \cdots + \begin{pmatrix} B_{i,v}^{11} & B_{i,v}^{12} \\ B_{i,v}^{21} & B_{i,v}^{22} \end{pmatrix} X^v \\ g_i^{'}(X) &= \begin{pmatrix} C_{i,0}^{11} & C_{i,0}^{12} \\ C_{i,0}^{21} & C_{i,0}^{22} \end{pmatrix} + \begin{pmatrix} C_{i,1}^{11} & C_{i,1}^{12} \\ C_{i,1}^{21} & C_{i,1}^{22} \end{pmatrix} X + \cdots + \begin{pmatrix} C_{i,v}^{11} & C_{i,v}^{12} \\ C_{i,v}^{21} & C_{i,v}^{22} \end{pmatrix} X^v. \end{aligned}$$

For $j = 1, 2$ $k = 1, 2$; with $B_{i,l}^{j,k}$, $C_{i,l}^{j,k} \in R_n$.

$$\begin{aligned} \text{Let } g_i^{jk}(X) &= B_{i,0}^{jk} + B_{i,1}^{jk}X + \cdots + B_{i,v}^{jk}X^v \text{ and} \\ g_i^{'jk}(X) &= C_{i,0}^{jk} + C_{i,1}^{jk}X + \cdots + C_{i,v}^{jk}X^v. \end{aligned}$$

Then $g_i^{jk}(X)$ and $g_i^{'jk}(X) \in R_n[X]$. Thus:

$$\begin{aligned} k(X) &= \sum_{i=0}^{r} \begin{pmatrix} g_i^{11}(X) & g_i^{12}(X) \\ g_i^{21}(X) & g_i^{22}(X) \end{pmatrix} \begin{pmatrix} f_i(X) & 0 \\ 0 & f_i(X) \end{pmatrix} \\ &= \sum_{i=0}^{s} \begin{pmatrix} g_i^{'11}(X) & g_i^{'12}(X) \\ g_i^{'21}(X) & g_i^{'22}(X) \end{pmatrix} \begin{pmatrix} f_i^{'}(X) & 0 \\ 0 & f_i^{'}(X) \end{pmatrix}. \end{aligned}$$

$$\text{Let } k_{jk}(X) = \sum_{i=1}^{r} g_i^{jk}(X)f_i(X) \ \left(= \sum_{i=1}^{s} g_i^{'jk}(X)f_i'(X) \right).$$

Then $k_{jk}(X) \in J_n$. But J_n is finitely generated by $h_1(X), \ldots, h_t(X)$, so there are $p_1^{j,k}(X), \ldots, p_t^{j,k}(X) \in R_n[X]$ such that:

$$k_{jk}(X) = p_1^{j,k}(X)h_1(X) + \cdots + p_t^{j,k}(X)h_t(X).$$

Then let $q_i(X) = \begin{pmatrix} p_i^{11}(X) & p_i^{12}(X) \\ p_i^{21}(X) & p_i^{22}(X) \end{pmatrix}$.

$$\text{Hence } k(X) = q_1(X)h_1(X) + \cdots + q_t(X)h_t(X)$$

and so $k(X) \in R_{n+1}[X]J_n$.

Then by induction we have that $I_1 \cap I_2 = R[X](h_1(X), \ldots, h_t(X))$. Hence $I_1 \cap I_2$ is finitely generated and so $R[X]$ is coherent.

Corollary 4.5.11 *Let R be the ring described above and let $T = T_R$ be the theory of all non-zero left R-modules. Then T is complete and non-ω-stable and there exists a model companion T_A, for the theory $T_0 = T \cup \{\sigma$ is an automorphism$\}$. This model companion is the theory of the $R[\sigma, \sigma^{-1}]$-module $\bigoplus_{j \in J} E_j^{(\aleph_0)}$ as described in the characterisation of model companions of modules following 4.5.3.*

To conclude this section we briefly show that it is certainly not the case that every suitable ring R has a coherent polynomial ring $R[X]$.

Definition 4.5.12 *A ring R is* (left) semihereditary *if every finitely generated (left) ideal is projective.*

If R is a regular ring then the weak global dimension of R, written w.gl.dim(R), is zero. This implies w.gl.dim($R[X]$) $= 1$ which in turn implies that $R[X]$ is (left) coherent iff $R[X]$ is (left) semihereditary. Thus, in the case of regular rings R, coherence of $R[X]$ is equivalent to $R[X]$ being semihereditary.

In [26] a characterisation of regular rings R, such that $R[X]$ is coherent/semihereditary is given when R belongs to one of two particular classes of regular rings: the N^*-complete rings and the $\aleph_0$-continuous rings. It is proved that if R is in one of these two classes of rings then $R[X]$ is left (equivalently right) coherent/semihereditary if and only if R has finite index if and only if

R is *strongly π-regular.* (R is strongly π-regular if for all $a \in R$ there exists $n \in \omega$ such that $Ra^n = Ra^{n+1}$).

A ring R is *(left) self-injective* if R is an injective object in the category of left R-modules. In [24] it is noted that any self-injective regular ring is continuous and hence $\aleph_0$-continuous. We now briefly give an example of a suitable ring which is (left) self-injective and not of finite index, thus demonstrating that there are suitable rings R such that $R[X]$ is not coherent.

Let K be a field and let V be an $\aleph_0$-dimensional K-vector space. Let $T = End_K(V)$. Obviously T is non-Artinian and does not have finite index. T has a unique maximal two sided ideal, M, the set of all K-linear transformations of V of finite rank. Let $R = T/M$. Then R is simple, regular and non-singular. Clearly R is non-Artinian and not of finite index. We can then construct Q, the maximal left quotient ring of R. (See [25] for the details of this construction.) We claim that Q is the required ring. Firstly, if Q is a (left) quotient ring of a ring R, then Q is non-singular if and only if R is non-singular. Secondly, (see e.g. [25] p.60) a quotient ring of a (left) non-singular ring is regular and (left) self injective. Also, (see [25] p83, ex. 3.17) if R is simple and non-singular, then the quotient ring Q is simple. As R is non-Artinian, Q is also non-Artinian. Finally, as $R \subseteq Q$ are regular rings with $_RR$ and essential submodule of $_QR$ then R and Q have the same index. Hence Q is a suitable ring of infinite index and thus by [26], $Q[X]$ is not coherent.

Chapter 5

Vector Spaces with Bilinear Forms

In this chapter we outline some definitions and results on vector spaces with bilinear forms which will be needed later on. We will be especially interested in forms over infinite dimensional spaces, where some of the familiar results from the finite dimensional case fail to generalise. Although our primary interest is in bilinear forms it is often convenient to give definitions and results for the more general sesquilinear forms. The basic theory, in the finite dimensional cases can be found in [60] or, with a more geometric emphasis, in [1]. For results on infinite dimensional spaces we refer to [27] which is mostly concerned with the countable dimensional case, and [37] which concentrates more on uncountable dimensional spaces.

5.1 Sesquilinear Forms

Definition 5.1.1 *Let K be a division ring and $\nu : K \to K$ an antiautomorphism of K. Let V be a (left) K-vector space. A map $\Phi : V \times V \to K$ is a* sesquilinear form *(with respect to ν) if Φ satisfies:*

1. $\Phi(x + y, z) = \Phi(x, z) + \Phi(y, z) \ \ \forall x, y, z \in V$

2. $\Phi(x, y+z) = \Phi(x, y) + \Phi(x, z) \ \ \forall x, y, z \in V$

3. $\Phi(\lambda x, y) = \lambda \Phi(x, y) \ \ \forall x, y \in V, \forall \lambda \in K$

4. $\Phi(x, \lambda y) = \Phi(x, y)\lambda^{\nu} \ \ \forall x, y \in V, \forall \lambda \in K$

For fixed K and ν and (left) K-vector space V, let $Sesq_{\nu}(V)$ be the set (additive group) of all ν-sesquilinear forms on V. We usually assume that K is a field, in which case ν is an automorphism of K. We say V is a *sesquilinear space* to mean that V is a K-vector space (for some division ring K) together with a sesquilinear form $\Phi : V \times V \to K$. We sometimes use square brackets to denote the form (i.e. $[x, y] = \alpha$ instead of $\Phi(x, y) = \alpha$).

Definition 5.1.2 *Let V be a sesquilinear space. The formula $\Phi(x, y) = 0$ defines a binary relation on $V \times V$ called* orthogonality*, denoted by $x \perp y$. A sesquilinear form Φ is* orthosymmetric *if the orthogonality relation is symmetric: i.e. $x \perp y \iff y \perp x$.* ***From now on we assume that all sesquilinear forms are orthosymmetric.***

Definition 5.1.3 *Let V be a vector space with sesquilinear form Φ and associated orthogonality relation $\perp$. Let X be a subset of V. Define $X^{\perp} = \{v \in V : v \perp x \ \forall x \in X\}$. Note that $X^{\perp}$ is necessarily a* subspace *of V.*

Lemma 5.1.4 *Let V be a sesquilinear space and $X, Y \subseteq V$. Then:*

1. $X \subseteq Y \Rightarrow Y^{\perp} \subseteq X^{\perp}$.

2. $X \subseteq (X^{\perp})^{\perp}$ *and hence* $X^{\perp} = X^{\perp\perp\perp}$ *(writing $(X^{\perp})^{\perp}$ as $X^{\perp\perp}$ etc.).*

3. $(X + Y)^{\perp} = X^{\perp} \cap Y^{\perp}$.

Theorem 5.1.5 *[27] Let $\Phi \in Sesq_{\nu}(V)$ be orthosymmetric and assume that* $\dim(V/V^{\perp}) > 1$.

1. *Then the square of ν is an inner automorphism of K, i.e. there is $\varepsilon \in K$ such that:*

   - $\nu^2(\lambda) = \varepsilon^{-1}\lambda\varepsilon \ \ \forall\lambda \in K$ *(So if K is a field or $\varepsilon \in Cen(K)$ then ν is an involution.)*
   - $\nu(\varepsilon) \cdot \varepsilon = \varepsilon \cdot \nu(\varepsilon) = 1_K$

2. $\Phi(y, x) = \varepsilon\, \Phi(x, y)^\nu \ \ \forall x, y \in V$

i.e. the (orthosymmetric) sesquilinear form Φ is ε-hermitian.

Definition 5.1.6 *Let $\Phi : V \times V \to K$ be an ε-hermitian form with respect to the antiautomorphism ν of K.*

- *If $\varepsilon = 1$ the form is called* hermitian.
- *If $\varepsilon = 1$ and ν is the identity on K then the form is* symmetric.
- *If $\varepsilon = -1$ the form is called* skew-hermitian.
- *If $\varepsilon = -1$ and ν is the identity on K then the form is* skew-symmetric.
- *If $\Phi(x, x) = 0 \ \forall x \in V$ then the form is* alternating *(or* alternate*).*

Note that (by Theorem 5.1.5) if ν is the identity on K, then K must be a field.

Lemma 5.1.7 *[27] Let V be a sesquilinear space. If the restriction of Φ to a subspace X of V is alternate but not identically zero then Φ is skew symmetric on the entire space V. In particular K is a field. If $Char(K) \neq 2$ then any skew symmetric form is alternate, and hence in this case, alternate and skew symmetric forms amount to the same thing.*

Definition 5.1.8 *A* bilinear form *is a sesquilinear form with respect to the identity automorphism. In particular, bilinear forms can only occur over fields.*

Lemma 5.1.9 *Let K be a field and V a K-vector space. Then if Φ is a bilinear form on V and $\dim(V/V^{\perp}) > 1$ then Φ is either symmetric or alternate.*

From now on (and in all subsequent chapters) we assume that $\mathbf{Char(K) \neq 2}$ (so in particular a bilinear form is either symmetric or skew-symmetric (= alternating)). If we do wish to consider forms over fields of characteristic 2 then there is a dichotomy between the so called 'trace valued' forms, which behave like forms over fields of characteristic not 2, and non traced valued forms, which may behave very differently. (See [27] for details.)

Definition 5.1.10 *Let V be a sesquilinear space and X, a subspace of V. The* radical of X, *is the set* $\mathrm{rad}(X) = X \cap X^{\perp}$. *$X$ is called* non-degenerate *if $rad(X) = 0$. In particular the space V is non-degenerate if and only if $V^{\perp} = 0$ (if and only if 0 is the only element of V orthogonal to every element of V). We assume that all sesquilinear spaces are non-degenerate. (This is no real loss of generality because if V is degenerate then we can fix a complement X, for $rad(V)$ in V and then V is the orthogonal sum of the non-degenerate space X and the space $rad(V)$ on which the form is identically zero.) Note also that a non-degenerate space may have degenerate subspaces.*

Definition 5.1.11 *Let V be a sesquilinear space.*

- *An element $x \in V$ is* isotropic *if $x \perp x$.*
- *The space V is* isotropic *if V contains a non-zero isotropic vector. (Otherwise V is* anisotropic.*)*
- *A subspace $X \subseteq V$ is* totally isotropic *if $X \subseteq X^{\perp}$ (i.e. the form Φ restricted to $X \times X$ is identically zero).*

Definition 5.1.12 *Let V be a sesquilinear space.*

1. *A basis $\mathcal{E} = \{e_i \mid i \in I\}$ for V is an* orthogonal basis *if whenever $i \neq j$ then $e_i \perp e_j$.*

2. *A basis* $\mathcal{E} = \{e_i \mid i \in I\}$ *for* V *is an* orthonormal basis *if* $\Phi(e_i, e_j) = \delta_{ij}$.

3. *A basis* $\mathcal{E} = \{e_i \mid i \in I\} \cup \{f_i \mid i \in I\}$ *for* V *is a* normal *(or* symplectic) *basis if* $e_i \perp e_j \quad \forall i, j \in I, \quad f_i \perp f_j \quad \forall i, j \in I$ *and* $\Phi(e_i, f_j) = \delta_{ij}$.

Suppose that (V, Φ) is a vector space with sesquilinear form Φ, over a field K. An element $\alpha \in K$ is *represented* by the form Φ if there exists $v \in V \backslash \{0\}$ such that $\Phi(v, v) = \alpha$. Thus a form is alternating if and only if the set of elements represented by the form is $\{0\}$, and the set of elements represented by a euclidean inner product on $\mathbb{R}^n$ is $\mathbb{R}^+$. If V is a vector space over a field K with symmetric bilinear form Φ such that every element represented by Φ has a square root in K, then any orthogonal basis for V can be converted to an orthonormal basis.

Lemma 5.1.13 *Let* V *be a (non-degenerate) isotropic, sesquilinear space (over a field of Char* $\neq 2$*). Then* V *has a basis consisting of isotropic vectors and every non-degenerate plane is spanned by two isotropic vectors. (Such a plane is called* hyperbolic.*)*

Note that if V is spanned by a normal basis $\mathcal{E} = \{e_i \mid i \in I\} \cup \{f_i \mid i \in I\}$ then each of the planes $\langle e_i, f_i \rangle$ is hyperbolic and the subspaces $E = \langle e_i \mid i \in I \rangle$ and $F = \langle f_i \mid i \in I \rangle$ are both totally isotropic.

5.2 Finite Dimensional Spaces

The classical case of finite dimensional vector spaces with a bilinear form is well known and well understood. If (V, Φ) is a symmetric bilinear space of finite dimension, then there exists an orthogonal basis for V. If V is non-degenerate and the field K has square roots for all field elements represented by the form Φ then V has an orthonormal basis and hence any two non-degenerate bilinear spaces of the same dimension over K are essentially the same (i.e. isometric, see Section 5.3). This follows from the fact that given any linear functional

$f : V \to V$ there exists $w \in V$ such that $f(v) = \Phi(v, w)$ for all $v \in V$, and from this it can be shown that any non-degenerate subspace of V has an orthogonal complement in V. Similarly any finite dimensional, non-degenerate, alternating (skew symmetric) space, which must be even dimensional, has a normal basis, and hence any two such spaces of the same dimension are isometric and all non-degenerate subspaces have orthogonal complements. Note that a non-degenerate symmetric bilinear space may have isotropic vectors and may even admit a normal basis - think of $\mathbb{C}^n$ with bilinear form defined by $\Phi((a_1, \dots, a_n), (b_1, \dots, b_n)) = a_1b_1 + \cdots + a_nb_n$. However a non-degenerate alternating space could never have an orthogonal basis.

In the case of infinite dimensional spaces the situation is, in general, much more complicated. However, some of the results from finite dimensional spaces generalize to results about finite dimensional subspaces of infinite dimensional spaces.

Lemma 5.2.1 *(See [27]) Let V be a non-degenerate sesquilinear space of any dimension. Let Z be any finite dimensional subspace of V. Let $r_1, \dots, r_m$ be a fixed basis of $rad(Z)$ and let Z_1 be any supplement of $rad(Z)$ in Z. Then there exist mutually orthogonal vectors $y_1, \dots, y_m \in Z_1^\perp$ with $\Phi(r_i, y_j) = \delta_{ij}$. Then*

$$V = K\langle r_1, y_1\rangle \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} K\langle r_m, y_m\rangle \overset{\perp}{\oplus} Z_1 \overset{\perp}{\oplus} V_1$$

where $K\langle r_i, y_i\rangle$ denotes the K-linear span of the vectors r_i and y_i, V_1 is a non-degenerate subspace of V and $V = U \overset{\perp}{\oplus} V$ means that V is the direct sum of the mutually orthogonal subspaces U and V.

Corollary 5.2.2 *Let V be a non-degenerate sesquilinear space of any dimension. Suppose $X \subseteq V$ is a* finite dimensional *subspace. Then $V = X \oplus X^\perp$ if and only if X is non-degenerate.*

The following lemma will prove very useful later on:

Lemma 5.2.3 *Let V be a non-degenerate sesquilinear space and let $a_1, \dots, a_n$ be a set of linearly independent vectors in V. Then given arbitrary $\lambda_1, \dots, \lambda_n \in K$ there exists $x \in V$ such that $\Phi(x, a_i) = \lambda_i \quad i = 1, \dots, n$.*

We may want to know when x can be found as in Lemma 5.2.3 belonging to a prescribed subspace $U \subseteq V$. This problem is answered by:

Lemma 5.2.4 *[27] Let V and $a_1, \dots, a_n$ be as in Lemma 5.2.3. Let $A = Span(a_1, \dots, a_n)$ and let U be a subspace of V. Then given arbitrary $\lambda_1, \dots, \lambda_n \in K$ there exists $x \in U$ such that $\Phi(x, a_i) = \lambda_i \quad i = 1, \dots, n$ if and only if $U^\perp \cap A = (0)$.*

5.3 Isometries and Similarities

Definition 5.3.1 *Let (V_1, Φ_1) and (V_2, Φ_2) be two sesquilinear (ε-hermitian) spaces with respect to the same field K, (anti)automorphism ν, and $\varepsilon \in K$. An* isometry *from V_1 to V_2 is a K-linear bijection $f : V_1 \to V_2$ satisfying $\Phi_1(x, y) = \Phi_2(f(x), f(y)) \ \ \forall x, y \in V_1$.*

If there exists an isometry between V_1 and V_2 then we say that V_1 and V_2 are isometric *and write $V_1 \cong V_2$ (or $(V_1, \Phi_1) \cong (V_2, \Phi_2)$).*

If V is a sesquilinear space then the set of isometries from V to V form a group under composition of maps, called the *orthogonal group* of V. In the case of an alternating form the orthogonal group is called the *symplectic group.*

Theorem 5.3.2 (Witt) *Let V be a non-degenerate sesquilinear space of any dimension. Let U_1 and U_2 be two finite-dimensional subspaces of V, and suppose there exists an isometry $f : U_1 \to U_2$. Then f can be extended to an isometry from V to V (i.e. f extends to a member of the orthogonal group of V).*

Proof. If V is finite dimensional (and $Char(K) \neq 2$) then the assertion is classical, see e.g. [60]. If V is infinite dimensional then we can find a finite dimensional, non-degenerate subspace, W of V containing both U_1 and U_2. Then, by applying the theorem in the finite dimensional case, f extends to an isometry f' of W. As W is finite dimensional, $V = W \oplus W^{\perp}$. Therefore we can define a map $g : V \to V$ by $g(v) = g(w + w') = f'(w) + w'$ (where $v \in V$ is written uniquely as $w + w'$ with $w \in W$ and $w' \in W^{\perp}$). Then clearly g is an isometry of V which extends f. ■

Corollary 5.3.3 (The Cancellation Theorem) *Let V be a non-degenerate sesquilinear space. Suppose that $V = F \oplus F^{\perp} = G \oplus G^{\perp}$, with F and G finite dimensional, isometric subspaces. Then $F^{\perp} \cong G^{\perp}$.*

For applications later on, we need to consider slightly more general structure preserving maps than isometries.

Definition 5.3.4 *Let (V_1, Φ_1) be a sesquilinear space over K_1 and (V_2, Φ_2) be a sesquilinear space over K_2. Then V_1 and V_2 are* semi-similar *if there is an isomorphism of fields $\sigma : K_1 \to K_2$, an element $\mu \in K_2 \backslash \{0\}$ and a σ-semilinear bijection $f : V_1 \to V_2$ (i.e. $f(\alpha v) = \sigma(\alpha) f(v)$) such that $\sigma(\Phi_1(x, y)) \cdot \mu = \Phi_2(f(x), f(y))$. Such a map f is called a* semi-similarity *or sometimes a* semi-similitude.

A semi-similarity between two spaces over the same field is a similarity *if the automorphism σ of the field is the identity. If there is a similarity between two spaces then they are called* similar.

Suppose that (V, Φ) is a sesquilinear space over a field K. Let $U(V, \Phi)$ be the group of isometries of V, $GU(V, \Phi)$, the group of similarities of V and $\Gamma U(V, \Phi)$ the group of semi-similarities of V. Clearly we have

$$U(V, \Phi) \leq GU(V, \Phi) \leq \Gamma U(V, \Phi).$$

It is also the case that, if $GL(V)$ is the general linear group of V, then $GU(V, \Phi) = \Gamma U(V, \Phi) \cap GL(V)$ and

$$\Gamma U(V, \Phi) = GU(V, \Phi) \rtimes Aut(K).$$

Theorem 5.3.5 (Witt's Theorem for Semi-similarities) *Let (V_1, Φ_1) and (V_2, Φ_2) be sesquilinear spaces over K_1 and K_2 respectively. Let $f : U_1 \to U_2$ be a semi-similarity (relative to a fixed isomorphism $\sigma : K_1 \to K_2$ and a fixed 'multiplier' μ) between finite dimensional subspaces $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$. Then there exists an extension of f to a semi-similarity between V_1 and V_2 relative to the isomorphism σ and the multiplier μ if and only if there exists at least one semi-similarity $h : V_1 \to V_2$ relative to σ with multiplier μ.*

5.4 Infinite Dimensional Spaces

As remarked earlier, many of the familiar properties of finite dimensional sesquilinear spaces do not generalise to infinite dimensional spaces. For instance, if V is a finite dimensional sesquilinear space, then for any subspaces X and Y of V we have that $X^\perp + Y^\perp = (X \cap Y)^\perp$. This properties fails in *all* infinite dimensional spaces. Similarly if X is a subspace of a finite dimensional space then $X^{\perp\perp} = X$, again, this property fails in *all* infinite dimensional spaces: given an infinite dimensional space V, there is an (infinite dimensional) subspace $X \subseteq V$ such that $X \neq X^{\perp\perp}$. In fact, we can characterise the infinite dimensional spaces as the class of spaces where certain familiar properties from finite dimensional spaces fail to hold.

Proposition 5.4.1 *([37]) Let V be a non-degenerate sesquilinear space with dim $V > 1$. Then the following are equivalent:*

1. *dim(V) is infinite;*
2. *There are subspaces $X, Y \subseteq V$ such that $X^\perp + Y^\perp \neq (X \cap Y)^\perp$;*
3. *There is a subspace $X \subseteq V$ such that $X^{\perp\perp} \neq X$;*
4. *There is a subspace $X \subseteq V$ with dim $X^{\perp\perp}/X =$dim V;*
5. *There is a proper subspace $X \subseteq V$ with $X^\perp = (0)$;*

6. *The lattice consisting of all subspaces $X \subseteq V$ such that $X = X^{\perp\perp}$ is not modular.*

It does turn out, however, that sesquilinear spaces of dimension $\aleph_0$ are significantly better behaved than spaces of uncountable dimension. The reason for this is that results on existence of orthogonal and normal bases in finite dimensions generalize to spaces of countably infinite dimension.

Theorem 5.4.2 (Gross) *Any sesquilinear space of dimension $\aleph_0$ can be decomposed into an orthogonal sum of lines and planes.*

The proof is just a modification of the Gram-Schmidt orthogonalization process, used to prove the corresponding theorem in finite dimensions, together with an application of Zorn's lemma. See [27] for the details.

Corollary 5.4.3 1. *Any non-degenerate sesquilinear space of dimension $\leq \aleph_0$ which is not alternate has an orthogonal basis.*

2. *A non-degenerate alternate space of dimension $\leq \aleph_0$ has a normal basis, and hence is an orthogonal sum of hyperbolic planes.*

In particular if V is a symmetric bilinear space of countable dimension over a field K with square roots, then V has an orthonormal basis.

The situation for spaces of uncountable dimension is very different. For example, let K be an uncountable field and let V be a K-vector space of dimension $\aleph_1$ with basis $(e_i \mid i \in I)$. (so $|I| = \aleph_1$). Define a symmetric bilinear form on V by $\Phi(e_i, e_j) = \alpha_{i,j}$ where $\{\alpha_{i,j} : i, j \in I\}$ is a set of elements of K algebraically independent over the prime subfield $K_0 \leq K$. Then (V, Φ) is a non-degenerate space and V has no decomposition into an orthogonal sum of finite dimensional subspaces. In fact if V has an orthogonal decomposition $V = X \oplus X^\perp$ then either X or $X^\perp$ is finite dimensional. This kind of construction can also be carried out over a countable or even finite field.

A sesquilinear space V is called *diagonal* if it can be decomposed as an orthogonal sum of finite dimensional subspaces. Thus all countable dimensional spaces are diagonal but there are uncountable dimensional spaces which are not. More surprisingly, it is not always the case that a subspace of a diagonal space is diagonal. This, and many other issues concerning uncountable dimensional spaces, are explored in detail in the various papers contained in [37].

5.5 Extensions of the Field of Scalars

In this section we show that given a sesquilinear space (V, Φ) over a field (division ring) K and a field extension F of K we can extend (V, Φ) to a sesquilinear space (V', Φ') over F. See the appendix of [37] for details.

Let $K \leq F$ be fields (division rings) and let V be a (left) K-vector space. The field F has a natural (right) K-vector space structure . Let $V' = F \otimes_K V$ (an abelian group), where $\otimes_K$ is the tensor product over K. V' can be given a (left) F-vector space structure (and hence also a K-vector space structure) by defining $\alpha(\beta \otimes_K v) = (\alpha\beta) \otimes_K v$ for $\alpha, \beta \in F$, $v \in V$. We have that $\dim_K V$ $=\dim_F V'$. The additive mapping $\imath : V \to V'$ given by, $\imath : v \longmapsto 1 \otimes_K v$ is a K-linear inclusion of vector spaces and therefore we can consider V to be a K-linear subspace of the K-vector space V'. It then follows that V' is simply the F-linear span of the set V. Similarly if U is a subspace of V then by defining $U' = F \otimes_K U$ we have a bijection between K-subspaces of V and F-subspaces of V'.

Now suppose that V has a non-degenerate bilinear form $\Phi : V \times V \to K$. Then, by fixing $v \in V$ and using the universality property of the tensor product, there is a unique additive group homomorphism $\Phi_v : V' \to F$ satisfying:

$$\forall \alpha \in F \;\; \forall w \in V \;\; \Phi_v(\alpha w) = \Phi_v(\alpha \otimes_K w) = \Phi(\alpha v, w) = \alpha \Phi(v, w).$$

By also varying v we can find a (unique) biadditive map $\Phi' : V' \times V' \to F$

such that

$$\Phi'(\alpha v, \beta w) = \alpha\beta\, \Phi'(v, w).$$

Thus Φ' extends Φ and is F-bilinear.

Let $\perp'$ denote orthogonality in V' then if U_1 and U_2 are subspaces of V then $U_1 \perp U_2$ iff $U_1' \perp' U_2'$. In fact we have that: $(U^{\perp})' = U^{\perp'} = U'^{\perp'}$. Thus in particular the space (V', Φ') is non-degenerate.

All of the construction and results of this section go through for the case when K and F are division rings, ν is an antiautomorphism of K which extends to F, and $\Phi : V \times V \to K$ is a sesquilinear form with respect ν and some $\varepsilon \in K$.

Chapter 6

Lie Geometries and Smoothly Approximated Structures

In this chapter we study a particular class of the structures, the *smoothly approximable structures*. These are natural generalisations of $\aleph_0$-categorical, ω-stable structures, which in turn generalise the totally categorical structures. Smoothly approximable structures were originally studied for their automorphism groups and from the resulting classification of these automorphism groups it turned out that the class of smoothly approximable structures coincides with the class of *lie coordinatisable structures*, roughly structures built up by gluing together various geometries over finite fields. Each of these geometries can be proved to have a simple theory, and because the gluing process is 'well-behaved' it follows that any lie coordinatisable (smoothly approximated) structure is simple.

The study of smoothly approximable structures was begun with Lachlan's work (with Cherlin and Harrington) on $\aleph_0$-categorical, ω-stable structures in [43], [19] and [17]. The classification of primitive automorphism groups for smoothly approximable structures was proved in [35], see also [47]. The proof of simplicity (and many other stability-like concepts) is due to Hrushovski and Cherlin and is reported in [15] and [33]. (See also [19].)

6.1 Smoothly Approximated Structures

Definition 6.1.1 *Let M be a finite or $\aleph_0$-categorical structure and let N be a finite substructure of M. Then N is a* finite homogenous substructure *of M (written $N \leq_{\text{hom}} M$) if whenever $\bar{a}$ and $\bar{b}$ are tuples from N, then there is an automorphism of M taking $\bar{a}$ to $\bar{b}$ if and only if there is such an automorphism of M fixing N setwise (i.e. $\bar{a}$ and $\bar{b}$ are in the same $Aut(M)$ orbit if and only if they are in the same $Aut_{\{N\}}(M)$ orbit).*

Definition 6.1.2 *A structure M is* smoothly approximated *(by finite substructures) if*

1. *M is countable and $\aleph_0$-categorical; and*
2. *there is a chain $N_0 \subseteq N_1 \subseteq N_2 \subseteq \ldots$ of finite substructures of M such that $M = \bigcup_{r \in \omega} N_r$ and each N_r is a finite homogenous substructure of M.*

Proposition 6.1.3 *[35] Let M be an $\aleph_0$-categorical structure. The following are equivalent:*

1. *M is smoothly approximated;*
2. *For every $\sigma \in Th(M)$ there is a finite $N \leq_{\text{hom}} M$ such that $N \models \sigma$.*

Corollary 6.1.4 *[35]*

- *If M is smoothly approximated, then $Th(M)$ is not finitely axiomatizable.*
- *If M is smoothly approximated then $Th(M)$ does not have the strict order property.*
- *Any group interpretable in a smoothly approximable structure is nilpotent-by-finite.*

In [35] the smoothly approximable structures with primitive automorphism groups are classified. From this classification it follows that these primitive structures are equivalent to Grassmanians of transitive structures with a 0-definable equivalence relation, each class of which has the same automorphism group as a classical geometry. It then follows that since any smoothly approximable structure is built up by primitive structures, a smoothly approximated structure is 'coordinatised' by classical geometries (also called *linear geometries*) in a sense to be explained in Section 6.3.

6.2 Lie Geometries

In this section we define linear geometries (referred to above as classical geometries) and investigate their properties. In Section 6.3 we show how these geometries can be glued together to make the class of lie coordinatised structures. All the results of this section and the following section can be found in [15], [33] and [19].

Definition 6.2.1 *A* (weak) linear geometry *is one of the following structures:*

1. *An infinite set with no structure other than '=';*

2. *An infinite dimensional vector space over a finite field K;*

3. *(The Symplectic Geometry) An infinite dimensional vector space with a non-degenerate alternating form over a finite field K;*

4. *(The Orthogonal Geometry) An infinite dimensional vector space with a non-degenerate quadratic form over a finite field K. (If $Char(K) \neq 2$ a non-degenerate quadratic form is equivalent to a symmetric bilinear form.);*

5. *(The Unitary Geometry) An infinite dimensional vector space with a non-degenerate hermitian form over a finite field K;*

6. *(The Polar Geometry) A pair of infinite dimensional vector spaces V and V^* over a finite field K, together with a non-degenerate bilinear map from $V \times V^*$ to K;*

7. *(The Quadratic Geometry) An infinite dimensional vector space V, over a finite field K of characteristic 2, together with a non-degenerate alternating form acting on a certain space of quadratic forms on V.*

In order to work at the level of greatest generality we use a multi-sorted language for these structures:

- If there is a vector space V (as in cases 2-7) there is a separate sort for V with an abelian group structure. In the polar case there is a sort for $V \cup V^*$ with an equivalence relation with two classes picking out V and V^* and an abelian group structure on each class.

- If there is a (finite) field K then K is treated as a separate sort with a field language and a function symbol for the action of K on V (and on V^*).

- If there is a form or quadratic map then there is a function symbol for this and a separate sort W, for the range of the map, carrying the structure of a 1-dimensional vector space over K.

- In the unitary geometry the involution of the field σ is part of the language.

We are mostly interested in cases (4) and (5) where the structure is a vector space over a finite field with a non-degenerate bilinear form (at least if $char(K) \neq 2$). For convenience we ignore case (7) which only occurs in characteristic 2. See [15] and [33] for details.

Definition 6.2.2 1. *A* linear geometry *is a weak linear geometry expanded by adding some constants in K and/or W (and predicates for V and V^* in the polar case).*

2. *A* basic linear geometry *is a weak linear geometry expanded by adding constants for all of K and W (and predicates for V and V^* in the polar case).*

A basic linear geometry is essentially the same thing as a one sorted vector space structure (with predicates picking out V and V^* in the polar case) with the sorts K and W added to the language, i.e. there are functions f_α for each $\alpha \in K$ for scalar multiplication ($f_\alpha(x) = \alpha x$) and relations $R_\alpha(x, y)$ for each $\alpha \in K$ to represent elements of W - the range of the form ($R_\alpha(x, y) \iff [x, y] = \alpha$). Note that we can only add the field to the language in this way because it is finite. In particular it is possible to axiomatize the statement '$\forall x \forall y \bigvee_{\alpha \in K} [x, y] = a$ '.

The difference between (weak) linear geometries and basic linear geometries is seen by the differences in their automorphism groups. Suppose, for example, that $M = (V, K, [\,,\,]\,)$ is a symplectic space, where K is a finite field, V is an infinite dimensional K-vector space and $[\,,\,] : V \times V \to K$ is a non-degenerate alternating form. Then the automorphism group of M considered as a basic linear geometry (i.e. by adding all the appropriate constants) is just the group $Sp(V)$, the group of all isometries of V, preserving the given bilinear form. If M is considered as a weak linear geometry then automorphisms of K are part of the automorphism group of M. In fact, in this case, $Aut(M)$ is the group of all semi-similarities of V, denoted $\Gamma Sp(V)$, the normaliser of $Sp(V)$ in $G\Gamma(V)$, where $G\Gamma(V)$ denotes the group of all semilinear transformations of V. Clearly, if M is a linear geometry then $Aut(M) = H$ where $Sp(V) \leq H \leq \Gamma Sp(V)$.

Proposition 6.2.3 *(See [33]) Each of the basic linear geometries has elimination of quantifiers.*

Proof. Consider the structure as a one sorted structure, V, as described above, by absorbing the field into the language. Let V be a countable structure.

Claim. Given any finite $B \subset V$, every quantifier-free type over B is realised in V.

Proof of claim. Let B be a finite subset of V. Any quantifier-free type over B has the form

$$[x, b] = f(b) \text{ for } b \in B \quad \text{and } [x, x] = c$$

where $f : B \to K$ is a linear functional. By non-degeneracy of the form, Lemma 5.2.3 and compactness, there is $z \in V$ such that $[z, b] = f(b)$ for all $b \in B$. It is then enough to find $y \in (B \cup \{z\})^{\perp}$ such that $[y, y] = c - [z, z]$ (since then $[y+z, b] = [y, b] + [z, b] = [z, b] = f(b)$ and $[y+z, y+z] = [y, y] + [z, z] = c$). So without loss of generality, it is sufficient to find $y \in B^{\perp}$ such that $[y, y] = c$. Let $F = \{\alpha : \alpha = [y, y] \text{ for some } y \in B^{\perp}\}$. Then F is non-empty, is closed under sums and closed under multiplication by squares from K. Thus $F = K$ and the claim holds.

Suppose $\bar{a}$ and $\bar{b}$ are finite tuples from V realising the same quantifier-free type. Then there is a partial automorphism of V taking $\bar{a}$ to $\bar{b}$. By Witt's theorem (as the tuples are finite) this automorphism extends to an automorphism (isometry) of V and so $tp(\bar{a}) = tp(\bar{b})$. By the claim, every quantifier-free type is realised in V and hence if p and q are any two types over the empty set with the same quantifier-free parts, then $p = q$. Together with compactness this is enough for quantifier elimination. (See Proposition 9.2.8.)

■

Corollary 6.2.4 *A basic linear geometry is $\aleph_0$-categorical.*

Proof. By quantifier elimination an n-type $p(x_1, \ldots, x_n)$ over $\emptyset$ is determined by the values of $[x_i, x_j]$ for $1 \leq i, j \leq n$ (and the pure vector space part of the type, but a pure vector space over a finite field is already totally categorical). Thus there are only finitely many n-types over $\emptyset$ for each n. Then by Ryll-Nardzewski the theory is $\aleph_0$-categorical. ■

Alternatively we could prove $\aleph_0$-categoricity by using Corollary 5.4.3 to show that any countable basic linear geometry has a normal basis (including the orthogonal case, as the field is finite), and hence any two countable models of the same basic linear geometry are isomorphic.

Proposition 6.2.5 *A basic linear geometry is simple. In fact it is supersimple of SU-rank 1. Moreover, non-forking in a basic linear geometry is just linear independence in the geometry considered as a vector space.*

Proof. This is proved in [19] by showing directly, using quantifier elimination, that every type does not divide over some finite subset of its domain. Here, we prove the proposition (using Theorem 2.3.11) by showing that vector space linear independence defines a notion of independence and satisfies the independence over a model theorem. This has the added advantage of showing that non-forking is just vector space linear independence.

Let T be the theory of a basic linear geometry (considered as a one-sorted vector space structure with the field K incorporated into the language). Let T_0 be the reduct of T to the language of pure K-vector spaces. Then T_0 is stable (in fact totally categorical which implies ω-stable) so non-forking on T_0 defines a notion of independence in any model of T_0, which turns out to be just K-linear independence; $tp_{T_0}(a_1, \dots, a_n/B)$ does not fork over A if and only if $K\text{-Span}(a_1, \dots, a_n) \cap K\text{-Span}(B) \subseteq K\text{-Span}(A)$. This is clearly a notion of independence on models of T, (call it Γ-non-forking) so in order to prove the simplicity of T it is sufficient to prove the independence over a model theorem for Γ-non-forking.

To simplify notation, we write $p\ dn\Gamma f$ over A, to mean that the type p does not Γ-fork over A and write 'C and B are Γ-independent over A', to mean $tp(C/B)\ dn\Gamma f$ over A.

Let $M \models T$ and suppose that $M = (V, K, [\,,\,])$ where K is a finite field, V a K-vector space, and $[\,,\,] : V \times V \to K$ a non-degenerate bilinear form. Let A and B be sets which are Γ-independent over M. Then, without loss of generality, A and B are K-subspaces (of a monster model $\tilde{M}$) and $A \cap B = M$. Let $p(x) \in S(M)$. Let $p_1(x) \in S(A)$ and $p_2(x) \in S(B)$ be such that

$p_1 \restriction M = p_2 \restriction M = p$ and p_1 $dn\Gamma f$ over M and p_2 $dn\Gamma f$ over M. We must find $q(x) \in S(A \cup B)$ extending p_1 and p_2 such that q $dn\Gamma f$ over A (equivalently q $dn\Gamma f$ over A and/or B).

Without loss of generality we can assume that $p_1(x) \vdash x \notin K\text{-Span}(A)$ and $p_2(x) \vdash x \notin K\text{-Span}(B)$, or else $p(x) \vdash x \in M$ and the proof is trivial. Then $p_1(x)$ defines a linear functional $g_1 : A \to K$ given by:

$$g_1(a) = \alpha \iff p_1(x) \vdash [x, a] = \alpha.$$

Similarly p_2 defines a linear functional $g_2 : B \to K$. Then, by quantifier elimination for T, it is sufficient to find c (in some saturated model $\tilde{M} \models T$) such that c is linearly independent from A and B, $[c, a] = g_1(a)\ \forall a \in A$, $[c, b] = g_2(b)\ \forall b \in B$ and $[c, c] = \gamma$, where $p(x) \vdash [x, x] = \gamma$.

Let $\Sigma(x)$ be the following set of formulas:

$$\begin{aligned} \Sigma(x) &= \{x \neq a + b \mid a \in A,\ b \in B\} \cup \{[x, a] = g_1(a) \mid a \in A\} \cup \\ \{[x, b] = g_2(b) \mid b \in B\} \cup \{[x, x] &= \gamma\} \end{aligned}$$

Claim. $\Sigma(x)$ is finitely satisfiable.

Proof of Claim. Let $\Sigma_0(x)$ be a finite subset of $\Sigma(x)$. Then there are finite subspaces $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $\Sigma_0(x) \subseteq \{x \neq a + b \mid a \in A_0,\ b \in B_0\}$ $\cup \{[x, a] = g_1(a) \mid a \in A_0\} \cup \{[x, b] = g_2(b) \mid b \in B_0\} \cup \{[x, x] = \gamma\}$. Then $g_1^0 = g_1 \restriction A_0 : A_0 \to K$ and $g_2^0 = g_2 \restriction A_0 : B_0 \to K$ are linear functionals on finite dimensional subspaces agreeing on their intersection. Hence there is a linear functional $h : A_0 \oplus B_0 \to K$ extending g_1^0 and g_2^0. Then, as $A_0 \oplus B_0$ is a finite dimensional space, there is $z \in A_0 \oplus B_0$ such that $[z, d] = h(d)$ $\forall d \in A_0 \oplus B_0$.

However, we want to find such a $z \notin A_0 \oplus B_0$ and with $[z, z] = \gamma$. Here we can use the same argument as in 6.2.3 to find a non-zero $t \in (A_0 \oplus B_0)^{\perp}$ such that $[t, t] = \gamma - [z, z]$ and then put $z' = z - t$. Then $[z', z'] = \gamma$ and $z' \notin A_0 \oplus B_0$. Hence z' satisfies $\Sigma_0(x)$ and the claim is proved.

By the claim and compactness there is some $c \in \tilde{M}$ satisfying $\Sigma(x)$. Put $q(x) = tp(c/A \cup B)$ and this is the required extension of p_1 and p_2. Hence the independence theorem over a model holds for Γ-non-forking, implying that T is simple and non-forking on T is exactly Γ-non-forking which is just K-linear independence. As T_0 is superstable of U-rank 1 and non-forking on T and T_0 are the same then T is supersimple of SU-rank 1. ■

Remark 6.2.6 *Any basic linear geometry other that a pure set or a pure vector space is unstable. The formula $[x, y] \neq 0$ (where $[\ ,\ ]$ is the bilinear or hermitian etc. form) has the order property and, in fact, also has the independence property.*

Proof. A formula $\phi(x, y)$ has the *order property* if there is $M \models T$ and tuples $(a_i, b_i : i \in \omega)$ such that $M \models \phi(a_i, b_j) \iff i < j$.

$\phi(x, y)$ has the *independence property* if there is $M \models T$ and tuples $(a_i : i \in \omega)$ and $(b_I : I \subseteq \omega)$ such that $M \models \phi(a_i, b_I) \iff i \in I$. (In fact, by compactness, it is enough to find such tuples b_I for all *finite* subsets $I \subset \omega$.)

A countable basic linear geometry has a normal basis $(e_i : i \in \omega)\hat{}(f_i : i \in \omega)$ with respect to the form $[\ ,\ ]$ on V. For the order property put $a_i = e_i$ and $b_i = f_1 + \cdots + f_i$. Then:

$$[a_i, b_j] \neq 0 \iff [e_i,\ f_1 + \cdots + f_j] \neq 0 \iff i < j.$$

For the independence property put $a_i = e_i$ and, for every finite $I \subset \omega$, put $b_I = \sum_{i \in I} f_i$. Then :

$$[a_i, b_I] \neq 0 \iff [e_i, \Sigma_{j \in I} f_j] \neq 0 \iff i \in I.$$

Thus any such basic linear geometry V is unstable. Note, that as V is simple, it cannot have the strict order property.

In [22] David Evans proves that basic linear geometries have the *small index property.* This means that if G is the group of automorphisms of a basic linear geometry then every subgroup of G of countable index is open in the permutation group topology on G. That is, given any subgroup H of countable index in G there is a finite subspace X of the geometry such that H contains the pointwise stabilizer of X, $G_{(X)}$. The reason for the interest in this property is that it means that the permutation group topology on G can be recovered from the abstract group structure of G.

The proof of the small index property for basic linear geometries actually gives that for any subgroup $H \leq G$ with $[G : H] \leq 2^{\aleph_0}$ there is a finite dimensional subspace X of the geometry V such that $G_{(X)} \leq H \leq G_{\{X\}}$ i.e. H lies between the pointwise and setwise stabilizers of X. In order to prove this it is first proved that if A and B are any two finite dimensional subspaces of V then $\langle G_{(A)}, G_{(B)} \rangle = G_{(A \cap B)}$. This implies that the basic linear geometries have weak elimination of imaginaries (every imaginary element is equi-algebraic with a finite set of real elements). Weak elimination of imaginaries for basic linear geometries can also be proved directly from the simplicity and finite rank of these geometries, see [33] and [19].

6.3 Lie Coordinatisable Structures

In this section we describe briefly how the geometries described in the previous section can be 'glued' together to make the class of lie coordinatised structures.

We must first define the projective and affine versions of the linear geometries described in the previous section.

Definition 6.3.1 1. *A* projective geometry *is obtained from a (basic) linear geometry by removing $acl(\emptyset)$ and factoring out by the equivalence relation $acl(x) = acl(y)$.*

2. *An* affine geometry *is obtained from a linear geometry as follows:*

Let V be (the vector space part of) a linear geometry (or one component of a polar geometry). Let (V, A) be the structure where V has all of its structure as a linear geometry and A is a second copy of V carrying no structure other than the regular action (addition map) $+ : V \times A \to A$.

Affine geometries, like linear geometries, have elimination of quantifiers and are supersimple of SU-rank 1. However, an affine geometry (V, A) does not quite have weak elimination of imaginaries, but if we add an extra imaginary sort A^* (described in [19] and [33]) then the resulting structure (V, A, A^*) does have weak elimination of imaginaries.

Definition 6.3.2 *Suppose that J is a projective geometry embedded in a structure M and let B be a finite subset of M. The* localisation *or* projectivisation *of J at B denoted $(J : B)$ is defined as follows:*

Let V be the linear geometry from which J is obtained. Let $V_1 = \{x \in V :$ for all $y \in acl(B) \cap V,\ [x, y] = 0\}$. Let $V_0 = V_1 \cap acl(B)$. Then $(J : B)$ is the projectivisation of the space V_1/V_0. It is the smallest $acl(B)$-definable projective subspace in $J/J \cap acl(B)$ of finite index.

Definition 6.3.3 *Let M and N be structures (over possibly different languages) such that the domain of M is a definable subset of N (with canonical parameter a).*

1. *M is* canonically embedded *in N if every subset of M^n definable in the structure (N, a) is $\emptyset$-definable in M.*

2. *M is* stably embedded *in N if for every formula $\theta(x, y) \in L_N$ there is a formula $\psi(x, z) \in L_M$ such that whenever $b \in N$ and $\theta(x, b)$ defines (in N) a subset X of M then there is $c \in M$ such that $\psi(x, c)$ defines X in M.*

3. *M is* fully embedded *in N if M is both canonically embedded in N and stably embedded in N.*

Lemma 6.3.4 *Suppose that M is fully embedded in N and suppose that M has weak elimination of imaginaries. Let $B \subseteq N$ and let $A = acl(B) \cap M$. Then for all $c_1, c_2 \in M$,*

$$tp_M(c_1/A) = tp_M(c_2/A) \Rightarrow tp_N(c_1/B) = tp_N(c_2/B).$$

Proof. See [19] or [33]. ■

The above lemma says that if M is fully embedded in N then, for any $c \in M$, in order to describe the type of c over some subset of N it is enough to describe the type of c over the intersection of the algebraic closure of that set with the structure M. Since affine geometries (after adding the extra imaginary sort) have weak elimination of imaginaries, any fully embedded affine geometry has the property described in Lemma 6.3.4 and this gives a powerful tool in studying the Lie coordinatised structures which we now define.

Definition 6.3.5 (Lie coordinatised structures) *A structure N is* Lie coordinatised *if N is a structure with finitely many 1-types such that for each $a \in N$ there is a finite sequence $a_0, \ldots, a_{n-1}$ in N with $a_i \in dcl(a)$ for each $i < n$ such that for each $i < n$, one of the following holds:*

1. *$a_i \in acl(a_{i-1})$; or*
2. *for some $j < i$, there exists an a_j-definable fully embedded (in N) basic projective linear geometry J with $a_i \in J$; or*
3. *for some $j < k < i$ there exists an a_j-definable fully embedded basic projective linear geometry J and an a_k-definable fully embedded basic affine space (V, A) such that $a_i \in A$ and (V, A) has projectivisation J (i.e. there is an a_k-definable bijection between $P(V)$ and $(J : a_k)$).*

There is also an orientation condition on quadratic geometries where they occur in this construction. We do not discuss it here.

Definition 6.3.6 *A structure M is* Lie coordinatisable *if it is bi-interpretable with a Lie coordinatised structure as described above. (In fact the class of Lie coordinatisable structures is closed under interpretations except for possibly problems caused by the orientation conditions on the quadratic geometries mentioned above.)*

As mentioned at the beginning of the chapter we have the following theorem, due to Hrushovski:

Theorem 6.3.7 *The class of lie coordinatisable structures is exactly the class of smoothly approximable structures.*

Smoothly approximable / lie coordinatisable structures are all structures with simple theories. We outline the proof. We first need the following definition from [30].

Definition 6.3.8 *A type p (in any theory T) is* simple *if given any extension q of p, there is a set A of cardinality $\leq |T|$ such that q does not divide over A.*

Definition 6.3.9 *Let T be a theory and let $\mathcal{P}$ be a class of types closed under automorphism. T is* coordinatised *by $\mathcal{P}$ if for every a there is $n \in \omega$ and a sequence $a_0, \ldots, a_{n-1}, a_n = a$ such that for each $i \leq n$, $tp(a_i/a_{i-1}) \in \mathcal{P}$. (a_{-1} is defined to be $\emptyset$.) The sequence $(a_0, \ldots, a_n)$ is called a* coordinatising sequence.

Proposition 6.3.10 *[30] Let T be a theory. Then if T is coordinatised by simple types T is a simple theory.*

Equivalently (according to [19]) T is *coordinatised by simple structures* if for each $a \in \tilde{M}$ there is $n \in \omega$ and a sequence $a_0, \ldots, a_{n-1}, a_n = a$ such that for each $i \leq n$, $a_{i-1} \in acl(a_i)$ or there is a simple structure J_i fully embedded in $\tilde{M}$ (over a_{i-1}) such that $a_i \in J_i$. Then, analogously to Proposition 6.3.10, if T is coordinatised by simple structures then T is simple. Therefore, by definition, a lie coordinatised structure is coordinatised by simple structures (since a basic lie geometry is simple) and is therefore simple.

Chapter 7

Choosing a Language

7.1 Problems with Infinite Fields

We now wish to study the model theory of vector spaces with bilinear forms over possibly infinite fields. In order to do this it is necessary to fix some language to work in, and, if the underlying field is allowed to be infinite, this choice is far from obvious. In the case when the field is finite, as described in the previous section, it makes little difference whether we consider the field elements as part of the language or as a separate sort, and in the latter case we can add the field to the language by adding finitely many constants for elements of the field sort.

If possible, in the case of an infinite field, we would like to avoid using what is perhaps the most natural formulation: a two sorted language with separate sorts for vector and field elements, with functions for vector addition, field addition and multiplication, scalar multiplication and for the bilinear form. The reason for this is that such a structure would contain at least the complete theory of the underlying field and so any properties of the structure (e.g. quantifier elimination, simplicity etc.) would be at least as complicated as those of the field. In fact we will eventually show that in some sense it is impossible to avoid using this full two sorted formulation. (See theorem 7.4.2.)

Consider the usual language, L, for vector spaces over a (fixed) field K. L is a one sorted language whose variables are vector elements and there are functions for vector addition, and for each $\alpha \in K$ there is a unary function $f_\alpha : V \to V$ corresponding to multiplication of vector elements by the element $\alpha \in K$. The complete theory of K-vector spaces in this language is strongly minimal, modular, and has elimination of quantifiers. In particular it does not interpret the underlying field K. This is clearly a much more basic structure in a model theoretic sense than the same vector space considered as a two sorted structure with a separate sort for the field with functions for field addition and multiplication. We can try to look at a vector space with a bilinear form as a one-sorted structure as follows:

Definition 7.1.1 *Let K be a field and let L_0 be the usual one-sorted language of K-vector spaces. For each $\alpha \in K$ let $R_\alpha(x, y)$ be a new binary relation and let $L = L_0 \cup \{R_\alpha(x, y) : \alpha \in K\}$.*

Suppose $M = (V, K, \Phi)$ is a K-vector space with a bilinear form. Then M can be considered as a L-structure by interpreting $R_\alpha(x, y) \iff \Phi(x, y) = \alpha$. Let T be the L-theory of M. Then, if K is finite, the models of T are all K-vector spaces with bilinear forms. However, if K is infinite (and so the range of Φ is infinite) then the set of sentences $\Gamma = \{\neg R_\alpha(x, y) : \alpha \in K\}$ is finitely satisfiable, and hence by compactness is consistent. Thus T has a model with elements x and y not in the domain of the (partial) function defined by the R_α's. So, although the R_α's define a K-bilinear form on the original model M, there are some models of $T = Th(M)$ on which they do not define a K-bilinear form. Note however, that all models of T are K-vector spaces.

The reason that this attempted axiomatization of K-bilinear forms fails is that compactness prevents us from axiomatizing, in a first order way, that '$\Phi : V \times V \to K$ is a total function', when K is infinite. So there will be some models in which $[\ ,\ ]$ is not a total function. There are two possible solutions to this problem:

1. **Use an infinitary language.** Instead of using the usual first order language $L_{\omega,\omega}$ we use the language $L_{\infty,\omega}$ which allows conjunctions and disjunctions of arbitrary (infinite) length, however a formula can still only have finitely many quantifiers. (See [31] pages 25-29 for an explanation of infinitary languages.) This approach is used in [29] to study vector spaces with bilinear (sesquilinear) forms over a fixed field K. The advantage is that it is then possible to axiomatize the statement:

$$\forall x\,,\forall y\ \bigvee_{\alpha\in K}[x,y]=\alpha$$

   expressing that $[\,,\,] : V \times V \to K$ is a total function. However a heavy price is paid for this extra expressibility. The compactness theorem, and hence many other familiar properties of first order logic, fail to hold in the language $L_{\infty,\omega}$. Because of this we do not pursue this approach further.

2. **Use a two-sorted language.** If we allow our language to have separate sorts for the vector and field elements it is then possible to write down formulas such as:

$$\forall x\,\forall y\,\exists\alpha\,[x,y]=\alpha$$

   without having to resort to using infinitary formulas. We then have a set-up where there are (at least) two sorts, called V (vectors) and K (scalars), a function $[\,,\,] : V \times V \to K$, and possibly other structure on the sorts V and K such as vector addition, field multiplication, scalar multiplication, etc. The problem that arises here is that it is impossible to have much control over the sort K without defining a lot of structure on K.

7.2 The Two-Sorted Vector Space Structure

Following the approach described above in (2) we define the following set-up:

Definition 7.2.1 *Fix a field K. Let L be the two sorted language with sorts V and W. L has binary functions $+_V : V \times V \to V$ and $+_W : W \times W \to W$, unary functions $f_\alpha : V \to V$ and $g_\alpha : W \to W$ for each $\alpha \in K$, constants 0_V and 0_W and a binary function $[\ ,\ ] : V \times V \to W$.*

Let T be the L-theory expressing that V and W are K-vector spaces, together with axioms expressing that $[\ ,\ ] : V \times V \to W$ is a non-degenerate bilinear map.

The idea is that the intended model of T is represented by the case where W is one-dimensional over K, and hence is essentially a copy of K , but without the full power of field multiplication defined on it. However, by compactness, there will be models of T in which the bilinear form takes values not restricted to a one dimensional subspace of W.

Proposition 7.2.2 *Let L and T be as described in Definition 7.2.1. Let M be a model of T such that W is one-dimensional over K. (so we can think of W as a copy of K.) Then field multiplication on K and scalar multiplication from $K \times V \to V$ is first-order definable in M.*

Proof. Consider the formula $\phi(x_1, x_2) := \exists y\, ([x_1, y] = 0 \wedge [x_2, y] \neq 0)$. Then $M \vDash \phi(a, b)$ if and only if the set $\{a, b\}$ is linearly independent or $a = 0$. Thus $M \vDash \phi(x_1, x_2) \wedge x_1 \neq 0$ if and only if $\{x_1, x_2\}$ is linearly independent. Consider the formula:

$$\psi(x_1, x_2, u) := \exists y\, ([x_1, y] = 1 \wedge [x_2, y] = u)$$

where x_1 and x_2 represent V-sort elements and u is a W-sort element. Then $M \vDash \psi(a, b, \alpha)$ if and only if a and b are linearly independent or $b = \alpha a$. (e.g. by lemma 5.2.3) Then by putting

$$\theta(x_1, x_2, u) := \psi(x_1, x_2, u) \wedge (x_1 = 0 \vee \neg\phi(x_1, x_2))$$

we have $M \vDash \theta(a, b, \alpha)$ if and only if $b = \alpha a$. Thus scalar multiplication is definable.

Clearly the formula $u(vx_1) = x_2$ (where $x_1, x_2 \in V$ and $u, v \in W$) is definable by using the formula θ. Then by defining:

$$\chi(u, v, w) := \forall x \, \forall y \, (u(vx) = y \leftrightarrow wx = y)$$

where $x_1, x_2 \in V$ and $u, v, w \in W$. Then $M \models \chi(\alpha, \beta, \gamma) \iff \alpha\beta = \gamma$ so field multiplication is definable. ■

The above proposition shows that as it is possible to define field multiplication on the W-sort of some model of T it is possible to do so on all models of T and hence the W-sort always carries the structure of a field extending K and elementarily equivalent to K. There is also a scalar multiplication function from the *field* W to the vector space V in all models of T. It follows that the V has the structure of a W-vector space and the bilinear map $[\ ,\ ] : V \times V \to W$ is a W-bilinear form on V.

Thus T axiomatizes the entire two sorted theory of vector spaces over fields elementarily equivalent to (and extending) K, with bilinear forms from $V \times V$ to the field extension of K. That is, T is the entire two sorted theory we were trying to avoid axiomatizing. In the next section we show that this is no coincidence: given almost any language, L, and any L-structure which is a vector space with bilinear form then the theory of that structure encodes the entire two-sorted theory as above.

7.3 Vector Spaces with Orthogonality Relations

In this section we start with a very basic language for discussing vector spaces with bilinear forms and prove that if there is a structure in this language which is a vector space with bilinear form over some given field K, then the theory of this structure contains the entire two-sorted theory of vector spaces with bilinear forms, as described above, and in particular contains the complete theory of the field K.

It is known (see e.g. [37]) that a non-degenerate bilinear (or even sesquilinear) form can be recovered (up to scalar multiples) by its orthogonality relation.

Proposition 7.3.1 *Let K be a division ring and V a (left) K-vector space of dimension > 2. Consider the following properties of a binary relation $R(x,y)$ on V :*

1. $R(x,y) \to R(y,x)$
2. $R(x,y) \to R(\lambda x, y) \quad$ *for all* $\lambda \in K$
3. $R(x,y) \wedge R(x,z) \to R(x, y+z)$
4. $(\forall x\, R(x,y) = 0) \to y = 0$
5. *For all $x \in V$ and for all planes (two-dimensional subspaces) $Y \subseteq V$, there is $y \in Y \backslash \{0\}$ such that $R(x,y)$ holds.*

Clearly if $\Phi : V \times V \to K$ is a sesquilinear form then the relation given by $R(x,y) \iff \Phi(x,y) = 0$ satisfies (1)-(5).

Conversely if $R(x,y)$ is a relation on V satisfying (1)-(5), then there is an antiautomorphism ν of K, $\varepsilon \in K\backslash\{0\}$, and a non-degenerate sesquilinear form, Φ, relative to (K,ν,ε) such that $\Phi(x,y) = 0 \iff R(x,y)$. (We say that Φ is compatible *with R.) Moreover the form Φ is unique up to scalar multiples. That is, if Ψ is another sesquilinear form compatible with R, then there is $\mu \in K$ such that $\forall x \forall y\, \Psi(x,y) = \mu\, \Phi(x,y)$. (The form $\mu\Phi$ is ε_1-hermitian with respect to the antiautomorphism $\mu^{-1}\nu\mu$ and $\varepsilon_1 = \varepsilon\, \nu(\mu^{-1})\mu$.) See [37] for a proof.*

Definition 7.3.2 *Let L be a one sorted language with a binary function, $+$, and a binary relation $R(x,y)$. Let K be a (skew)field, V a K-vector space of dimension ¿2, and $\Phi : V \times V \to K$ a non-degenerate sesquilinear form. Let M_0*

be the L-structure with domain V, with $+$ interpreted as vector addition and $R(x,y)$ interpreted as the orthogonality relation induced by Φ (i.e. $R(x,y) \iff \Phi(x,y) = 0$ on M_0).

Let T be the complete first-order L-theory of M_0.

We will eventually prove, amongst other things, that T contains the complete theory of the field K. Although all of the following works with a general sesquilinear form over a division ring, in order to simplify the arguments, we assume that K is a field and $R(x,y)$ is the orthogonality relation induced by a bilinear form. Note that any model of T is an abelian group under $+$, and we may as well include the constant 0 for the identity element. However, since K is not part of the language, a model of T is not necessarily a K-vector space.

Definition 7.3.3 *For $n \in \omega$ let χ_n be the L-formula defined by:*

$$\chi_n(x_1, \ldots, x_n, y) := \forall z \left(\bigwedge_{i=1}^{n} R(x_i, z) \to R(y, z) \right).$$

Lemma 7.3.4 *For $n \in \omega$ and $a_1, \ldots, a_n, b \in M_0$ we have:*

$$M_0 \models \chi_n(a_1, \ldots, a_n, b) \iff b \in K\text{-}span\,(a_1, \ldots, a_n).$$

Proof. ($\Leftarrow$) Suppose $a_1, \ldots, a_n, b \in M_0$ and $b \in K$-span $(a_1, \ldots, a_n)$. Then there are $\lambda_1, \ldots, \lambda_n \in K$ such that $b = \lambda_1 a_1 + \cdots + \lambda_n a_n$. Then given $z \in M_0$, if $R(a_i, z)$ holds for $i = 1, \ldots, n$, then by linearity of R we have: $R(\lambda_1 a_1 + \cdots + \lambda_n a_n, z)$, i.e. $R(b, z)$.

($\Rightarrow$) Suppose $a_1, \ldots, a_n, b \in M_0$ and $M_0 \models \chi_n(a_1, \ldots, a_n, b)$, i.e. $M_0 \models \forall z \, (\bigwedge_{i=1}^{n} R(a_i, z) \to R(b, z))$. Let $A = K$-span $(a_1, \ldots, a_n)$ and $B = K$-span (b). Then $M_0 \models \bigwedge_{i=1}^{n} R(a_i, z) \iff z \in A^{\perp}$ and $M_0 \models R(b, z) \iff z \in B^{\perp}$. Therefore $A^{\perp} \subseteq B^{\perp}$ and hence $B^{\perp\perp} \subseteq A^{\perp\perp}$. But any finite dimensional subspace of a vector space is equal to its bi-orthogonal. Thus $B = B^{\perp\perp} \subseteq A^{\perp\perp} = A$, i.e. $B \subseteq A$. In particular $b \in A = K$-span $(a_1, \ldots, a_n)$. ■

By using the formulas χ_n we can define a generalised independence relation, on any model $M \models T$, which behaves like linear independence with respect to some, as yet unspecified, field.

Definition 7.3.5 *For $M \models T$ define a closure operation on M by: for $A \subseteq M$ and $c \in A$ then $c \in cl(A)$ if there is $n \in \omega$ and $a_1, \ldots, a_n \in A$ such that $M \models \chi_n(a_1, \ldots, a_n, c)$.*

It is easy to prove that $cl(-)$ really does define a closure operation on any $M \models T$. That is:

- $A \subseteq cl(A)$
- $cl(cl(A)) = cl(A)$
- The Exchange Property: $c \in cl(A \cup \{b\}) \backslash cl(A) \Rightarrow b \in cl(A \cup \{c\})$
- Finite Character: If $c \in cl(A)$ then there is a finite $A_0 \subseteq A$ such that $c \in cl(A_0)$

The proof follows from the fact that $cl(-)$ defines a closure operation on M_0 (since it defines linear independence on M_0) and then axiomatizing that it is a closure operation. E.g. for the exchange axiom we have that for each $n \in \omega$:

$$\begin{aligned} M_0 \models \forall x_1, \ldots, \forall x_n \forall y \forall z \qquad & (\chi_{n+1}(x_1, \ldots, x_n, y, z) \wedge \neg\chi_n(x_1, \ldots, x_n, z) \\ \rightarrow \quad & \chi_{n+1}(x_1, \ldots, x_n, z, y)). \end{aligned}$$

Hence T proves this formula also, and so the exchange axiom holds for $cl(-)$ in any model of T.

Note that if $M \models T$ and $A = cl(A)$ is any closed set, then A is a subgroup of the abelian group $(M, +)$. Of course, in M_0, the cl-closed sets are exactly the K-linear subspaces. We will eventually prove that $cl(-)$ defines linear span on any model of T, but with respect to a field, possibly other than K.

Definition 7.3.6 *Let $M \models T$. Let E be the relation on $M \setminus \{0\}$ defined by $E(x,y) \iff \chi_1(x,y)$. Then clearly E is an equivalence relation. For $a \in M$ let $E_a := \{x \in M \mid E(x,a)\} \cup \{0\}$. Thus, on M_0 the set E_a is just the K-linear span of the element a. In any model $M \models T$ a set of the form E_a is a subgroup of $(M,+)$.*

The following easy lemma, which is essentially a rephrasing of (5) of Proposition 7.3.1, is the vital ingredient for defining multiplication functions on any model of T.

Lemma 7.3.7 *Let x,u,v be any three elements in a non-degenerate sesquilinear space. Then either $x \perp v$ or there is a unique $v' \in K$-span(v) such that $x \perp u+v'$.*

Proof. Suppose that $x \not\perp v$. Then either u and v are linearly dependent, in which case $x \perp u+v'$ if and only if $v' = -u$, or $\{u,v\}$ is a linearly independent set, in which case it defines a plane, $Y = \{\lambda u + \mu v \mid \lambda, \mu \in K\}$. By (5) of proposition 7.3.1 there is some $y \in Y \setminus \{0\}$ such that $x \perp y$. Let $y = \lambda u + \mu v$. Then as $x \not\perp v$ it follows that $\lambda \neq 0$. Thus $x \perp u + \frac{\mu}{\lambda} v$. It follows that $\frac{\mu}{\lambda}$ is unique, since if $x \perp u + \eta_1 v$ and $x \perp u + \eta_2 v$ then $x \perp (\eta_1 - \eta_2) v$ which implies $\eta_1 = \eta_2$. Put $v' = \frac{\mu}{\lambda} v$. ■

Corollary 7.3.8 *The following is an axiom of T :*

$$\forall x \, \forall u \, \forall v \; [R(x,v) \vee \exists! v' (\chi_1(v,v') \wedge R(x, u+v'))] \, .$$

Proof. This is simply by using the fact that $M_0 \models \chi_1(a,b) \iff b \in K$-span (a) and $T = Th(M_0)$. Note that the formula $\chi_1(v,v')$ is equivalent to the statement: $v' \in E_v$. ■

Corollary 7.3.8 enables us to construct a partial ternary function on any model of T which will eventually allow us to construct field and scalar multiplication functions on all models of T.

Definition 7.3.9 *Let $M \models T$. Define a partial function $\gamma : M^3 \to M$ (on all triples (x, u, v) such that $\neg R(x, v)$) by: for $x, u, v \in M$ with $\neg R(x, v)$, $\gamma(x, u, v)$ is the (necessarily unique) $v' \in E_v$ such that $R(x, u + v')$.*

The important point is that γ is a function definable in the language L:

$$\gamma(x, u, v) = w \iff \neg R(x, v) \wedge \chi_1(v, w) \wedge R(x, u + w).$$

Then, given any $x \in M$, let $X = M \backslash \{y : R(x, y)\}$ and define $\gamma_x : M \times X \to M$ by: $\gamma_x(u, v) = \gamma(x, u, v)$. (Then each function γ_x is definable in L with the parameter x.) Note that in M_0 , $x \perp u + \lambda v \iff [u + \lambda v, x] = 0 \iff [u, x] = -[\lambda v, x] \iff [u, x] = -\lambda[v, x] \iff \lambda = \frac{-[u,x]}{[v,x]}$. Thus, in M_0 : $\gamma_x(u, v) = \frac{-[u,x]}{[v,x]} v$.

Lemma 7.3.10 *The function γ_x satisfies the following on all models of T :*

1. $\gamma_x(u_1 + u_2, v) = \gamma_x(u_1, v) + \gamma_x(u_2, v)$
2. $E(u, u') \to E(\gamma_x(u, v), \gamma_x(u', v))$
3. $E(v, v') \to \gamma_x(u, v) = \gamma_x(u, v')$
4. $E(x, y) \to \gamma_x(u, v) = \gamma_y(u, v)$
5. $\gamma_x(u, v) = 0 \iff R(u, x)$
6. $E(u, u') \to \gamma_x(u, u') = -u$
7. $\gamma_x(\gamma_x(u, v), w) = -\gamma_x(u, w)$

Proof. By the characterisation of $\gamma_x(u, v)$ in M_0 it is clear that (1)-(7) hold in M_0. Since they are all expressible in L they must hold in all models $M \models Th(M_0)$. ∎

Lemma 7.3.11 *Let $M \vDash T$. Let $u, v \in M\backslash\{0\}$ such that $M \vDash \neg E(u,v)$. Let $\Gamma(u,v) := \{x : \neg R(x,u) \wedge \neg R(x,v) \wedge R(x, u+v)\}$. Then:*

1. $\Gamma(u,v) \neq \emptyset$.

2. $\forall u' \in E_u \, \forall x, y \in \Gamma(u,v) \;\; \gamma_x(u',v) = \gamma_y(u',v)$.

Proof. (1) Suppose $\forall x\,(\; R(x,u+v) \to R(x,u) \vee R(x,v)\,)$. Then if both $R(x,u+v)$ and $R(x,u)$ hold, by additivity of R we have $R(x,v)$. Thus by symmetry $R(x,u+v) \to R(x,u) \wedge R(x,v)$. This implies $E(u,v)$ which contradicts the choice of u and v.

(2) By definition of $\Gamma(u,v)$, $\gamma_x(u,v) = v = \gamma_y(u,v)$ for all $x, y \in \Gamma(u,v)$. Working in M_0, if $u' \in E_u$ then $u' = \lambda u$ for some $\lambda \in K$. Then by the characterisation of γ_x in M_0 it follows that $\gamma_x(u',v) = \gamma_x(\lambda u, v) = \lambda\gamma_x(u,v) = \lambda v = \lambda\gamma_y(u,v) = \gamma_y(u',v)$. Thus (2) holds in M_0 and hence in all models of T. ■

Definition 7.3.12 *Let $M \vDash T$. Let $u, v \in M\backslash\{0\}$ such that $M \vDash \neg E(u,v)$. Define $\delta_{u,v} : E_u \to E_v$ by:*

$$\textit{For } u' \in E_u \, : \, \delta_{u,v}(u') = \gamma_x(u',v) \textit{ where } x \in \Gamma(u,v).$$

Lemma 7.3.11 implies that there is some $x \in \Gamma(u,v)$ and that $\delta_{u,v}$ is well defined.

Since $\delta_{u,v}(\lambda u) = \lambda v$ in M_0, it is easy to see that, in any model of T, $\delta_{u,v}$ is a definable bijection between E_u and E_v. In fact, as E_u and E_v both have abelian group structures it follows that $\delta_{u,v}$ is an Abelian group isomorphism.

Definition 7.3.13 *Let $M \vDash T$ and let $v \in M\backslash\{0\}$. Define a function σ_v : $E_v \times M \to M$ by*

$$\textit{For } v' \in E_v \, : \, \sigma_v(v',u) = \delta_{w,u} \circ \delta_{v,w}\,(v') \quad \textit{where } \nvDash \chi_2(u,v,w).$$

In M_0, $\sigma_v(\lambda v, u) = \lambda u$ (see e.g. 7.3.10) and hence σ_v is well defined in M_0 and therefore well defined (independent of the choice of w) in any model of T. Note that this definition requires the original assumption that $\dim(M_0) \geq 3$.

Remark 7.3.14 *Instead of defining the functions σ_v for each $v \in M \backslash \{0\}$ we could define a function $\sigma : M^3 \to M$ defined on triples (v, w, u) whenever $v \neq 0$ and $w \in E_v$. Then given any $M \models T$, any $v \in M \backslash \{0\}$, and any $w \in E_v$, $\sigma(v, w, u) = \sigma_v(w, u)$. The advantage of this is that σ is definable without parameters whereas σ_v is defined using the parameter v and therefore only makes sense in a model of T, containing the element v.*

Definition 7.3.15 *Let $M \models T$ and let $v \in M \backslash \{0\}$. Define $\tau_v : E_v \times E_v \to E_v$ by:*

$$\tau_v := \sigma_v \restriction E_v \times E_v.$$

Note that the functions $\delta_{v,u}$, σ_v , τ_v are all definable in the language L.

Theorem 7.3.16 *Let $M \models T$. Let $v \in M \backslash \{0\}$. Then:*

1. *$(E_v, \tau_v, +)$ is a field with multiplicative identity v and additive identity 0.*

2. *If $w \in M \backslash \{0\}$, then the fields $(E_w, \tau_w, +)$ and $(E_v, \tau_v, +)$ are definably isomorphic (via the isomorphism $\sigma^*_{w,v}$ defined for $w' \in E_w$ by $\sigma^*_{w,v}(w') = \sigma_w(w', v)$)*

3. *The field $(E_v, \tau_v, +)$ is elementarily equivalent (in the language of fields) to the field K.*

4. *M has the structure of an $(E_v, \tau_v, +)$-vector space, with scalar multiplication function $\sigma_v : E_v \times M \to M$. Then, for $n \in \omega$ and $a_1, \ldots, a_n, b \in M$ we have: $M \models \chi_n(a_1, \ldots, a_n, b) \iff b \in E_v$-span$(a_1, \ldots, a_n)$ and in particular the closure operation $cl(-)$ defines E_v-linear span on M.*

Proof. (1) In M_0, if $v_1, v_2, v_3 \in E_v$, then there are $\lambda, \mu, \nu \in K$ such that $v_1 = \lambda v$, $v_2 = \mu v$ and $v_3 = \nu v$. Then in M_0, $\tau_v(\tau_v(v_1, v_2), v_3) = \tau_v(\tau_v(\lambda v, \mu v), \nu v) = \tau_v(\lambda\mu v, \nu v) = (\lambda\mu)\nu v = \lambda(\mu\nu)v = \tau_v(\lambda v, \mu\nu v) = \tau_v(\lambda v, \tau_v(\mu v, \nu v)) = \tau_v(v_1, \tau_v(v_2, v_3))$. Thus $M_0 \vDash \forall v_1 \forall v_2 \forall v_3 [\tau_v(\tau_v(v_1, v_2), v_3) = \tau_v(v_1, \tau_v(v_2, v_3))]$. Hence $T \vDash$'τ_v is associative '. Similarly T proves the rest of the field axioms for $(E_v, \tau_v, +)$ by showing that $(E_v, \tau_v, +)$ is a field in M_0 with identities v and 0.

(2) Let $v, w \in M\backslash\{0\}$. Then by working in M_0, it follows that $\sigma^*_{w,v}$ is an isomorphism between the fields $(E_w, \tau_w, +)$ and $(E_v, \tau_v, +)$. This fact can be expressed in L. Hence $T \vDash$'$\sigma^*_{w,v} : E_w \to E_v$ is an isomorphism of fields'.

(3) Let $(K, \cdot, +)$ be the base field of M_0 and $u \in M_0\backslash\{0\}$. Then, working in M_0, the fields $(K, \cdot, +)$ and $(E_u, \tau_u, +)$ are isomorphic. Suppose $M \vDash T$ and $v \in M\backslash\{0\}$. Then as $M \equiv M_0$ the definable substructures $(E_v, \tau_v, +)$ of M and $(E_u, \tau_u, +)$ of M_0 must be elementarily equivalent. Therefore $(E_v, \tau_v, +) \equiv (K, \cdot, +)$.

(4) Again, by working in M_0, $\sigma_v : E_v \times M_0 \to M_0$ is simply the function taking $(\lambda v, m)$ to λm. Thus M_0 proves that σ_v is a faithful action of the field $(E_v, \tau_v, +)$ on the Abelian group $(M_0, +)$. This is expressible in L. Hence T proves that the function σ_v is a scalar multiplication function from $(E_v, \tau_v, +)$ to M. Thus M has the structure of an $(E_v, \tau_v, +)$-vector space. The rest follows immediately. ■

Thus given any model $M \vDash T$ and any $v \in M\backslash\{0\}$ it is possible to construct a definable field $(E_v, \tau_v, +)$ and a definable scalar multiplication of E_v on M such that M has the structure of an E_v-vector space. The relation R then satisfies conditions (1)-(5) of 7.3.1 with respect to the new vector space structure on M. Then proposition 7.3.1 says that there is a unique (up to scalar multiples) sesquilinear form $\Phi : M \times M \to E_v$ compatible with the relation R. Hence we can recover the sesquilinear form as well as the vector space structure. This can be done definably as follows: (To simplify the proof we assume that R is the orthogonality relation induced by a bilinear form.)

Proposition 7.3.17 *There is an L-definable function $\Omega : M^5 \to M$, defined on quintuples (v,a,b,c,d) whenever $v \neq 0$ and $R(a,b) \vee \neg R(c,d)$, such that if Ω_v is the function given by: $\Omega_v(a,b,c,d) = \Omega(v,a,b,c,d)$, then $Ran(\Omega_v) = E_v$ and for any $M \models T$, and any $v \in M\backslash\{0\}$ the function $\Omega_v : M^4 \to E_v$ has the following property:*

Given any *function $\Psi : M \times M \to E_v$, then Ψ is a sesquilinear form on M (with M considered as an E_v-vector space) compatible with the relation R if and only if, for all $a,b,c,d \in M$ (with $\neg R(c,d)$) $\frac{\Psi(a,b)}{\Psi(c,d)} = \Omega_v(a,b,c,d)$.*

Proof. To begin with we work in M_0. Let $v \in M_0\backslash\{0\}$. We show how to define Ω_v on a quadruple (a,b,c,d) from M_0 with $R(a,b) \vee \neg R(c,d)$. If $R(a,b)$ put $\Omega_v(a,b,c,d) = 0$. So assume $\neg R(a,b)$. Then there is some $e \in M_0$ such that $\neg R(e,b) \wedge \neg R(e,d)$. Let $\Phi : M_0 \times M_0 \to K$ be the original bilinear form on M_0. Then, for any such e, $\frac{\Phi(a,b)}{\Phi(c,d)} = \frac{\Phi(a,b)}{\Phi(e,b)} \times \frac{\Phi(e,b)}{\Phi(e,d)} \times \frac{\Phi(e,d)}{\Phi(c,d)}$. Hence $\frac{\Phi(a,b)}{\Phi(c,d)} = \alpha \iff \exists \beta, \gamma, \delta$ such that $\frac{\Phi(a,b)}{\Phi(e,b)} = \beta$, $\frac{\Phi(e,b)}{\Phi(e,d)} = \gamma$, $\frac{\Phi(e,d)}{\Phi(c,d)} = \delta$ and $\beta\gamma\delta = \alpha$. We can then axiomatize $\frac{\Phi(a,b)}{\Phi(e,b)} = \beta$ in L as follows: $\frac{\Phi(a,b)}{\Phi(e,b)} = \beta \iff \Phi(a,b) = \beta\,\Phi(e,b) \iff \Phi(a - \beta e, b) = 0 \iff R(a - \beta e, b) \iff R(a - \sigma_v(\beta v, e)\,,\, b)$. Similarly $\frac{\Phi(e,b)}{\Phi(e,d)} = \gamma \iff R(e,\, b - \sigma_v(\gamma v, d))$ and $\frac{\Phi(e,d)}{\Phi(c,d)} = \delta \iff R(e - \sigma_v(\delta v, c)\,, d\,)$.

Then let $\phi(a,b,c,d,\alpha)$ be the formula:

$$\begin{aligned}&\exists e[\neg R(e,b) \wedge \neg R(e,d)] \wedge (\forall e\,[\neg R(e,b) \wedge \neg R(e,d)] \to \\ &\qquad \forall \beta,\gamma,\delta \in E_v\,[\,R(a - \sigma_v(\beta v, e)\,,\, b) \wedge R(e,\, b - \sigma_v(\gamma v, d)) \\ &\qquad\qquad \wedge R(e - \sigma_v(\delta v, c)\,, d\,) \to \tau_v(\tau_v(\beta,\gamma),\delta) = \alpha])\end{aligned}$$

Then $M_0 \models \forall\, a,b,c,d\ [\,\neg R(a,b) \wedge \neg R(c,d) \ \to \exists! \alpha\, \phi(a,b,c,d,\alpha)\,]$, and thus T proves this sentence also. Let Ω_v be the function defined on M_0 (and hence also defined on all models $M \vDash T$), by:

$$\Omega_v(a,b,c,d) = \begin{cases} \bullet \text{ undefined if } \neg R(a,b) \wedge R(c,d) \\ \bullet \ 0 \qquad\qquad \text{if } R(a,b) \\ \bullet \text{ the unique } \alpha \in E_v \text{ such that } M \vDash \phi(a,b,c,d,\alpha) \\ \qquad\qquad\quad \text{if } \neg R(a,b) \wedge \neg R(c,d)\,. \end{cases}$$

Clearly, in M_0, $\Omega_v(a,b,c,d) = \frac{\Phi(a,b)}{\Phi(c,d)}$. If M is any model of T and $\Phi : M \times M \to E_v$ is some E_v-bilinear form on M, compatible with R, then for $a,b,c,d \in M$ with $\neg R(a,b) \wedge \neg R(c,d)$ it is clear by the definition of ϕ that $\frac{\Phi(a,b)}{\Phi(c,d)} = \alpha \iff M \vDash \phi(a,b,c,d,\alpha)$ and hence that $\Omega_v(a,b,c,d) = \frac{\Phi(a,b)}{\Phi(c,d)}$ on M.

Since the functions σ_v are uniformly definable we can define $\Omega : M_0^5 \to M_0$ by $\Omega(v,a,b,c,d) = \Omega_v(a,b,c,d)$. Then since this is a function compatible with R on M_0 then Ω is a function on all models $M \vDash T$ and is compatible with R.

Let $M \vDash T$ and $v \in M \backslash \{0\}$. Then by proposition 7.3.1 there is some E_v-bilinear form $\Phi : M \times M \to E_v$ compatible with the relation R. By the definition of the function Ω_v (since Ω_v is compatible with R) we must have that for all $a,b,c,d \in M$ $\frac{\Phi(a,b)}{\Phi(c,d)} = \Omega_v(a,b,c,d)$. Now suppose that $\Psi : M \times M \to E_v$ is any binary function. If $\frac{\Psi(a,b)}{\Psi(c,d)} = \Omega_v(a,b,c,d)$ for all $a,b,c,d \in M$ then $\frac{\Psi(a,b)}{\Psi(c,d)} = \frac{\Phi(a,b)}{\Phi(c,d)}$ which implies there is $\mu \in E_v$ such that $\Psi(a,b) = \mu\,\Phi(a,b)$ all $a,b \in M$ and hence $\Psi : M \times M \to E_v$ is the bilinear form $\mu\,\Phi$ which must be compatible with R. Conversely, if $\Psi : M \times M \to E_v$ is a bilinear form compatible with R then by proposition 7.3.1 there is $\mu \in E_v$ such that $\Psi = \mu\,\Phi$ and so $\frac{\Psi(a,b)}{\Psi(c,d)} = \frac{\Phi(a,b)}{\Phi(c,d)} = \Omega_v(a,b,c,d)$ for all $a,b,c,d \in M$.

Suppose that we have fixed $v \in M$ to obtain a field E_v. Then, since v can be considered as both an element of M and of the field E_v, we can choose (by non-degeneracy of $\perp$) some $u \in M$ such that $\neg R(u,v)$. We can then define a distinguished bilinear form, Φ_u, on M, by $\Phi_u(a,b) = \Omega_v(a,b,u,v)$. Then Φ_u is the unique E_v-bilinear form compatible with R such that $\Phi_u(u,v) = v$. In other words, by fixing two non-zero, non-orthogonal, constants u and v , we fix a distinguished copy of the field and a unique bilinear form $\Phi_{u,v}$ compatible with R (such that $\Phi_{u,v}(u,v) = v$).

However, the choice of v and u is arbitrary, and any non-zero $v \in M$ could be chosen to give an equally valid (and different) copy of the underlying field of M and scalar multiplication function $\sigma_v : E_v \times M \to M$. In the structure M_0, the function $\sigma_v : E_v \times M \to M$ is just the map $(\lambda v, u) \mapsto \lambda u$. Thus, in M_0 and hence in all models of T, these scalar multiplication functions are related by:

$$\sigma_w(w', u) = \sigma_v(\sigma_w(w', v), u) \quad \text{where } w' \in E_w.$$

Therefore, if we know the E_v-scalar multiplication function σ_v , and the isomorphism $\sigma^*_{w,v}$ between E_w and E_v , then we know the full E_w-scalar multiplication function σ_w.

By Theorem 7.3.16, given any $M \vDash T$, there is a field, F, defined up to isomorphism, such that M has the structure of an F-vector space. This structure is induced via any of the scalar multiplications functions $\sigma_v : E_v \times M \to M$, and each field $(E_v, \tau_v, +)$ is isomorphic to F. If we fix some non-zero $v \in M$ and some isomorphism $\phi_v : F \to V$, then there is a scalar multiplication function $\rho_v : F \times M \to M$, given by $\rho_v(\alpha, m) = \sigma_v(\phi_v(\alpha), m)$. This then induces maps $\rho_w : F \times M \to M$ defined by $\rho_w(\alpha, m) = \sigma_w(\sigma_v(\phi(\alpha), w), m)$ and hence $\rho_w(\alpha, m) = \sigma_w(\phi_w(\alpha), m)$ where $\phi_w : F \times M \to M$ is given by $\phi_w(\alpha) = \sigma_v(\phi_v(\alpha), w)$. Thus, an isomorphism between F and some E_v induces isomorphisms between F and each E_w for $w \in M \backslash \{0\}$.

We now show formally, that any model $M \vDash T$ interprets (and is interpretable in) a model of the full two-sorted theory of a vector space with bilinear form, and that if we add two non-orthogonal constants to M then the structures are bi-interpretable.

Definition 7.3.18 *Let L' be the two sorted language with sorts V (vectors), and K (scalars).*

L' has the following functions:

$+_V : V \times V \to V$ *(vector addition)*

$+_K : K \times K \to K$ *(field addition)*

$\circ_K : K \times K \to K$ *(field multiplication)*

$\gamma : K \times V \to V$ *(scalar multiplication)*

$[\ ,\ ] : V \times V \to K$ *(the bilinear form, also written $\Phi : V \times V \to K$)*

and constants 0_V, 0_K and 1_K.

Let L be the one sorted language $\{+, R\}$, defined in 7.3.2. Let (V, K, Φ) be a vector space V, over a field K, with a bilinear form Φ. Let M be the L-structure with domain V, with $+$ interpreted as vector addition and R interpreted as the orthogonality relation induced by the bilinear form Φ. Let T be the L-theory of M. Let N be the two-sorted L'-structure with domain $V \cup K$ and with all the L' functions and constants having their obvious interpretations. Let T' be the L'-theory of N.

Then, in some sense, T and T' are almost the same theory. All the structure of M is clearly definable in N and conversely, by theorem 7.3.16 and proposition 7.3.17 all the structure on N can be recovered in the apparently simpler structure M.

We know (by Theorem 7.3.16) that given $v \in M \backslash \{0\}$ and $u \in M$ such that $M \models \neg R(u, v)$, there are functions τ_v, σ_v, and $\Phi_{u,v}$ definable in $L \cup \{u, v\}$ such that the structures $N = (V, K, +_v, +_K, \circ_K, \gamma, [\ ,\ ])$ and $(M, E_v, +, + \restriction_{E_v}, \tau_v, \sigma_v, \Phi_{u,v})$ are isomorphic. Hence, if we add to both L and L', two new constant symbols u and v, interpreted in both T and T' as non-orthogonal elements, then the expanded structures M and N are bi-interpretable.

Define an equivalence relation $\sim$ on M^2 by:

$$(a,b) \sim (a',b') \iff \begin{cases} a = a' \text{ and } b, b' \notin E_v, \text{ or} \\ b = b' \in E_v \end{cases} .$$

Therefore $M^2/\sim$ is isomorphic to (M, E_v) via the map:

$$(a,b)/\sim \longmapsto \begin{cases} a & \text{if } b \notin E_v \\ b & \text{if } b \in E_v \end{cases}$$

Then the structures N and $M^2/\sim$ (with the additional constants u and v) are bi-definable. This says that the structures M and N (with the additional constants) are bi-interpretable. Thus the one-sorted theory T, really is equivalent (after adding two constants) to the seemingly more complicated two-sorted theory T'. That is, if T is any theory of bilinear forms, in a language strong enough to interpret orthogonality and vector addition then T contains the full two sorted theory of bilinear forms, T'.

If we do not add the two constant symbols then it is still the case that M is interpretable in N and M is interpretable in N but in this case there is no bi-interpretation.

7.4 Non-Simplicity

The original idea behind much of this work had been to try to find some language for discussing vector spaces with bilinear forms over infinite fields and then to prove that this theory is simple in that language. The previous section shows that if we want to use a language where vector addition and the orthogonality relation derived from the form are definable then we are forced into using the language L' described in definition 7.3.18. In particular the structure of the underlying field and the bilinear form are definable. Then clearly if K is field whose theory is non-simple then the theory of the structure (V, K, Φ) is also non-simple, as it contains the complete theory of K.

If the dimension of V over K is infinite then by Remark 6.2.6 the theory of the structure (V, K, Φ) is unstable. If we have also that K is infinite then we can prove that the theory of the structure (V, K, Φ) is not simple.

Proposition 7.4.1 *Let K be an infinite field, V an infinite dimensional K-vector space. Let $\Phi : V \times V \to K$ be a non-degenerate sesquilinear form. Let M be a structure with domain containing V and such that for some $\alpha \in K \backslash \{0\}$ the formula $\Phi(u, v) = \alpha$ is definable in M. Then $\Phi(u, v) = \alpha$ has the tree property and hence the theory $T = Th(M)$ is non-simple.*

Proof. Recall that a formula $\phi(x, y)$ has the *tree property* if there is $k \in \omega$ and tuples $(\bar{a}_\eta : \eta \in \omega^{<\omega})$ such that :

1. $\{\phi(x, \bar{a}_{\eta\hat{\ }r}) : r \in \omega\}$ is k-inconsistent for all $\eta \in \omega^{<\omega}$.

2. $\{\phi(x, \bar{a}_{\beta\restriction r}) : r \in \omega\}$ is consistent for all $\beta \in \omega^\omega$.

We show that for $\alpha \in K\backslash\{0\}$ the formula $\Phi(u, v) = \alpha$ has the tree property. Let $\{a_i : i \in \omega\}$ be a linearly independent set of elements of V (possible as V is infinite dimensional). Let $\{\lambda_i : i \in \omega\}$ be a set of distinct elements from K. For $\eta \in \omega^{<\omega}$, let $\eta = (k_1, \dots k_n)$ where $n = \lg(\eta)$ and $k_n = \eta(n)$. Put $a_\eta = \lambda_{k_n} a_n$. Let $\phi(x, a_\eta) :=$‘ $\Phi(x, \lambda_{k_n} a_n) = \alpha$’ .

1. Claim: For $\eta \in \omega^{<\omega}$ $\{\phi(x, a_{\eta\hat{\ }r}) : r \in \omega\}$ is 2-inconsistent. Let $\eta \in \omega^{<\omega}$ and consider the set of formulas $\Gamma(x) = \{\phi(x, a_{\eta\hat{\ }r}) : r \in \omega\}$. Then $\Gamma(x) = \{\Phi(x, \lambda_{k_r} a_{n+1}) = \alpha : r \in \omega\} = \{\Phi(x, a_{n+1}) = \mu_r : r \in \omega\}$ where $\mu_r = \alpha/\lambda_r$. Clearly the μ_r are distinct. Then since $\Phi(x, a_{n+1}) : V \to K$ is a function, the set $\Gamma(x)$ must be pairwise inconsistent.

2. Claim: For $\beta \in \omega^\omega$, $\{\phi(x, a_{\beta\restriction r}) : r \in \omega\}$ is consistent. Using the notation in claim (1), we must prove that $\Delta(x) = \{\phi(x, a_{\beta\restriction r}) : r \in \omega\} = \{\Phi(x, a_r) = \mu_{\beta(r)} : r \in \omega\}$ is consistent. But by lemma 5.2.3, since the a_r are linearly independent, given any $n \in \omega$ and $r_1, \dots , r_n \in \omega$, the

set $\{\Phi(x, a_{r_i}) = \mu_{\beta(r_i)} : i = 1, \dots n\}$ is consistent. Hence $\Delta(x)$ is finitely satisfiable and so by compactness is consistent.

Thus $\phi(x, y)$ has the tree property and hence T is *not* a simple theory.

Note that it is crucial for this proof that both K and the dimension of V are infinite.

In fact we can also prove the non-simplicity of vector spaces over infinite fields with bilinear forms by using Theorem 3.1.4. Recall that this theorem says that a group definable in a simple theory has the i.c.c. on uniformly definable subgroups up to commensurability. In particular if G is a group definable in a simple theory there is no family of uniformly definable subgroups $\mathcal{H} = \{H_i : i \in \omega\}$ such that $\left|\bigcap_{i \leq j} H_i : \bigcap_{i<j} H_i\right|$ is infinite for each $j \in \omega$. Now let $(V, +, \perp)$ be a vector space over an infinite field with orthogonality relation $\perp$ induced by some non-degenerate bilinear form on V. Let $\{a_i : i \in \omega\}$ be an infinite linearly independent set in V. Let H_i be the set $H_i = \{v \in V : v \perp a_i\}$. Then H_i is a subgroup of $(V, +)$ and since the underlying field is infinite, $\left|\bigcap_{i \leq j} H_i : \bigcap_{i<j} H_i\right|$ is infinite for each $j \in \omega$. Therefore $(V, +)$ cannot be a group definable in a simple theory. Hence the structure $(V, +, \perp)$ is not simple.

The following theorem summarises the main results of this chapter:

Theorem 7.4.2 *Let K be an infinite field. Let V be a K-vector space of dimension m where $m \in \mathbb{N} \cup \{\infty\}$ and $m > 2$. Let $[\ ,\ ] : V \times V \to K$ be a non-degenerate bilinear form. Let $\mathfrak{B} = (V, +_V, \perp)$ be the structure with domain V, binary function $+_V : V \times V \to V$ corresponding to vector addition and binary relation $\perp$ such that $x \perp y \iff [x, y] = 0$.*

Let L be any language and T be any L-theory. Suppose that T has a model, M, with the following properties:

M has a definable set D, a definable binary function $+ : D \times D \to D$ and a definable binary relation R on D such that the structures $\mathfrak{D} = (D, +, R)$ and $\mathfrak{B} = (V, +_V, \perp)$ are isomorphic.

Then:

- T *contains the complete theory of the field* K.

- *For each model* $N \models T$, *there are definable (with parameters) sets* $X, Y \subseteq N$, *and definable functions* $+_X : X \times X \to X$, $\circ : X \times X \to X$, $+_Y : Y \times Y \to Y$, $\gamma : X \times Y \to Y$, *and* $\Phi : Y \times Y \to X$ *such that*

  1. $(X, \circ, +_X)$ *is a field elementarily equivalent to* K

  2. $(Y, +_Y)$ *is an abelian group*

  3. *The function* γ *defines a scalar multiplication of* X *on* Y, *and* Y *is an* X*-vector space with dim*(Y/X) =*dim*(V/K).

  4. $\Phi : Y \times Y \to X$ *is a non-degenerate bilinear form (of the same type - alternating or symmetric - as* $[\ ,\ ]$*)*

- *If dim*(V/K) *is infinite then the formula* $\Phi(u, v) = w$ *has the tree property and hence* T *is* not *a simple theory.*

Chapter 8

Vector Spaces with Bilinear Forms over Fields of Finite Characteristic

Suppose that F is a finite field and V is an infinite dimensional F-vector space with a non-degenerate bilinear form $\Phi : V \times V \to F$, then, by the results of Chapter 6, the first order theory of (V, Φ) has elimination of quantifiers in the language $L = \{+, (\lambda_\alpha)_{\alpha \in F}, \Phi\}$ and is simple, unstable. However, by the results of Chapter 7, if the field, F, is allowed to be infinite then the theory of (V, Φ) (in any language containing $\perp$ and $+$) is at least as complicated as the theory of the field F, and, if V is infinite dimensional over F, then the theory of (V, Φ) is never simple.

Suppose that K is a possibly infinite field with a finite subfield $F \leq K$. (In particular K has finite characteristic.) Then K is an F-vector space. Let V be a K-vector space (so V is also an F-vector space) and $\Phi : V \times V \to K$, a non-degenerate bilinear form. Let $\mathfrak{h} = \{h_i : i \in I\}$ be an F-basis for K. We can then define F-bilinear forms $\Phi_i : V \times V \to F$ for each $i \in I$ by taking $\Phi_i(x, y)$ to be the h_i-component of $\Phi(x, y)$. We consider the structure V as a K-vector space with the (infinitely many) F-bilinear forms Φ_i (but *not* Φ

itself), and show that under certain conditions on K, F, and the basis $\mathfrak{h}$, this structure has elimination of quantifiers and is simple (unstable).

The original bilinear form Φ and its associated orthogonality relation are then type-definable but not definable, giving an example of a simple theory with a partial type having the tree property.

8.1 The Structure $(V, (\Phi_i)_{i \in I})$

Definition 8.1.1 *Let K be an infinite field of finite characteristic $p \neq 2$. Let F be a finite subfield of K. (e.g. $F \cong \mathbb{Z}_p$.) Let $\mathfrak{h} = \{h_i : i \in I\}$ be an F-basis for K. Let V be a K-vector space. Let $\Phi : V \times V \to K$ be a non-degenerate bilinear form. Define functions $\Phi_i : V \times V \to F$ for $i \in I$ by:*

$$\Phi(x, y) = \sum_{i \in I} \lambda_i h_i \Rightarrow \Phi_i(x, y) = \lambda_i.$$

Thus $\Phi_i(x, y)$ is the i^{th} component of $\Phi(x, y)$. We write $x \perp_i y$ to mean $\Phi_i(x, y) = 0$.

Remark 8.1.2 *Let $x, y \in V$, then:*

1. *$\Phi(x, y) = \sum_{i \in I} \Phi_i(x, y) h_i$.*
2. *$\Phi(x, y) = 0 \iff \Phi_i(x, y) = 0$ for all $i \in I$.*
3. *$\Phi_i(x, y) = 0$ for all but finitely many $i \in I$.*

Lemma 8.1.3 *For each $i \in I$, $\Phi_i : V \times V \to F$ is a non-degenerate F-bilinear form. Moreover, each Φ_i is K-stable, i.e. if $\alpha \in K$, then $\Phi_i(\alpha x, y) = \Phi_i(x, \alpha y)$.*

Proof. For $x_1, x_2, y \in V$, $\lambda_1, \lambda_2 \in F$, $\Phi_i(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 \Phi_i(x_1, y) + \lambda_2 \Phi_i(x_2, y)$ is immediate from the bilinearity of Φ. To prove Φ_i non-degenerate, choose $x \in V \setminus \{0\}$ and (by non-degeneracy of Φ) $y_0 \in V$ such that $\Phi(x, y_0) = \alpha \neq 0$. Then $\beta = \frac{h_i}{\alpha} \in K$. Put $y = \beta y_0$. Then $\Phi(x, y) = \Phi(x, \beta y_0) =$

$\frac{h_i}{\alpha}\Phi(x, y_0) = \frac{h_i}{\alpha}\alpha = h_i$. Hence $\Phi_i(x, y) = 1$ and so Φ_i is non-degenerate. Note that if Φ is symmetric (resp. alternating) then so is each Φ_i. Since Φ is K-bilinear $\Phi(\alpha x, y) = \Phi(x, \alpha y)$ and hence the h_i components are equal, i.e. $\Phi_i(\alpha x, y) = \Phi_i(x, \alpha y)$. However, if $\alpha \in K \backslash F$ then it is possible that $\Phi_i(\alpha x, y) \neq \alpha \Phi_i(x, y)$. ∎

Notation 8.1.4 *For $i \in I$ and $\mu \in F$ let $R_i^\mu(x, y)$ be the binary relation on V given by $R_i^\mu(x, y) \iff \Phi_i(x, y) = \mu$. Then, for each $i \in I$, the set $\{R_i^\mu : \mu \in F\}$ is a finite set of mutually exclusive set of relations with $\models \forall x, y \bigvee_{\mu \in F} R_i^\mu(x, y)$ and so we can think of each function $\Phi_i : V \times V \to F$ as this finite set of relations.*

Definition 8.1.5 *Let L_0 be the usual (1-sorted) language of K-vector spaces. Let $L = L_0 \cup \{\Phi_i : i \in I\}$, where each Φ_i is really the finite set of relations $\{R_i^\mu : \mu \in F\}$. Let M_0 be the L-structure with domain V and functions $\{\Phi_i : i \in I\}$ as described in Definition 8.1.1. Let T be the L-theory of the structure M_0.*

Remark 8.1.6 *Note that any model of $M \models T$ is a K-vector space with non-degenerate bilinear forms $\Phi_i : M \times M \to F$. However (by compactness) there may be no bilinear form $\Phi : M \times M \to K$ compatible with all the Φ_i as in M (although we do have that each of the forms Φ_i is K-stable as described above).*

Theorem 8.1.7 *Let F be a finite field and suppose that K is an infinite algebraic extension of F. Then there is a vector space basis $\mathfrak{h} = \{h_i : i \in I\}$ for K over F such that the theory T of the structure $M_0 = (V, (\Phi_i)_{i \in I})$, as described in Definitions 8.1.1 and 8.1.5, admits elimination of quantifiers in the language L and is simple, unstable.*

The proof of simplicity is similar to the proof of simplicity of basic linear geometries in Chapter 6. After proving the quantifier elimination result we show that the notion of independence given by K-linear independence satisfies the independence over a model theorem (Theorem 2.3.9) and hence is non-forking in the necessarily simple theory T. However, in this case we need to work harder to prove the quantifier elimination result, the difficulty being that T has models with no bilinear form $\Phi : V \times V \to K$ compatible with all the Φ_i. We must therefore look for properties connecting the Φ_i in the model M_0 which *are* axiomatizable in L and hence hold in all models of T.

The crucial property turns out to be the following extension of Lemma 5.2.3. For convenience we assume that the index set I is countable of order type ω. For $n, k \in \omega$ consider the axiom **(n,k)**:

(n,k): Given F-linearly independent elements $b_1, \ldots, b_n$,
and any elements $(\lambda_{i,l} : i \leq n, l \leq k)$ from F,
there is some element c such that
$\Phi_l(c, b_i) = \lambda_{i,l}$ for $i \leq n, l \leq k$.

(For a general index set I we would require a separate axiom $(\mathbf{n},\mathbf{r}_1, \ldots, \mathbf{r}_k)$ for each subset $\{r_1, \ldots, r_k\} \subseteq I$. All the following proofs would go through identically for such an axiom scheme but with much more complicated notation.) Note that **(n,k)** is axiomatizable in L: $b_1, \ldots, b_n$ are F-linearly independent if and only if $\bigwedge_{(\lambda_1,\ldots\lambda_n) \in F^n \setminus \{0\}} (\lambda_1 b_1 + \ldots + \lambda_n b_n \neq 0)$ and since F is finite this is a finite conjunction. In order to prove the quantifier elimination part of Theorem 8.1.7 we prove that if M_0 satisfies **(n,k)** for each $n, k \in \omega$ then T has elimination of quantifiers in L, and then if K is algebraic over F there is some basis $\mathfrak{h} = \{h_i : i \in I\}$ for K as a vector space over F, such that **(n,k)** holds for each $n, k \in \omega$.

8.2 Proof of Quantifier Elimination

Proposition 8.2.1 *Suppose that M_0 satisfies **(n,k)** for each $n, k \in \omega$. Then T has elimination of quantifiers in L.*

Proof. Assume that the original form Φ on M_0 is alternating. We use a back and forth argument. Let M be a sufficiently saturated model of T, and let A_1, A_2 be finitely generated substructures of M, with a partial isomorphism $f : A_1 \to A_2$. Let $c \in M \setminus A_1$. Then to prove quantifier elimination for T, it suffices to find some element $c' \in M$ such that f extends to a partial isomorphism f', between the substructures of M generated by (A_1, c) and (A_2, c').

A finitely generated substructure of M is simply a K-linear subspace. A function $f : A_1 \to A_2$ is a partial isomorphism if and only if f is K-linear and $\Phi_i(f(a), f(a')) = \Phi_i(a, a')$ for all $i \in I$ and $a, a' \in A_1$. As M is a non-degenerate Φ_i-space for each $i \in I$, without loss of generality we can assume that both A_1 and A_2 are non-degenerate Φ_i-spaces for each $i \in I$.

Let $a_1, \ldots, a_m$ be a K-basis for A_1 and $a'_1, \ldots, a'_m$ a K-basis for A_2. Then $\{h_i a_j : i \in I, j = 1, \ldots, m\}$ is an F-basis for A_1 and similarly for A_2. Then by the condition **(n,k)** and compactness we can find $c' \in M$ such that $\Phi_l(c', h_i a'_j) = \Phi_l(c, h_i a_j)$ for all $i, l \in I$ and $j = 1, \ldots m$. (Without loss of generality $c \notin A_1$ and $c' \notin A_2$.) Define $f' : (A_1, c) \to (A_2, c')$ by:

$$f'(a + \alpha c) = f(a) + \alpha c' \quad \text{for } a \in A_1,\ \alpha \in K.$$

Then f' is K-linear: Let $d_1, \ldots, d_s \in (A_1, c)$ and $\lambda_1, \ldots, \lambda_s \in K$. For $i = 1, \ldots, s$ let $d_i = \alpha c + a_i$. Then:

$$\begin{aligned}
f'(\lambda_1 d_1 + \cdots + \lambda_s d_s) &= f'(\,\lambda_1(\alpha c + a_1) + \cdots + \lambda_n(\alpha c + a_s)\,) \\
&= f'(\,\lambda_1 \alpha c + \lambda_1 a_1 + \cdots + \lambda_n \alpha c + \lambda_s a_s) \\
&= f'(\,(\lambda_1 \alpha + \cdots + \lambda_n \alpha) c + \lambda_1 a_1 + \cdots + \lambda_s a_s) \\
&= (\lambda_1 \alpha + \cdots + \lambda_n \alpha) c' + \lambda_1 f(a_1) + \cdots + \lambda_s f(a_s)) \\
&= \lambda_1 \alpha c' + \lambda_1 f(a_1) + \cdots + \lambda_n \alpha c' + \lambda_s f(a_s) \\
&= \lambda_1(\alpha c' + f(a_1)) + \cdots + \lambda_n(\alpha c' + f(a_s)) \\
&= \lambda_1 f'(d_1) + \cdots + \lambda_s f'(d_s).
\end{aligned}$$

To finish the proof that f' is a partial isomorphism we must prove that f' is a partial isometry with respect to each Φ_i. Let $a, b \in A_1$ and let $d_1, d_2 \in (A_1, c)$ be given by $d_1 = a + \alpha c$, $d_2 = b + \beta c$. Suppose that Φ, and hence each Φ_i, is alternating. So $\Phi_i(\alpha x, \beta x) = 0$ and $\Phi_i(x, y) = -\Phi_i(y, x)$ all $x, y \in M$, $\alpha, \beta \in K$. Then:

$$\begin{aligned}
\Phi_i(d_1, d_2) &= \Phi_i(\alpha c + a\,,\ \beta c + b) \\
&= \Phi_i(\alpha c, \beta c) + \Phi_i(\alpha c, b) + \Phi_i(a, \beta c) + \Phi_i(a, b) \\
&= 0 + \Phi_i(\alpha c, b) - \Phi_i(\beta c, a) + \Phi_i(a, b) \\
&= 0 + \Phi_i(c, \alpha b) - \Phi_i(c, \beta a) + \Phi_i(a, b) \\
&= 0 + \Phi_i(c', \alpha f(b)\,) - \Phi_i(c', \beta f(a)\,) + \Phi_i(f(a), f(b)) \\
&= \Phi_i(\alpha c', \beta c') + \Phi_i(\alpha c', f(b)) + \Phi_i(\beta c', f(a)) + \Phi_i(f(a), f(b)) \\
&= \Phi_i(\alpha c' + f(a)\,, \beta c' + f(b)) \\
&= \Phi_i(f'(d_1), f'(d_2)).
\end{aligned}$$

Thus M has the back-and-forth property and hence its theory has elimination of quantifiers. If the form is symmetric instead of alternating more or less the same argument as above goes through, except that we must find c' with the additional property that $\Phi_i(c', c') = \Phi_i(c, c)$ for all $i \in I$. This can be done in the same way as the corresponding proof (of Proposition 6.2.3) in the finite field case.

Before going on to show that property **(n,k)** holds in particular circumstances we prove its equivalence to an apparently weaker condition. This shows that whether or not the structure M_0 satisfies **(n,k)** depends only on the choice of basis $\mathfrak{h} = \{h_i : i \in I\}$ for K as a vector space over F.

Lemma 8.2.2 *The structure M_0 satisfies* ***(n,k)*** *for each pair $n, k \in \omega$ if and only if it satisfies* ***(n,k)**** *for each $n, k \in \omega$, where* ***(n,k)**** *is the property:*

$$\textbf{\textit{(n,k)*}} : \textit{ For each } b \in M_0\backslash\{0\} \textit{ and each set}$$
$$(\lambda_{i,l} : i \leq n, l \leq k) \textit{ from } F\textit{, there is some element}$$
$$c \textit{ such that } \Phi_l(c, h_i b) = \lambda_{i,l} \textit{ for } i \leq n, l \leq k.$$

Proof. Left to right is obvious since for any (non-zero) element b the set $\{h_i b : i \in I\}$ is F-linearly independent. To prove the converse we first prove:

Claim. Suppose that **(n,k)*** holds for $n, k \in \omega$. Then given K-linearly independent elements $a_1, \ldots, a_m \in M_0$, and $\{\lambda^j_{i,l} : i \leq n, l \leq k, j \leq m\} \subseteq F$, there exists $c \in M_0$ such that $\Phi_l(c, h_i a_j) = \lambda^j_{i,l}$ for all $i \leq n$, $l \leq k$, $j \leq m$.

Proof of Claim. By **(n,k)*** we can find, for each fixed $j = 1, \ldots, m$, an element c_j such that $\Phi_l(c_j, h_i a_j) = \lambda^j_{i,l}$ for all $i \leq n, l \leq k$. Let $\Phi(c_j, a_j) = \alpha_j$ where Φ is the original K-bilinear form on M_0. Then $\Phi(c_j, h_i a_j) = h_i\, \Phi(c_j, a_j) = h_i \alpha_j$. Therefore any c'_j such that $\Phi(c'_j, a_j) = \alpha_j$ will work instead of c_j. Since $a_1, \ldots, a_m$ are K-linearly independent, we can use Lemma 5.2.3 to find $c \in M_0$ such that $\Phi(c, a_j) = \alpha_j$ for each $j = 1, \ldots, m$ and then c satisfies the conditions of the claim.

Now suppose $b_1, \ldots, b_n$ are any F-linearly independent elements in M_0. Let $k \in I$ and let $\{\lambda_{i,l} : i \leq n, l \leq k\} \subseteq F$. Choose K-linearly independent elements $a_1, \ldots, a_m \in M_0$ such that $b_1, \ldots b_n \in K\text{-Span}(a_1, \ldots, a_m)$ and choose $r \in \omega$ such that $b_1, \ldots, b_n \in F\text{-Span}(h_i a_j : i \leq r, j \leq m)$. Then, by the claim, we can choose $c \in M_0$ such that $\Phi_l(c, h_i a_j) = \mu^j_{i,l}$ for all $i \leq r, l \leq k, j \leq m$ and arbitrarily chosen $\mu^j_{i,l} \in F$. Since $b_1, \ldots, b_n$ are F-linearly independent elements contained in $F\text{-Span}(h_i a_j : i \leq r, j \leq m)$, this is sufficient. ■

Lemma 8.2.3 *The structure M_0 satisfies **(n,k)** for each pair $n, k \in \omega$ if and only if for each finite set $b_1, \ldots, b_n$ of F-linearly independent elements with $B = F$-Span$(b_1, \ldots, b_n)$ the following holds:*

$$\left(\bigcap_{l=1}^{k-1} B^{\perp_l}\right)^{\perp_k} \bigcap \ (B) = (0) \tag{8.1}$$

where $\perp_l$ and $\perp_k$ are the orthogonality relations induced by the F-bilinear forms Φ_l and Φ_k.

Proof. By the previous lemma we can assume that $b_i = h_i b$ for some (fixed) non-zero $b \in M_0$. By an easy induction we can assume, without loss of generality, that $\lambda_{i,l} = 0$ for $l = 1, \ldots, k-1$ and $i = 1, \ldots n$. Then apply Lemma 5.2.4 to the set $U = \left(\bigcap_{l=1}^{k-1} B^{\perp_l}\right)$ and the F-bilinear form Φ_k. Thus M_0 has the property that given arbitrary $\lambda_1, \ldots \lambda_n \in F$ we can find $c \in U$ such that $\Phi_k(c, b_i) = \lambda_i$; if and only if $U^{\perp_k} \cap B = (0)$. The converse is immediate from 5.2.4. ■

Remark 8.2.4 *In order to prove that **(n,k)** holds for each pair $n, k \in \omega$ it is sufficient to prove that **(n,k)** holds for each pair n, k with $k \leq n$. Since, if $k > n$, then we simply choose some $N \geq k$ and since **(N,k)** holds it follows that **(n,k)** holds. The important point here is that in order to prove **(n,k)** in Lemma 8.2.3 we first prove **(m,k)** for each $m < n$.*

Thus, to prove quantifier elimination for the structure M_0, it is sufficient to prove that: for all $n, k \in \omega$ (with $k \leq n$) and all $b \in M_0 \backslash \{0\}$, if $B = F$-Span$(h_1 b, \ldots, h_n b)$, then

$$\left(\bigcap_{l=1}^{k-1} B^{\perp_l}\right)^{\perp_k} \bigcap \ (B) = (0)\,.$$

Lemma 8.2.5 *Let $n, k \in \omega$ and let $B = F$-Span$(h_1 b, \ldots, h_n b)$, then $\left(\bigcap_{l=1}^{k-1} B^{\perp_l}\right)^{\perp_k} \bigcap \ (B) = (0)$ if and only if for each $i = 1, \ldots, n$ there is $c \in M_0$ such that $c \perp_l B$ for $l = 1, \ldots, k-1$ and $c \not\perp_k h_i b$.*

Proof. $\left(\bigcap_{l=1}^{k-1} B^{\perp_l}\right)^{\perp_k} \bigcap \ (B) = (0) \iff$ there is no $d \in B\backslash\{0\}$ such that $d \perp_k \left(\bigcap_{l=1}^{k-1} B^{\perp_l}\right) \iff$ there is no $d \in B\backslash\{0\}$ such that $\left(\bigcap_{l=1}^{k-1} B^{\perp_l}\right) \subseteq (d)^{\perp_k} \iff$ for all $d \in B\backslash\{0\}$ there is $c \in \left(\bigcap_{l=1}^{k-1} B^{\perp_l}\right)$ such that $c \not\perp_k d$ $\iff$ for each $i = 1, \ldots, n$ there is c such that $c \perp_l B$ for $l = 1, \ldots, k-1$ and $c \not\perp_k h_i b$. ■

Therefore it is sufficient to find a basis $\mathfrak{h} = \{h_i : i \in I\}$ with the property that for any $b \neq 0$ (and any $n, k \in \omega$) if $d \in F\text{-Span}(h_1 b, \ldots, h_n b)$ then there is some element c such that $\Phi(c, h_i b)$ has zero l-component for each $l = 1, \ldots k-1$, but $\Phi(c, d)$ has non-zero k-component. This is all independent of the choice of b since the values of $\Phi(c, h_i b)$ and $\Phi(c, d)$ are determined by the value of $\Phi(c, b)$, and given any $b \in K\backslash\{0\}$ and $\alpha \in K$, there is some element c such that $\Phi(c, d) = \alpha$.

As $\mathfrak{h} = \{h_i : i \in I\}$ is an F-vector space basis for the field K, for each $i, j \in I$ there exist $\{\nu_q^{i,j} : q \in I\}$ such that $h_i h_j = \sum_{q \in I} \nu_q^{i,j} h_q$. (Note that $\nu_q^{ij} = \nu_q^{ji}$ and for any fixed $i, j \in \omega$, $\nu_q^{i,j} = 0$ for all but finitely many $q \in I$.) We assume that $h_1 = 1_k$ and therefore $\nu_q^{1j} = \delta_{(j,q)}$.

Suppose that $b \in M_0\backslash\{0\}$, n, $k \in \omega$ and $d \in F\text{-Span}(h_1 b, \ldots, h_n b)$ ($d = (\mu_1 h_1 + \cdots + \mu_n h_n)b$ with $\mu_1, \ldots, \mu_n \in F$). Then for any $\alpha \,(= \sum_{i \in I} \lambda_i h_i) \in K$ we can find $c \in M_0$ such that $\Phi(c, b) = \alpha$. Then

$$\begin{aligned}
\Phi(c, h_j b) &= h_j \, \Phi(c, b) = h_j \alpha = h_j \left(\sum_{i \in I} \lambda_i h_i\right) = \sum_{i \in I} \lambda_i (h_i h_j) \\
&= \sum_{i \in I} \lambda_i \left(\sum_{q \in I} \nu_q^{i,j} h_q\right) = \sum_{q \in I} \left(\sum_{i \in I} \lambda_i \nu_q^{i,j}\right) h_q
\end{aligned}$$

and

$$\begin{aligned}
\Phi(c, d) &= \left(\sum_{j=1}^{n} \mu_j h_j\right) \Phi\,(c, b) = \left(\sum_{j=1}^{n} \mu_j h_j\right) \left(\sum_{i \in I} \lambda_i h_i\right) \\
&= \sum_{j=1}^{n} \left(\mu_j \left(\sum_{q \in I} \left(\sum_{i \in I} \lambda_i \nu_q^{i,j}\right) h_q\right)\right) = \sum_{q \in I} \left(\sum_{i \in I} \left(\sum_{j=1}^{n} \mu_j \nu_q^{i,j}\right) \lambda_i\right) h_q.
\end{aligned}$$

Thus $\Phi(c, h_j b)$ has h_q-component $\sum_{i \in I} \lambda_i \nu_q^{i,j}$, and $\Phi(c,d)$ has h_q-component $\sum_{i\in I} \left(\sum_{j=1}^n \mu_j \nu_q^{i,j} \right) \lambda_i$.

Hence we require, exactly, that the basis $\mathfrak{h} = \{h_i : i \in I\}$ has the following property:

For any $n, k \in \omega$ and $\mu_1, \ldots, \mu_n \in F$ there exist $(\lambda_i : i \in I) \subseteq F$ such that:

(1) $\displaystyle\sum_{i\in I} \lambda_i \nu_q^{i,j} = 0$ for $q = 1, \ldots k-1$ and $j = 1, \ldots, n$; and

(2) $\displaystyle\sum_{i\in I} \left(\sum_{j=1}^n \mu_j \nu_k^{i,j} \right) \lambda_i \neq 0.$

i.e. $\Phi(c, h_i b)$ has zero h_q-component for each $q = 1, \ldots k-1$, but $\Phi(c,d)$ has non-zero h_k-component. We can put this in matrix notation as follows:

For $j \in I$, let A_j be the $I \times k$ -matrix:

$$A_j = \begin{pmatrix} \nu_1^{j1} & \nu_1^{j2} & \cdots & \nu_1^{ji} & \cdots \\ \nu_2^{j1} & \nu_2^{j2} & \cdots & \nu_2^{ji} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ \nu_{k-1}^{j1} & \nu_{k-1}^{j2} & \cdots & \nu_{k-1}^{ji} & \cdots \end{pmatrix}$$

Thus A_j is the matrix whose i^{th} column is the vector consisting of the first $k-1$ components of the vector $h_j h_i$. We express this vector as $h_j h_i \restriction_{k-1}$.

Let B be the matrix

$$B = \begin{pmatrix} \nu_k^{11} & \nu_k^{12} & \cdots & \nu_k^{1i} & \cdots \\ \nu_k^{21} & \nu_k^{22} & \cdots & \nu_k^{2i} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ \nu_k^{n1} & \nu_k^{n2} & \cdots & \nu_k^{ni} & \cdots \end{pmatrix}$$

and let $\bar{b}$ be the vector $(\mu_1, \ldots, \mu_n)\, B$. Then we must find $(\lambda_i : i \in I)$ such that: $A_j \bar{\lambda} = \bar{0}$ for $j = 1, \ldots, n$ and $\bar{b}\bar{\lambda} \neq 0$ where $\bar{\lambda} = (\lambda_i : i \in I)^T$. Equivalently we must show that for some $\beta \neq 0$ the system of equations

$$\begin{pmatrix} A_1 \\ \vdots \\ A_n \\ \bar{b} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_i \\ \vdots \end{pmatrix} = \begin{pmatrix} \bar{0} \\ \vdots \\ \bar{0} \\ \beta \end{pmatrix}$$

has a solution $(\lambda_i : i \in I)$. This is the case if and only if the vector $\begin{pmatrix} \bar{0} \\ \vdots \\ \bar{0} \\ 1 \end{pmatrix}$ is in the F-linear span of the columns of the matrix, $\begin{pmatrix} A_1 \\ \vdots \\ A_n \\ \bar{b} \end{pmatrix}$. It is clearly equivalent to carry out the above argument for each of the vectors $(1, 0, \ldots, 0)B$, $(0, 1, \ldots, 0)B, \ldots, (0, \ldots, 0, 1)B$ in place of $\bar{b}$.

Hence let A'_j be the matrix:

$$A'_j = \begin{pmatrix} \nu_1^{j1} & \nu_1^{j2} & \cdots & \nu_1^{ji} & \cdots \\ \nu_2^{j1} & \nu_2^{j2} & \cdots & \nu_2^{ji} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ \nu_{k-1}^{j1} & \nu_{k-1}^{j2} & \cdots & \nu_{k-1}^{ji} & \cdots \\ \nu_k^{j1} & \nu_k^{j2} & \cdots & \nu_k^{ji} & \cdots \end{pmatrix}$$

Thus A'_j is the matrix whose i^{th} column is the vector consisting of the first k components of the vector $h_j h_i$ (i.e. the i^{th} column is $h_j h_i \restriction_k$). Let A be the matrix $\begin{pmatrix} A'_1 \\ \vdots \\ A'_n \end{pmatrix}$. Therefore A is the matrix with i^{th} column equal to the vector $((h_1 h_i \restriction_k), \ldots, (h_n h_i \restriction_k))^T$. The first column (in the case $n = k$) is simply the vector $((1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots (0, \ldots, 0, 1))^T$. Therefore if we can prove that for each $j = 1, \ldots, n$ the vector e_{kj} with all components zero except for the kj^{th} component which is equal to 1, (i.e. $e_{kj} :=$ $((0,0,\ldots,0),\ldots,(0,\ldots,0,1),\ldots,(0,\ldots,0))^T$) is in the F-linear span of the columns of A then we are done. In fact we prove that a slightly stronger property holds. Namely that given any $n, k \in \omega$ (with $n \geq k$) then any vector of length $n \times k$ with one non-zero entry and all other entries equal to zero, is in the F-linear span of the columns of A (which is clearly the case if and only if the columns of A span a vector space of dimension $n \times k$).

We now show that if F is a finite field and K is an algebraic extension of F, then we can construct a basis $\mathfrak{h} = \{h_i : i \in I\}$ for K as a vector space over F with this property.

Assume that K is countable and let $(\gamma_i : i \in \omega)$ be some enumeration of K. Construct $\mathfrak{h} = \{h_i : i \in I\}$ as follows:

Let $h_1 = 1_K$.

Suppose we have constructed $h_1, \ldots, h_m$. Extend this to an F-vector space basis $h_1, \ldots, h_m, h_{m+1}, \ldots h_n$ for the field $F_m = F(h_1, \ldots, h_m)$. (We can do this since K is algebraic over F, so the field F_m is finite and in particular is a finite dimensional F-vector space.) Choose the least $\gamma_{i_1} \notin F_m$, then the least $\gamma_{i_2} \notin F_m(\gamma_{i_1})$, and continue up to $\gamma_{i_r} \notin F_m(\gamma_{i_1}, \ldots, \gamma_{i_{r-1}})$ until $F_m(\gamma_{i_1}, \ldots, \gamma_{i_r})$ is a field extension of F_m of degree $\geq n^2$. Then as $F_m(\gamma_{i_1}, \ldots, \gamma_{i_r})$ is a finite extension of the finite field F, by the theorem of the primitive element, there is some $\beta \in K$ such that $F_m(\gamma_{i_1}, \ldots, \gamma_{i_r}) = F_m(\beta)$. Let $[F_m(\beta) : F_m] = N \geq n^2$. Then the set $\{\beta^j h_i : i \leq n, j \leq N-1\}$ is an F-vector space basis for $F_m(\beta)$. We slightly modify this basis (keeping the h_i ($i \leq n$) and the $\beta^j h_1$ ($j \leq N-1$) fixed) to obtain a basis with the required property.

Note that if $\beta^k h_i$ is a basis element (of index $> n$) then the basis element h_i multiplied by the basis element β^k is the basis element $\beta^k h_i$ and hence has zero h_q-component for all $q \leq n$. However, if h_i and β^k remain basis elements, but if for some $j \leq n$, we replace the basis element $\beta^k h_i$ with $\beta^k h_i - h_j$ (which clearly leaves the linear span the same) then the product of basis elements

$\beta^k \times h_i$ written as a linear combination of basis elements is $(\beta^k h_i - h_j) + h_j$. Thus $\beta^k h_i$ has h_j component equal to 1, and all other h_q-components zero for $q \leq n$.

Suppose, for $k > 0$ and $2 \leq s \leq n$, we replace the set of basis elements

$$\beta^k \quad \beta^k h_2 \quad \beta^k h_3 \quad \ldots \quad \beta^k h_s \quad \ldots \quad \beta^k h_n$$

with the set of elements

$$\beta^k \quad \beta^k h_2 \quad \beta^k h_3 \quad \ldots \quad \beta^k h_s - h_t \quad \ldots \quad \beta^k h_n.$$

Then for $i \leq n$ and $i \neq s$, the product $\beta^k h_i$ of the basis elements β^k and h_i has zero q-component for all $q \leq n$, but the product of basis elements $\beta^k h_s = (\beta^k h_s - h_t) + h_t$ has t-component $= 1$ and all other q-components (for $q \leq n$) equal to zero.

As $N \geq n^2$, without loss of generality, we can replace the index set N with an index set $\{(s,t) : s \leq n, t \leq n\}$ (identifying $\beta^{(1,1)}$ with $\beta^0 = 1_K$). We then modify the basis $\{\beta^{(s,t)} h_i : i \leq n, s, t \leq n\}$ by replacing each $\beta^{(s,t)} h_i$, where $i = s$, with the element $\beta^{(s,t)} h_s - h_t$ (except for $\beta^{(1,1)} h_i$ which all stay fixed) and leaving all other elements fixed. (We are using the set of basis elements $\{\beta^{(s,t)} h_i : i \leq n\}$ to map h_s to h_t.)

We then extend the original basis $h_1, \ldots, h_m, h_{m+1}, \ldots h_n$ for the field $F_m = F(h_1, \ldots, h_m)$ to the basis for $F_m(\beta)$, modified from $\{\beta^{(s,t)} h_i : i \leq n, s, t \leq n\}$ as described above.

We can continue constructing the basis by choosing another β' with $[F_m(\beta)(\beta') : F_m(\beta)] \geq (nN)^2$ and repeat the process.

Proposition 8.2.6 *The basis $\mathfrak{h} = \{h_i : i \in I\}$ constructed above has the following property: given any $n \in \omega$ there is $N \in \omega$ such that for any $s, t \leq n$ there is $k_{(s,t)} \in F$-Span$(h_i : i \leq N)$ such that for each $j \leq n$ the vector $k_{(s,t)} h_j$ has zero q-component for all $q, j \leq n$ except for $j = s$ and $q = t$ when $k_{(s,t)} h_s$ has t-component 1.*

Proof. By the above construction we can choose $n_0 \geq n$ such that F-Span$(h_1, \ldots, h_{n_0}) = F(h_1, \ldots, h_{n_0}) = F(h_1, \ldots, h_n)$. We can then choose $N \geq n_0$ such that for each pair (s,t) with $2 \leq s \leq n_0$ and $t \leq n_0$ there is a basis element $h_{(s,t)}$ with $n_0 < (s,t) \leq N$ such that $h_{(s,t)}h_s$ has h_t-component 1 and h_q-component zero for all other $q \leq n_0$, and for $j \leq n_0$ with $j \neq s$, $h_{(s,t)}h_j$ has h_q-component zero for all $q \leq n_0$. Put $k_{(s,t)} = h_{(s,t)}$. This only leaves the case $s = 1$. In this case we can choose some s' with $2 \leq s' \leq n_0$ and $h_{(s',t)}$ as above. Then the vector $k_{(s,t)} = h_{(s',t)}h_{s'}$ has the required properties. ■

Now choose any $k \leq n < \omega$. Let A be the $(n \times k) \times I$ matrix with i^{th} column equal to $((h_1 h_i \restriction_k), \ldots, (h_n h_i \restriction_k))^T$. For $s = 1, \ldots, n$ and $t = 1, \ldots, k$, let $e_{s,t}$ be the vector of length $n \times k$ with the $(s-1)k + t$ component equal to 1 and all other entries equal to zero. Then, by the above proposition, $e_{s,t}$ is in the F-linear span of the columns of A. Thus the dimension of the space spanned by the columns of the matrix A is $n \times k$. Hence M_0 with basis $\mathfrak{h} = \{h_i : i \in I\}$ satisfies **(n,k)**.

Therefore we have proved the first part of Theorem 8.1.7, namely that if K is algebraic over the finite subfield F, then there is a basis $\mathfrak{h} = \{h_i : i \in I\}$ for K as an F-vector space, such that the associated structure $(V, (\Phi_i)_{i \in I})$ has elimination of quantifiers in its natural language.

8.3 Simplicity of $(V, (\Phi_i))$

We now prove the remainder of Theorem 8.1.7, that given the basis $\mathfrak{h} = \{h_i : i \in I\}$ chosen in the previous section, the theory, T, of the structure $(V, (\Phi_i)_{i \in I})$ is simple (unstable). The proof is very similar to the proof of simplicity of basic linear geometries given in Proposition 6.2.5. We 'guess' that the correct notion of independence on T is K-linear independence on the models of T considered as K-vector spaces and than show that this really is a notion of independence on T, and that it satisfies the independence over a model theorem.

Let T_0 be the theory of the reducts of models of T to the language of K-vector spaces. Then T_0 is just the theory of K-vector spaces. Hence T_0 is stable (in fact $\aleph_1$-categorical) and non-forking on T_0 is just K-linear independence. Thus K-linear independence is a notion of independence on models of T.

Definition 8.3.1 *Let $M \models T$ and $A, B, C \subseteq M$. Then tp(C/B) Γ-forks over A if tp(C/B) forks over A in the theory T_0. C and B are Γ-independent over A if tp(C/B) does not Γ-fork over A.*

Proposition 8.3.2 *Let $M \models T$ and let $A, B \supseteq M$ with A and B Γ-independent over M. Let $p_1 \in S(A)$, $p_2 \in S(B)$ be types which do not Γ-fork over M and such that $p_1 \restriction M = p_2 \restriction M$. Then there is some type $q \in S(A \cup B)$ which does not Γ-fork over M and which extends both p_1 and p_2.*

Proof. Without loss of generality A and B are both K-vector spaces and hence $A \cap B = M$. Working in some sufficiently saturated model $\tilde{M} \models T$ choose $\bar{c}_1$ realising p_1 and $\bar{c}_2$ realising p_2. As p_1 does not Γ-fork over M we have $\bar{c}_1 \cap A = \bar{c}_1 \cap M$ and similarly $\bar{c}_2 \cap B = \bar{c}_2 \cap M$. Therefore we may assume without loss of generality that $\bar{c}_1 \cap A = \emptyset$ and $\bar{c}_2 \cap B = \emptyset$. Note that $tp(\bar{c}_1/M) = tp(\bar{c}_2/M)$.

We must find $\bar{c} \notin K$-span(A, B) such that $tp(\bar{c}/A) = tp(\bar{c}_1/A)$ and $tp(\bar{c}/B) = tp(\bar{c}_2/B)$.

Let $\bar{c}_1 = (c_1^1, \ldots, c_n^1)$, $\bar{c}_2 = (c_1^2, \ldots, c_n^2)$. By quantifier elimination it is sufficient to find $\bar{c} = (c_1, \ldots, c_n)$ such that $\bar{c} \cap K$-span$(A, B) = \emptyset$ and

$$\Phi_l(c_i, a) = \Phi_l(c_i^1, a) \text{ for all } a \in A,\ i = 1, \ldots, n \text{ and all } l \in I \text{ and}$$
$$\Phi_l(c_i, b) = \Phi_l(c_i^2, b) \text{ for all } b \in B,\ i = 1, \ldots, n \text{ and all } l \in I.$$

But if $m \in M = A \cap B$ then $\Phi_l(c_i^1, m) = \Phi_l(c_i^2, m)$ for all $l \in I$.

Without loss of generality we may assume that $\bar{c}_1$ and $\bar{c}_2$ are F-linearly independent (but not necessarily K-linearly independent). We can also assume that $\bar{c}_1 = (c_1^1, \ldots, c_m^1, c_{m+1}^1, \ldots, c_n^1)$ where $\{c_1^1, \ldots, c_m^1\}$ is K-linearly independent and for each k with $m < k \leq n$ there exist $\alpha_i^k \in K$ (for $i = 1, \ldots, m$) such that $c_k^1 = \sum_{i=1}^m \alpha_i^k c_i^1$. (Note that for each $k > m$ there is some $\alpha_i^k \in K \backslash F$ or else the tuple $\bar{c}_1$ is F-linearly dependent.) Suppose that $c_k^1 = \sum_{i=1}^m \alpha_i^k c_i^1$ for some $k \leq n$. Then since $tp(\bar{c}_1/M) = tp(\bar{c}_2/M)$ it follows that $c_k^2 = \sum_{i=1}^m \alpha_i c_i^2$. For $k \leq m$ define $\alpha_i^k = \delta_{(i,k)}$ (Kronecker) so $c_k^j = \sum_{i=1}^m \alpha_i c_i^j$ for any $k \leq n$ and $j = 1, 2$.

Choose $k \in \omega$ and $a_1, \ldots, a_r, b_1, \ldots, b_s, m_1, \ldots, m_t$, with $a_i \in A \backslash M$, $b_i \in B \backslash M$, $m_i \in M$. Since each Φ_l is F-linear we can assume that the set $\{a_1, \ldots, a_r, b_1, \ldots, b_s, m_1, \ldots, m_t\}$ is F-linearly independent.

Thus we must find $c_1, \ldots, c_m$ such that $k = 1, \ldots, n$ and all $l \in I$

$$\Phi_l\left(\sum_{i=1}^m \alpha_i^k c_i, a_j\right) = \Phi_l\left(\sum_{i=1}^m \alpha_i^k c_i^1, a_j\right) \quad \text{for all } j = 1, \ldots, r$$

$$\Phi_l\left(\sum_{i=1}^m \alpha_i^k c_i, b_j\right) = \Phi_l\left(\sum_{i=1}^m \alpha_i^k c_i^2, b_j\right) \quad \text{for all } j = 1, \ldots, s$$

$$\Phi_l\left(\sum_{i=1}^m \alpha_i^k c_i, m_j\right) = \Phi_l\left(\sum_{i=1}^m \alpha_i^k c_i^1, m_j\right) = \Phi_l\left(\sum_{i=1}^m \alpha_i^k c_i^2, m_j\right) \quad \text{for all } j = 1, \ldots, t$$

We could immediately apply property **(n,k)** to find $c_1, \ldots, c_m$ satisfying the first three parts of the above condition for all $k = 1, \ldots m$, but this would not guarantee that the $c_k = \sum_{i=1}^m \alpha_i^k c_i$ for $k > m$ have the required properties. The difficulty being that although the forms Φ_l are F-linear, they are not in general K-linear. However we can exploit the fact that they are K-stable - i.e. $\Phi_l(\alpha x, y) = \Phi_l(x, \alpha y)$ for all $x,\ y \in \widetilde{M},\ \alpha \in K$ and $l \in I$.

Suppose that $\alpha \in K/F$ and given an element a and $\lambda_l, \mu_l \in F$ for $l \in I$ then as the elements a and αa are F-linearly independent we can use property **(n,k)** (and compactness) to find and element c such that $\Phi_l(c, a) = \lambda_l$ and $\Phi_l(\alpha c, a) = \Phi_l(c, \alpha a) = \mu_l$ for $l \in I$.

In general $c_{m+1}, \ldots, c_n$ are F-linearly independent so we may assume without loss of generality that the set $\{(\alpha_1^k a_{j_1}, \ldots, \alpha_m^k a_{j_1}), (\alpha_1^k b_{j_2}, \ldots, \alpha_m^k b_{j_2}), (\alpha_1^k m_{j_3}, \ldots, \alpha_m^k m_{j_3}) : j_1 \leq r, j_2 \leq s, j_3 \leq t : k \leq n\}$ is F-linearly independent.

We can then apply **(n,k)** m times to the system of equations

$$\sum_{i=1}^{m} \Phi_l\left(c_i, \alpha_i^k a_j\right) = \Phi_l\left(c_k^1, a_j\right) \quad \left(= \Phi_l\left(\sum_{i=1}^{m} \alpha_i^k c_i^1, a_j\right)\right) \quad k \leq n, j \leq r$$

$$\sum_{i=1}^{m} \Phi_l\left(c_i, \alpha_i^k b_j\right) = \Phi_l\left(c_k^1, b_j\right) \quad \left(= \Phi_l\left(\sum_{i=1}^{m} \alpha_i^k c_i^1, b_j\right)\right) \quad k \leq n, j \leq s$$

$$\sum_{i=1}^{m} \Phi_l\left(c_i, \alpha_i^k m_j\right) = \Phi_l\left(c_k^1, m_j\right) \quad \left(= \Phi_l\left(\sum_{i=1}^{m} \alpha_i^k c_i^1, m_j\right)\right) \quad k \leq n, j \leq t$$

to find $(c_1, \ldots c_m)$ with each $c_k \notin K$-span(A, B) for $k = 1, \ldots, m$ such that:

$$\Phi_l(c_k, a_i) = \Phi_l(c_k^1, a_i) \text{ for all } i = 1, \ldots, n \text{ and all } l \in I,$$

$$\Phi_l(c_k, b_i) = \Phi_l(c_k^2, b_i) \text{ for all } i = 1, \ldots, m \text{ and all } l \in I, \text{ and}$$

$$\Phi_l(c_k, m_i) = \Phi_l(c_k^1, m_i) = \Phi_l(c_k^2, m_i) \text{ for all } i = 1, \ldots r \text{ and all } l \in I.$$

Therefore by compactness and quantifier elimination there is $\bar{c} \notin K$-span(A, B) such that $tp(\bar{c}/A) = tp(\bar{c}_1/A)$ and $tp(\bar{c}/B) = tp(\bar{c}_2/B)$.

In the case of a symmetric bilinear form we must also check that $\Phi_l(c_i, c_i) = \Phi_l(c_i^1, c_i^1)$ $(= \Phi_l(c_i^2, c_i^2))$ for each $l \in I$. But as the field F is finite and forms over finite fields are always universal this is no problem. (See e.g. the proof of 6.2.3 or use Lemma 9.5.2.)

We must also choose the c_i so that $\Phi_l(c_i, c_j) = \Phi_l(c_i^1, c_j^1)$ $(= \Phi_l(c_i^2, c_j^2))$ for $i, j = 1, \ldots$ and $i \neq j$. However, since the c_i (for $i = 1, \ldots, n$) are F-linearly independent this is not a problem, by another application of **(n,k)**. ∎

Corollary 8.3.3 *The theory T is simple.*

Remark 8.3.4 *If either the field K is infinite, or the dimension of the vector space V over K is infinite then the theory T is unstable. This is because if*

V is infinite dimensional over F then, since each form $\Phi_l : V \times V \to F$ is a non-degenerate F-bilinear form, it is possible to construct a countably infinite sequence $(e_i : i \in \omega) \subseteq V$ such that $\Phi_l(e_i, e_j) = 0 \iff i = j$ and hence the formula $\Phi_l(x, y) = 0$ has the order property. Therefore the structure $(V, (\Phi_i)_{i \in I})$ is stable if and only if it is finite.

The proof of Theorem 8.1.7 is complete.

We show that if M_0 is infinite dimensional over K then the theory T has the 'type definable tree property'. That is, we give a partial type $P(x, y)$ such that there is $k \in \omega$ and tuples $(\bar{a}_\eta : \eta \in \omega^{<\omega})$ such that $\{P(x, \bar{a}_{\eta\hat{}r}) : r \in \omega\}$ is k-inconsistent for all $\eta \in \omega^{<\omega}$ and $\{P(x, \bar{a}_{\beta\restriction r}) : r \in \omega\}$ is consistent for all $\beta \in \omega^\omega$.

Let $\alpha \in F^I$. Let $P_\alpha(x, y)$ be the partial type $\{\Phi_i(x, y) = \alpha(i) : i \in I\}$. Then given $\alpha \in F^I$ with $\alpha(i) = 0$ for all but finitely many $i \in I$, the type $P_\alpha(x, y)$ is realised in the structure M_0 (i.e., in M_0, $P_\alpha(x, y)$ is the type-definable relation $\Phi(x, y) = \alpha$). Thus, by compactness, for every $\alpha \in F^I$, $P_\alpha(x, y)$ is consistent and hence realised in some $M \models T$.

Suppose that M_0 is infinite dimensional over K and let $\alpha \in F^I$ be such that $\alpha(i) = 0$ for all but finitely many $i \in I$. Then the type $P_\alpha(x, y)$ has the tree property. Choose $(a_i : i \in \omega)$ K-linearly independent in M_0. Choose $(\mu_i : i \in \omega)$ distinct elements of K. Then, exactly as in the proof of 7.4.1, for $\eta = (k_1, \ldots k_n)$ let $\bar{a}_\eta = \mu_{k_n} a_k$ and then $\{P_\alpha(x, \bar{a}_{\eta\hat{}r}) : r \in \omega\}$ is 2-inconsistent for all $\eta \in \omega^{<\omega}$ and $\{P(x, \bar{a}_{\beta\restriction r}) : r \in \omega\}$ is consistent (in fact realised in M_0) for all $\beta \in \omega^\omega$.

Chapter 9

The Model Theory of Vector Spaces With Bilinear Forms

9.1 The Two Sorted Language

In light of the results of Section 7, if we wish to study the model theory of vector spaces with bilinear forms using any language strong enough to interpret vector addition and the orthogonality relation induced by the bilinear form, then we are forced into using the two sorted language described in Definition 7.3.18. For convenience we restate the definition of this language:

Definition 9.1.1 *Let L be the two sorted language with sorts V (vectors), and K (scalars).*

L has the following functions:

$+_V : V \times V \to V$ *(vector addition)*

$+_K : K \times K \to K$ *(field addition)*

$\circ_K : K \times K \to K$ *(field multiplication)*

$\gamma : K \times V \to V$ *(scalar multiplication)*

$[\ ,\ ] : V \times V \to K$ *(the bilinear form, also written $\Phi : V \times V \to K$)*

and constants 0_V , 0_K and 1_K.

Notation 9.1.2 *Vector variables (variables of sort V) are denoted by capital letters: X, Y, Z etc. Vector constants are denoted by lower case letters: a, b, c etc. Field sort variables (variables of sort K) are denoted by lower case letters: x, y, z etc. and field sort constants are denoted by small Greek letters α, β, γ etc.*

If M is an L-structure, $M = (V, K)$, and $A \subseteq M$ then $A = A_V \cup A_K$ where A_V is the set of V-sort elements in A and A_K is the set of K-sort elements in A.

Definition 9.1.3 *Let K be a field (or a complete theory of fields) and let $m \in \mathbb{N} \cup \{\infty\}$.*

1. *Let T be the L-theory expressing that K is a field, V is a K-vector space, and $\Phi : V \times V \to K$ is a non-degenerate bilinear form.*

2. *Let T^K be the L-theory extending T expressing that the field sort is a model of the theory (of the field) K.*

3. *Let T_m be the L-theory extending T expressing that V is a K-vector space of dimension m.*

4. *T_m^K has the obvious meaning: a vector space of dimension m over the field K.*

5. *For $F \in \{S, A\}$, let ${}_FT$ denote the L-theory extending T expressing that Φ is a bilinear form of type F, where a form of type S is a symmetric form, and a form of type A is an alternating form.*

6. *T is called a* good theory *(of bilinear forms) if either T is a theory of the type ${}_AT_m^K$ for any field K, or a theory of the type ${}_ST_m^K$ where K is any field with square roots for all its elements. (Note that any theory ${}_FT_m^K$, where K is an algebraically closed field, is a good theory.)*

Note that although the use of K is ambiguous in that it refers to a sort in L, a field or the complete theory of that field it should be clear from the context exactly what is meant. Usually, when we use T , T^K, T^K_m, etc., we really mean ${}_FT$, ${}_FT^K$, ${}_FT^K_m$, etc. for $F = S$ or $F = A$ but without necessarily specifying which. In many situations, the (theory of the) field K will be fixed (usually algebraically closed of some fixed characteristic) and we will denote T^K, T^K_m etc. by T, T_m etc. Also, the parameter m will often be fixed and where this is clear, T_m, or even T^K_m, will be denoted simply by T. Exactly what is meant should be clear in each context.

Remark 9.1.4 *If $M_1 = (V_1, K_1)$ and $M_2 = (V_2, K_2)$ are models of T with bilinear forms Φ_1 and Φ_2 respectively, then $\sigma := (\sigma_V , \sigma_K)$ is an isomorphism (of L-structures) from M_1 to M_2 if and only if $\sigma_K : K_1 \to K_2$ is an isomorphism of fields and $\sigma_V : V_1 \to V_2$ is a semilinear map (with respect to the isomorphism σ_K) such that*

$$\forall a, b \in V_1 \;\; \Phi_2(\sigma_V(a) \, , \, \sigma_V(b)) = \sigma_K(\, \Phi_1(a, b))$$

i.e. σ_V is a semi-similarity from V_1 to V_2 with respect to the isomorphism σ_K and with 'multiplier' 1_{K_2}.

Proposition 9.1.5 *Let $M_1 = (V_1, K_1)$ and $M_2 = (V_2, K_2)$ be models of T such that the fields K_1 and K_2 are isomorphic and the forms on M_1 and M_2 are of the same type (i.e. either both have alternating forms or both have symmetric forms). Suppose that $\dim(V_1) = \dim(V_2) \leq \aleph_0$. Then if either*

1. *the forms on V_1 and V_2 are both alternating, or*

2. *the field K_1 has square roots for all its elements*

then the structures M_1 and M_2 are isomorphic as L-structures.

Proof. Let $\sigma_K : K_1 \to K_2$ be a field isomorphism. Let Φ_1 be the bilinear form on V_1 and let Φ_2 be the bilinear form on V_2. Suppose that the forms are both alternating. Then by Corollary 5.4.3 there is a normal basis $\{e_i : i \in I\} \cup \{f_i : i \in I\}$ for V_1 and a normal basis $\{e'_i : i \in I\} \cup \{f'_i : i \in I\}$ for V_2 where $2\,|I| = \dim(V_1) = \dim(V_2)$. Define a map $\sigma_V : V_1 \to V_2$ by:

$$\sigma_V(\sum_{i\in I} \alpha_i e_i + \sum_{i\in I} \beta_i f_i) = \sum_{i\in I} \sigma_K(\alpha_i) e'_i + \sum_{i\in I} \sigma_K(\beta_i) f'_i.$$

Then it is clear that σ_V is semi-linear with respect to the isomorphism σ_K and

$$\forall a, b \in V_1 \;\; \Phi_2(\sigma_V(a)\,,\,\sigma_V(b)\,) = \sigma_K(\,\Phi_1(a,b)).$$

Hence $\sigma = (\sigma_V\,, \sigma_K) : M_1 \to M_2$ is an isomorphism of L-structures.

In case (2) when the forms are symmetric and the fields have square roots then by Corollary 5.4.3 there are orthogonal bases for V_1 and V_2 which can easily be transformed into orthonormal bases. As in the first case we can construct an isomorphism of L-structures by extending any map between orthonormal bases. ∎

Proposition 9.1.5 quickly leads to a proof that any of the theories T^K_m to which it applies (i.e. any good theory) is complete. However, it is easier to give this proof after the quantifier elimination result of the next section.

Proposition 9.1.5 also shows that if T is the theory of infinite dimensional vector spaces over a (fixed theory of an) algebraically closed field K with a non-degenerate bilinear form (either symmetric or alternating) then, since K has $\aleph_0$ many countable models, it follows that T has $\aleph_0$ many countable models also.

Instead of restricting ourselves to theories of good bilinear forms as described in Definition 9.1.3 (6) we could consider the larger class of theories of bilinear forms, T, where in any model $(V, K) \models T$ with bilinear form Φ, any element of K represented by Φ has a square root in K. This property is clearly axiomatizable by the formula: $\forall y\,((\exists X\,(X \neq 0) \wedge \Phi(X,X) = y) \to \exists z\,(y = z^2))$. Anything we say about good theories of bilinear forms will also apply to this larger class of theories.

9.2 Quantifier Elimination

We show that after adding some new relations to the language L and, if necessary, an elimination set for the theory K in the language of fields, then (for any field K and any $m \in \mathbb{N} \cup \{\infty\}$) the theory T_m^K has elimination of quantifiers in this expanded language.

Let L and T, the extensions of T, and all other notation be as defined as in the previous section.

Definition 9.2.1 *For $n \in \omega$ let $\theta_n(X_1, \ldots X_n)$ be the following formula:*

$$\theta_n(X_1, \ldots X_n) := \forall y_1, \ldots, \forall y_n((y_1 X_1 + \cdots + y_n X_n = 0) \to (\bigwedge_{i=1}^{n} y_i = 0)).$$

Let L_θ be the expansion of L resulting from adding each formula θ_n to the language L as a new atomic formula.

Note that given a model $M = (V, K) \models T$ and any $a_1, \ldots a_n \in V$ we have $M \models \theta_n(a_1, \ldots, a_n)$ if and only if the set $\{a_1, \ldots a_n\}$ is K-linearly independent in V. Also changing the language from L to L_θ does not alter the definable sets, it just alters the quantifier-free definable sets.

Definition 9.2.2 *Let K be some fixed field. (Equivalently K is some fixed complete theory of fields.) Let L_θ^K (resp. L^K) be the language obtained by adding an elimination set for K (in the language of fields) to the set of atomic formulas of L_θ (resp. to L). In particular if K is an algebraically closed field then $L_\theta^K = L_\theta$.*

A θ-formula *is a quantifier free formula in the language L_θ^K. If M is an L-structure, $c \in M$ and $A \subseteq M$ then $\theta qftp(c/A)$ is the set of θ-formulas with parameters in A and satisfied by c.*

If M and N are L- (or L_θ-, or L_θ^K-) structures then a θ-isomorphism *from M to N is just a bijection from M to N, preserving θ-formulas.*

For $n \in \omega$, let $\chi_n(X_1, \ldots, X_n, Z)$ be the formula '$\exists y_1, \ldots, \exists y_n(y_1X_1 + \cdots + y_nX_n = Z)$'. Given $M = (V, K) \models T$ and any $a_1, \ldots a_n, b \in V$ we have $M \models \chi_n(a_1, \ldots, a_n, b)$ if and only if b is in the K-linear span of the set $\{a_1, \ldots a_n\}$. Note that the formula $\chi_n(X_1, \ldots, X_n, Z)$ is equivalent to $\bigvee_{\{i_1, \ldots i_r\} \subseteq \{1, \ldots, n\}} (\theta_r(X_{i_1}, \ldots, X_{i_r}) \wedge \neg\theta_{r+1}(X_{i_1}, \ldots, X_{i_r}, Z))$, and so each formula χ_n is equivalent to a θ-formula.

Theorem 9.2.3 *Let K be a fixed (complete theory of a) field and let $m \in \mathbb{N} \cup \{\infty\}$. Let $F \in \{A, S\}$. Then the theory ${}_FT^K_m$ has elimination of quantifiers in the language L^K_θ. In particular, if T is the theory of vector spaces (of some fixed dimension) over algebraically closed fields (of some fixed characteristic) with a bilinear (either alternating or symmetric) form, then T has elimination of quantifiers in the language L_θ.*

Notation 9.2.4 *Let T be some theory ${}_FT^K_m$ as described above, with bilinear form Φ. We work in a very saturated model of $M \models T$, written $M = (V, \tilde{K})$. In particular $\tilde{K}$ is a very saturated model of K. For $A \subseteq \tilde{M}$ let $A = A_V \cup A_K$ with $A_V \subseteq V$ and $A_K \subseteq \tilde{K}$.*

Let K_A be the subfield of $\tilde{K}$ generated by the following three sets:

- A_K
- $\{\Phi(a_1, a_2) : a_1, a_2 \in A_V\}$
- *For each $n \in \omega$ the set $\{\alpha_1, \ldots \alpha_n\}$ such that there are $a_1, \ldots a_n, a \in A_V$ with $M \models \theta_n(a_1, \ldots a_n)$ and $a = \alpha_1 a_1 + \cdots + \alpha_n a_n$.*

Let $\overline{A_V}$ be the K_A-linear span of the set A_V. Let K^{alg}_A denote the (field theoretic) algebraic closure of the field K_A. Let $\overline{A^{alg}_V}$ denote the K^{alg}_A-linear span of the set A_V. Finally, let $\langle A \rangle$ denote the $\tilde{K}$-linear span of the set A_V.

By $\dim(A)$ *we mean the dimension of $\langle A \rangle$ as a $\tilde{K}$-subspace of V.*

Remark 9.2.5 *Note that it is only necessary to carry out these closure operations once. If* a_1 *,* $a_2 \in \overline{A_V}$ *then* $\Phi(a_1, a_2) \in K_A$ *and if* $a_1, \ldots a_n, a \in \overline{A_V}$ *with* $M \models \theta_n(a_1, \ldots a_n)$ *and* $a = \alpha_1 a_1 + \cdots + \alpha_n a_n$ *then* $\alpha \in K_A$ *i.e.* $K_{(\overline{A_V}, K_A)} = K_{(A_V, A_K)}$*.*

Lemma 9.2.6 *Suppose that* $\alpha \in K_A$*, then there is a* θ*-formula* $\phi(x)$*, with parameters in* A *with unique solution* α*. If* $a \in \overline{A_V}$ *then there is a* θ*-formula* $\psi(x)$*, with parameters in* A *with unique solution* a*. Thus* $(\overline{A_V}, K_A) \subseteq \theta qf$*-dcl*$(A)$*.*

Proof. If $\alpha \in A_K$ then this is obvious. If $\alpha = \Phi(a_1, a_2)$ with $a_1, a_2 \in A_V$ then α is the unique element satisfying the formula '$x = \Phi(a_1, a_2)$'.

Suppose that $a_1, \ldots a_n, a \in A_V$ with $\tilde{M} \models \theta_n(a_1, \ldots a_n)$ and $a = \alpha_1 a_1 + \cdots + \alpha_n a_n$. Note that, for example, α_1 is the unique solution to the formula '$\exists y_2, \ldots \exists y_n(x a_1 + y_2 a_2 + \cdots + y_n a_n) = a$'. However this formula is not θ-quantifier-free. If $a = \alpha_1 a_1 + \cdots + \alpha_n a_n$ then $a - \alpha_1 a_1 = \alpha_2 a_2 + \cdots + \alpha_n a_n$. Since $a_1, \ldots, a_n$ are linearly independent with respect to the big field, $\tilde{K}$, it follows that α_1 is the unique element such that the set $\{a - \alpha_1 a_1, a_2, \ldots, a_n\}$ is linearly dependent. Hence α_1 is the unique element satisfying the θ-formula $\neg\theta_n(a - x a_1, a_2, \ldots, a_n)$ and similarly for each α_i for $i = 1, \ldots, n$.

Thus, every element in the generating set of K_A is θqf-definable and hence K_A is θqf-definable.

If $a \in \overline{A_V}$ then $a = \alpha_1 a_1 + \cdots + \alpha_n a_n$ with $\alpha_i \in K_A$ and $a_i \in A_V$. Since each α_i is θqf-definable it is clear that a is θqf-definable. So $(\overline{A_V}, K_A) \subseteq \theta qf$-*dcl*$(A)$ as required. ∎

Remark 9.2.7 *Similarly* $(\overline{A_V^{alg}}, K_A^{alg}) \subseteq \theta qf$*-acl*$(A)$*.*

Proof of theorem 9.2.3. We work in some sufficiently saturated model $M = (V, \tilde{K})$ with a non-degenerate bilinear form $[\ ,\ ] : V \times V \to \tilde{K}$. The aim is to prove that if $(\bar{a}, \bar{\alpha})$ and $(\bar{b}, \bar{\beta})$ are finite tuples from M with the property that $\theta qftp(\bar{a},\bar{\alpha}) = \theta qftp(\bar{b},\bar{\beta})$ then we have $tp(\bar{a},\bar{\alpha}) = tp(\bar{b},\bar{\beta})$. Compactness then gives quantifier elimination.

Suppose, inductively, that if $(\bar{a},\bar{\alpha})$ and $(\bar{b},\bar{\beta})$ are finite tuples from M of length r, and g_0 is a partial θ-isomorphism taking $(\bar{a},\bar{\alpha})$ to $(\bar{b},\bar{\beta})$ then there is an automorphism of M extending g_0, taking $(\bar{a},\bar{\alpha})$ to $(\bar{b},\bar{\beta})$. We prove the assertion for tuples of length $r+1$.

Case 1. Suppose that $lg(\bar{a},\bar{\alpha}) = lg(\bar{b},\bar{\beta}) = r$ and add an extra vector element to each of these tuples. Assume that $\theta qftp(a,\bar{a},\bar{\alpha}) = \theta qftp(b,\bar{b},\bar{\beta})$. We aim to prove that there is some automorphism $\sigma \in Aut(M)$ taking $(a,\bar{a},\bar{\alpha})$ to $(b,\bar{b},\bar{\beta})$. By the inductive hypothesis, we can assume, without loss of generality, that $(\bar{a},\bar{\alpha}) = (\bar{b},\bar{\beta})$ and that $\theta qftp(a/\overline{A_V} \cup K_A) = \theta qftp(b/\overline{A_V} \cup K_A)$, where $A = (\bar{a},\bar{\alpha})$, and hence there is a partial isomorphism g_0 of M, fixing $\overline{A_V} \cup K_A$, and taking a to b. Then $\langle A \rangle$ is a finite dimensional $\tilde{K}$-vector space and we can choose a basis $e_1, \dots, e_n$ for $\langle A \rangle$ contained in $\overline{A_V}$. In fact, $e_1, \dots, e_n$ is a basis for $\overline{A_V}$ as a vector space over K_A: clearly $e_1, \dots, e_n$ is K_A-linearly independent, and if $a' \in \overline{A_V}$ then $a' = \eta_1 e_1 + \dots + \eta_n e_n \in \overline{A_V}$ (where $\eta_1, \dots \eta_n \in \tilde{K}$) then, essentially by the third component of the construction of K_A, it follows that $\eta_1, \dots \eta_n \in K_A$ and hence $\overline{A_V} = K_A\text{-span}(e_1, \dots, e_n)$. Case 1. then splits into two further cases.

Case 1(a) $a \notin \langle A \rangle$. Then $M \vDash \neg\chi_n(e_1, \dots, e_n, a)$ and as $\theta qftp(a/A) = \theta qftp(b/A)$, it follows that $M \vDash \neg\chi_n(e_1, \dots, e_n, b)$, i.e. $b \notin \langle A_V \rangle$. Let $tp_K(-)$ denote the type of a tuple of field sort elements with respect to the theory of the field K and similarly for $qftp_K(-)$. Note that for $a' \in \overline{A_V}$ $qftp_K([a,a']/K_A) \subseteq \theta qftp(a/A)$. Therefore:

$$qftp_K([a,a],[a,e_1],\dots,[a,e_n]/K_A) = qftp_K([b,b],[b,e_1],\dots,[b,e_n]/K_A).$$

As the language L contains an elimination set for K (in particular if K is algebraically closed then K has quantifier elimination in its natural language) we have that:

$$tp_K([a,a],[a,e_1],\dots,[a,e_n]/K_A) = tp_K([b,b],[b,e_1],\dots,[b,e_n]/K_A).$$

Let

$$\begin{aligned} K'_A &= K_A([a,a],[a,e_1],\dots,[a,e_n]) \text{ and} \\ K'_B &= K_A([b,b],[b,e_1],\dots,[b,e_n]). \end{aligned}$$

Then, since $\tilde{K}$ is sufficiently saturated and hence homogenous, there is an automorphism σ_K of $\tilde{K}$, fixing the subfield K_A, taking $[a,a]$ to $[b,b]$ and taking $[a,e_i]$ to $[b,e_i]$ for $i=1,\dots,n$. In particular σ_K takes K'_A to K'_B and fixes the common subfield K_A.

Now let $A_1 = K'_A\text{-span}(a,e_1,\dots,e_n)$ and $B_1 = K'_B\text{-span}(b,e_1,\dots,e_n)$. We aim to show that $(A_1, K'_A, \Phi\restriction_{A_1})$ and $(B_1, K'_B, \Phi\restriction_{B_1})$ are semi-similar bilinear spaces via the field isomorphism $\sigma_K\restriction_{K'_A} : K'_A \to K'_B$.

$$\begin{aligned} A_1 &= \{\lambda_0 a + \sum_{i=1}^{n} \lambda_i e_i : \lambda_i \in K'_A\} \\ B_1 &= \{\lambda_0 b + \sum_{i=1}^{n} \lambda_i e_i : \lambda_i \in K'_B\} \end{aligned}$$

and each element of A_1 (resp. B_1) has a unique representation in terms of the basis elements $a, e_1, \dots e_n$ (resp. $b, e_1, \dots, e_n$). Define $g_V : A_1 \to B_1$ by

$$g_V\,(\lambda_0 a + \sum_{i=1}^{n} \lambda_i e_i) = \sigma_K(\lambda_0) b + \sum_{i=1}^{n} \sigma_K(\lambda_i) e_i.$$

To show that g_V is a semi-similitude we must prove: for all $a_1, a_2 \in A_1$ and all $\alpha \in K'_A$

- $g_V(a_1 + a_2) = g_V(a_1) + g_V(a_2)$
- $g_V(\alpha a_1) = \sigma_K(\alpha) g_V(a_1)$
- $[g_V(a_1), g_V(a_2)] = \sigma_K([a_1, a_2])$

The first of these is trivial. For the second,

$$\begin{aligned}
g_V(\alpha a_1) &= g_V\left(\alpha\left(\lambda_0 a + \sum_{i=1}^{n}\lambda_i e_i\right)\right)\\
&= g_V\left(\alpha\lambda_0 a + \sum_{i=1}^{n}\alpha\lambda_i e_i\right)\\
&= g_V(\alpha\lambda_0 a) + \sum_{i=1}^{n} g_V(\alpha\lambda_i e_i)\\
&= \sigma_K(\alpha\lambda_0)b + \sum_{i=1}^{n}\sigma_K(\alpha\lambda_i)e_i\\
&= \sigma_K(\alpha)\,(\sigma_K(\lambda_0)b + \sum_{i=1}^{n}\sigma_K(\lambda_i)e_i)\\
&= \sigma_K(\alpha)g_V(a_1).
\end{aligned}$$

Thus g_V is semilinear with respect to the isomorphism σ_K.

To prove that g_V is a semi-similarity:

$$\begin{aligned}
[g_V(a_1), g_V(a_2)] &= \left[g_V(\lambda_0 a + \sum_{i=1}^{n}\lambda_i e_i)\,,\ g_V(\lambda_0' a + \sum_{i=1}^{n}\lambda_i' e_i)\right]\\
&= \left[\sigma_K(\lambda_0)b + \sum_{i=1}^{n}\sigma_K(\lambda_i)e_i\,,\ \sigma_K(\lambda_0')b + \sum_{i=1}^{n}\sigma_K(\lambda_i')e_i\right]\\
&= \sigma_K(\lambda_0)\,\sigma_K(\lambda_0')[b,b] + \sum_{i=1}^{n}\sigma_K(\lambda_0)\sigma_K(\lambda_i')[b,e_i]\\
&\quad + \sum_{i=1}^{n}\sigma_K(\lambda_i)\sigma_K(\lambda_0')[e_i,b] + \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_K(\lambda_i)\sigma_K(\lambda_i')[e_i,e_j]\\
&= \sigma_K(\lambda_0\lambda_0'[a,a]) + \sigma_K(\sum_{i=1}^{n}\lambda_0\lambda_i'[a,e_i])\\
&\quad +\sigma_K(\sum_{i=1}^{n}\lambda_i\lambda_0'[e_i,a]) + \sigma_K(\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_i'[e_i,e_j])\\
&= \sigma_K\left(\left[(\lambda_0 a + \sum_{i=1}^{n}\lambda_i e_i)\,,\ (\lambda_0' a + \sum_{i=1}^{n}\lambda_i' e_i)\right]\right)\\
&= \sigma_K([a_1,a_2]).
\end{aligned}$$

Hence the spaces $(A_1, K_A', \Phi\restriction_{A_1})$ and $(B_1, K_B', \Phi\restriction_{B_1})$ (which are substructures of M) are semi-similar via the semi-similarity g_V and the field isomorphism $\sigma_K\restriction_{K_A'}\colon K_A' \to K_B'$.

Let $\overline{A_1} = \langle A_1 \rangle$ and $\overline{B_1} = \langle B_1 \rangle$. Then by the results of Section 5.5 we have that $\overline{A_1} \cong \tilde{K} \otimes_{K'_A} A_1$ and $\overline{B_1} \cong \tilde{K} \otimes_{K'_B} B_1$ (since there is a unique way of extending the bilinear form from $\overline{A_1}$ to $\tilde{K} \otimes_{K'_A} A_1$ and from $\overline{B_1}$ to $\tilde{K} \otimes_{K'_B} B_1$). Since we already know that the isomorphism $\sigma_K{\restriction}_{K'_A}\colon K'_A \to K'_B$ extends to an automorphism σ_K of $\tilde{K}$ and that the structures $(A_1, K'_A, \Phi{\restriction}_{A_1})$ and $(B_1, K'_B, \Phi{\restriction}_{B_1})$ are isomorphic (as substructures of M) via g_V, the partial automorphism g_V, naturally extends to a semi-similarity $g'_V : \overline{A_1} \to \overline{B_1}$.

Then, by Witt's Theorem for Semi-similarities (Theorem 5.3.5), the semi-similarity $g'_V : \overline{A_1} \to \overline{B_1}$ (between finite dimensional $\tilde{K}$-subspaces) can be extended to a semi-similarity $\sigma_V : V \to V$ with respect to $\sigma_K \in Aut(\tilde{K})$. Thus $\sigma = (\sigma_V, \sigma_K)$ is an automorphism of $M = (V, \tilde{K})$ taking $\overline{A_1}$ to $\overline{B_1}$ and, in particular, taking a to b and fixing (A_V, K_A). Therefore $tp(a/A) = tp(b/A)$ and case 1(a) is proved.

Case 1(b) $a \in \langle A_V \rangle$. Thus $M \vDash \chi_n(e_1, \ldots, e_n, a)$ and therefore, since $\theta qftp(a/A) = \theta qftp(b/A)$, we have that $M \vDash \chi_n(e_1, \ldots, e_n, b)$, i.e. $b \in \langle A_V \rangle$.

Let $a = \lambda_1 e_1 + \cdots + \lambda_n e_n$ and $b = \mu_1 e_1 + \cdots + \mu_n e_n$. Without loss of generality we can assume that $A = (\overline{A_V}, K_A)$. By assumption there is a partial automorphism (elementary map) f_0 taking a to b and fixing A. Then by the proof of Lemma 9.2.6 each λ_i is the unique element satisfying $\neg\theta_n(a - xe_i\,,\, e_1, \ldots, \hat{e}_i, \ldots, e_n)$ and each μ_i is the unique element satisfying $\neg\theta_n(b - xe_i\,,\, e_1, \ldots, \hat{e}_i, \ldots, e_n) = f_0(\neg\theta_n(a - xe_i\,,\, e_1, \ldots, \hat{e}_i, \ldots, e_n)\,)$. Thus f_0 takes each λ_i to μ_i, and hence $\theta qftp(\lambda_1, \ldots, \lambda_n/A) = \theta qftp(\mu_1, \ldots, \mu_n/A)$. As in case 1(a) we have that

$$qftp_K(\lambda_1, \ldots, \lambda_n/K_A) = qftp_K(\mu_1, \ldots, \mu_n/K_A).$$

As L^K_θ contains amongst its atomic formulas, an elimination set for K we have that $tp_K(\lambda_1, \ldots, \lambda_n/K_A) = tp_K(\mu_1, \ldots, \mu_n/K_A)$ and hence there is an automorphism σ_K of $\tilde{K}$ fixing K_A and taking $(\lambda_1, \ldots, \lambda_n)$ to $(\mu_1, \ldots, \mu_n)$.

We can then construct an automorphism g_V of $\langle A_V \rangle$ taking a to b by defining

$$g_V(\eta_1 e_1 + \cdots + \eta_n e_n) = \sigma_K(\eta_1)e_1 + \cdots + \sigma_K(\eta_n)e_n \quad \text{for } \eta_1, \ldots, \eta_n \in \tilde{K}.$$

Then, as in case 1(a), g_V is easily seen to be a semi-similarity of $\langle A \rangle$ and thus by Witt's Theorem, extends to a semi-similarity σ_V of V. Then $\sigma = (\sigma_V, \sigma_K)$ is an automorphism of $M = (V, \tilde{K})$ taking a to b and fixing A. Hence $tp(a/A) = tp(b/A)$ and case 1(b) is proved. This completes the proof of case 1.

Case 2. Suppose that $\theta qftp(\alpha, \bar{a}, \bar{\alpha}) = \theta qftp(\beta, \bar{b}, \bar{\beta})$ with α and β field elements and $lg(\bar{a}, \bar{\alpha}) = lg(\bar{b}, \bar{\beta}) = r$. Then, using the inductive hypothesis, as in case 1, we can assume that $(\bar{a}, \bar{\alpha}) = (\bar{b}, \bar{\beta})$ and

$$\theta qftp(\alpha/A) = \theta qftp(\beta/A)$$

where $A = \overline{A_V} \cup K_A$. As in the previous cases it follows that $qftp_K(\alpha/K_A) = qftp_K(\beta/K_A)$ and hence $tp_K(\alpha/K_A) = tp_K(\beta/K_A)$. Therefore we can find an automorphism σ_K of $\tilde{K}$ fixing K_A and taking α to β. We then proceed exactly as in case 1(b) and produce an automorphism $\sigma = (\sigma_V, \sigma_K)$ of $M = (V, \tilde{K})$ taking a to b and fixing A. This completes the proof of case 2.

Thus, the induction goes through in all cases and so for any finite tuples $(\bar{a}, \bar{\alpha})$, $(\bar{b}, \bar{\beta})$ from M :

$$\theta qftp(\bar{a}, \bar{\alpha}) = \theta qftp(\bar{b}, \bar{\beta}) \Rightarrow tp(\bar{a}, \bar{\alpha}) = tp(\bar{b}, \bar{\beta}).$$

Together with compactness (see Proposition 9.2.8) this is enough for quantifier elimination for the theory T in the language L_θ^K and hence Theorem 9.2.3 is proved.

Since we could not find it anywhere explicitly in the literature, we give a proof that if a theory has the property that the equality of two quantifier free types implies the equality of the full type then the theory has elimination of quantifiers.

Proposition 9.2.8 *Let T be an L-theory and let Δ be some set of L-formulas closed under boolean combinations. Suppose that for all finite tuples $\bar{a}$ and $\bar{b}$ (in some sufficiently saturated model $M \models T$) we have that:*

$$\Delta tp(\bar{a}) = \Delta tp(\bar{b}) \Rightarrow tp(\bar{a}) = tp(\bar{b}).$$

Then, given any L-formula $\phi(\bar{x})$ there is a Δ-formula $\psi(\bar{x})$ such that $T \models \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.

Proof. Suppose, for a contradiction, that there is some L-formula $\phi(\bar{x})$ such that for all Δ-formulas $\psi(\bar{x})$ $T \models \neg\forall \bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.

Claim. Given any Δ-formulas $\psi_1, \dots \psi_n$ there exist tuples $\bar{a}$ and $\bar{b}$ such that:

$$T \models \phi(\bar{a}) \wedge \neg\phi(\bar{b}) \wedge \bigwedge_{i=1}^{n} \left(\psi_i(\bar{a}) \leftrightarrow \psi_i(\bar{b})\right).$$

Proof of claim. For $\alpha \in {}^n 2$ let $\chi_\alpha := \left(\bigwedge_{\alpha(i)=1} \psi_i(\bar{x})\right) \wedge \left(\bigwedge_{\alpha(i)=0} \neg\psi_i(\bar{x})\right)$. Then

1. $T \models \forall \bar{x} \bigvee_{\alpha \in {}^n 2} \chi_\alpha(\bar{x})$.

2. If $\alpha \neq \beta$ then $T \models \forall \bar{x} \neg \left(\chi_\alpha(\bar{x}) \leftrightarrow \chi_\beta(\bar{x})\right)$

3. Each χ_α is a Δ-formula.

By assumption ϕ is not equivalent to a Δ-formula and hence it cannot be equivalent to any union of the χ_α's (or to $\top$ or $\bot$). Thus, for some $\alpha \in {}^n 2$ we have $\chi_\alpha \cap \phi \neq \emptyset$ and $\chi_\alpha \cap \neg\phi \neq \emptyset$. Hence we can find $\bar{a}$ and $\bar{b}$ such that $\bar{a}, \bar{b} \in \chi_\alpha$, $\bar{a} \in \phi$ and $\bar{b} \notin \phi$. But $\bar{a}, \bar{b} \in \chi_\alpha \Rightarrow \bigwedge_{i=1}^{n} \left(\psi_i(\bar{a}) \leftrightarrow \psi_i(\bar{b})\right)$. Thus $T \models \phi(\bar{a}) \wedge \neg\phi(\bar{b}) \wedge \bigwedge_{i=1}^{n} \left(\psi_i(\bar{a}) \leftrightarrow \psi_i(\bar{b})\right)$ and the claim holds.

By the claim and compactness the set:

$$\Gamma(\bar{x}, \bar{y}) = \{\phi(\bar{x}) \wedge \neg\phi(\bar{y})\} \cup \{\psi_i(\bar{x}) \leftrightarrow \psi_i(\bar{y}) : \psi_i \in \Delta\}$$

is consistent. Therefore we can find tuples $\bar{a}$ and $\bar{b}$ such that $T \models \phi(\bar{a}) \wedge \neg\phi(\bar{b})$ and $T \models (\psi_i(\bar{a}) \leftrightarrow \psi_i(\bar{b}))$ for all $\psi_i \in \Delta$. Hence $\Delta tp(\bar{a}) = \Delta tp(\bar{b})$ but $tp(\bar{a}) \neq tp(\bar{b})$ contradicting the original assumption. ■

Corollary 9.2.9 *Let K be the complete theory of a field and $m \in \mathbb{N} \cup \{\infty\}$. Then*

1. *The theory ${}_AT^K_m$ is complete.*
2. *If K is a field with square roots for each element then the theory ${}_ST^K_m$ is complete.*

That is, any good theory, *in the sense of Definition 9.1.3 (6) is complete.*

Proof. Let T be the theory described in (1) or (2). Since T is countable it is sufficient to prove (by the downward Löwenheim-Skolem Theorem) that if M_1 and M_2 are countable models of T then there is some model $N \models T$ and elementary embeddings $f_1 : M_1 \preccurlyeq N$ and $f_2 : M_2 \preccurlyeq N$. So let $M_1 = (V_1, K_1)$ and $M_2 = (V_2, K_2)$ be countable models of T with bilinear forms Φ_1 and Φ_2 respectively. By assumption $K_1 \equiv K_2$ and hence there is some field $F \equiv K_1 \equiv K_2$ with elementary embeddings $\sigma_1 : K_1 \preccurlyeq F$ and $\sigma_2 : K_2 \preccurlyeq F$. (Thus if K_1 and K_2 are fields with square roots then so is F.) F is a field extension of both K_1 and K_2. Let $V_1' = F \otimes_{K_1} V_1$ and let $V_2' = F \otimes_{K_2} V_2$. Then by Section 5.5 (for $i = 1, 2$) V_i' is an F-vector space naturally extending V_i ($V_i' = F$-span(V_i) and so $\dim(V_i'/F) = \dim(V_i/K_i)$) and the bilinear form $\Phi_i : V_i \times V_i \to K_i$ extends uniquely to a bilinear form $\Phi_i' : V_i' \times V_i' \to F$. Note that the forms Φ_i' are alternating (resp. symmetric) if and only if the forms Φ_i are. Hence each of the structures (V_1', F, Φ_1') and (V_2', F, Φ_2') are also models of T.

We can now apply Proposition 9.1.5 to the countable dimensional F-spaces (V_1', Φ_1') and (V_2', Φ_2') to obtain an isometry $\varphi : V_1' \to V_2'$. So let N be the structure (V_1', F) with bilinear form Φ_1' which is well defined up to isomorphism. Choose an orthonormal basis $(e_i : i \in I)$ (or normal basis in the alternating case) for V_i. Then $(e_i : i \in I)$ is also an orthonormal (or normal) basis for V_i'. Thus, by quantifier elimination, the embedding of $M_i = (V_i, K_i)$ into $N = (V_i', \Phi_i')$ fixing the basis $(e_i : i \in I)$ is elementary (it preserves linear independence and hence preserves θ-formulas) and the proof is complete. ∎

Suppose that T is any good theory of bilinear forms. Then T is complete and the countable models of T are in a one-to-one correspondence with the theory of a field. In [64] it is proved that theories of fields satisfy Vaught's conjecture. That is, if T_0 is the theory of field then (without assuming the continuum hypothesis) the number of countable models of T_0 is either $\leq \aleph_0$ or is equal to $2^{\aleph_0}$. This is proved by noting that if T_0 has less that $2^{\aleph_0}$ countable models then T_0 is small, and then showing that the only small fields are the finite fields and the algebraically closed fields which are known to satisfy Vaught's conjecture. It then follows immediately that if T is any good theory of bilinear forms then T satisfies Vaught's conjecture and has less that the continuum number of countable models if and only if T is a theory of bilinear forms over a finite or algebraically closed fields.

9.3 Consequences of Quantifier Elimination

In this section T is a theory of the form ${}_FT^K$ where either F is an alternating form or F is a symmetric form and K is a field with square roots. By Corollary 9.2.9, any extension of T which specifies the dimension of its models is complete. We usually assume that K is an algebraically closed field (of some fixed characteristic). We work sometimes in the language L, and sometimes in the language L_θ.

Definition 9.3.1 *Let Θ_n be the sentence $\Theta_n := \exists X_1, \dots, \exists X_n\, \theta_n(X_1, \dots, X_n)$.*

The formula Θ_n says 'there exist n linearly independent elements'. Note that for $m \geq n$, $T \models \Theta_m \rightarrow \Theta_n$. Then, for $m \in N$, $T_m = T \cup \{\Theta_m \wedge \neg\Theta_{m+1}\}$ and $T_\infty = T \cup \{\Theta_m : m \in \omega\}$. As mentioned above, for each $m \in \mathbb{N} \cup \{\infty\}$, T_m is a complete theory. In particular note that $T_\infty = \bigcup_{m<\omega} \left(\bigcap_{m \leq n < \omega} T_n\right)$.

Remark 9.3.2 *In the case where T is a theory of an alternating form and $m < \omega$, then T_m only has models when m is even, since any finite dimen-*

sional, non-degenerate, alternating space is even dimensional. Therefore, in the alternating case, by T_m (for $m < \omega$) we really mean T_{2m}.

Proposition 9.3.3 *1. For any L-sentence, σ, there is $R \in \omega$ such that $T_\infty \models \sigma \iff T_m \models \sigma$ for all $m \geq R$.*

2. Given any L-formula ψ, there is a θ-formula ϕ and some $R < \omega$, such that $T_m \models \phi \leftrightarrow \psi$ for all $m \geq R$. (In particular $T_\infty \models \phi \leftrightarrow \psi$.)

3. Every L-formula is equivalent, modulo T, to a boolean combination of θ-formulas and sentences Θ_n.

Proof. (1) ($\Rightarrow$) Suppose $T_\infty \models \sigma$. Then, by compactness, since $T_\infty = T \cup \{\Theta_m : m \in \omega\}$, there is $R < \omega$ such that $T \cup \{\Theta_R\} \models \sigma$. If $m \geq R$ then $T_m \models \Theta_R$ and hence $T_m \models \sigma$.

($\Leftarrow$) Suppose that $T_m \models \sigma$ for all $m \geq R$. Suppose for a contradiction, that $T_\infty \not\models \sigma$. Then, as T_∞ is complete, $T_\infty \models \neg\sigma$. Hence by ($\Rightarrow$) there is $S < \omega$ such that $T_m \models \neg\sigma$ for all $m \geq S$. This contradicts the consistency of the T_m.

(2) Let $\psi(x)$ be any L-formula. By quantifier elimination in T_∞ there is a θ-formula $\phi_\infty(x)$ such that $T_\infty \models \forall x(\psi(x) \leftrightarrow \phi_\infty(x))$. Let σ be the L-sentence, $\sigma := \forall x(\psi(x) \leftrightarrow \phi_\infty(x))$. Then by (1) there is some $R < \omega$ such that $T_m \models \forall x(\psi(x) \leftrightarrow \phi_\infty(x))$ for all $m \geq R$.

(3) Let $\psi(x)$ and $\phi_\infty(x)$ be as in the proof of (2). Then $T \cup \{\Theta_R\} \models \forall x(\psi(x) \leftrightarrow \phi_\infty(x))$. By quantifier elimination in each T_m, for each $m < R$, there is some θ-formula ϕ_m, such that $T_m \models \forall x(\psi(x) \leftrightarrow \phi_m(x))$. Thus $T \cup \{\Theta_m \wedge \neg\Theta_{m+1}\} \models \forall x(\psi(x) \leftrightarrow \phi_m(x))$. Let $\phi(x)$ be the following formula:

$$\phi(x) := \left(\bigvee_{m=1}^{R-1} (\Theta_m \wedge \neg\Theta_{m+1} \wedge \phi_m(x)) \right) \vee (\Theta_R \wedge \phi_\infty(x)) .$$

Then $T \models \forall x(\psi(x) \leftrightarrow \phi(x))$ and ϕ is clearly of the required form. ■

Let K be an algebraically closed field and let T be the theory of K-vector spaces with bilinear forms (either symmetric or alternating). For any $m \in$

$\mathbb{N} \cup \{\infty\}$ the theory T_m is complete and has elimination of quantifiers in the language L_θ. Hence if M and N are any two models of T_m with $M \subseteq N$ (considered as L_θ-structures) then by quantifier elimination it follows that M is an elementary substructure of N, so T_m considered as an L_θ-theory is model complete. Here we prove that the theory T_m has the stronger property of being model complete in the language L.

Proposition 9.3.4 *Let $T = {}_F T_m^K$ where K is an algebraically closed field (of some fixed characteristic). Then T is model complete in the language L.*

Proof. By the above remarks it is sufficient to prove that if M and N are models of T and M is an L-substructure of N then, M is, in fact, an L_θ-substructure N. i.e. $M \subseteq_L N \Rightarrow M \subseteq_{L_\theta} N$.

The crucial point is that, by definition, each formula θ_n is an $\forall_1$-formula (in the language L) and therefore is preserved downwards (in L-substructures). However, by using properties of the bilinear form, we can also express each formula θ_n as an existential ($\exists_1$-) L-formula and hence θ_n is preserved upwards and this will give the desired result.

Recall that for $n \in \omega$, $\chi_n(X_1, \ldots, X_n, Z)$ is the formula '$\exists y_1, \ldots, \exists y_n(y_1 X_1 + \cdots + y_n X_n = Z)$'. Given $M = (V, K) \models T$ and any $a_1, \ldots a_n, b \in V$, $M \models \chi_n(a_1, \ldots, a_n, b)$ if and only if b is in the K-linear span of the set $\{a_1, \ldots a_n\}$. But, since the bilinear form is non-degenerate, $b \in K\text{-span}(a_1, \ldots, a_n)$ if and only if $\langle a_1, \ldots, a_n\rangle^\perp \subseteq \langle b\rangle^\perp$ and hence:

$$T \models \chi_n(X_1, \ldots, X_n, Z) \leftrightarrow \forall Z \left(\bigwedge_{i=1}^{n} Z \perp X_i \rightarrow Z \perp Y \right).$$

Write $\chi_{n-1}(X_1, \ldots, \hat{X}_i, \ldots, X_n, Z)$ for the formula
$\chi_{n-1}(X_1, \ldots, X_{i-1}, X_{i+1} \ldots, X_n, Z)$. Let $M \models T$ and $a_1, \ldots a_n \in M$. Then :

$$M \models \neg\theta_n(a_1, \ldots, a_n) \iff \text{the set } \{a_1, \ldots, a_n\} \text{ is linearly dependent } \iff M \models \bigvee_{i=1}^{n} \chi_{n-1}(a_1, \ldots, \hat{a}_i, \ldots, a_n, a_i).$$

Therefore:

$$M \models \forall X_1, \dots, \forall X_n[\neg\theta_n(X_1, \dots, X_n) \leftrightarrow \bigvee_{i=1}^{n} \chi_{n-1}(X_1, \dots, \hat{X}_i, \dots, X_n, X_i)]$$

Now, let $M \subseteq_L N$ be models of T and let $a_1, \dots a_n \in M$. If $N \models \theta_n(a_1, \dots, a_n)$ then since θ_n is, by definition, a universal formula, we have $M \models \theta_n(a_1, \dots, a_n)$. Conversely, suppose that $N \models \neg\theta_n(a_1, \dots, a_n)$. Thus $N \models \bigvee_{i=1}^{n} \chi_{n-1}(a_1, \dots, \hat{a}_i, \dots, a_n, a_i)$. Therefore, for some $i = 1, \dots, n$, $N \models \chi_{n-1}(a_1, \dots, \hat{a}_i, \dots, a_n, a_i)$ and then $M \models \chi_{n-1}(a_1, \dots, \hat{a}_i, \dots, a_n, a_i)$ as χ_{n-1} is also a universal formula. Hence $M \models \neg\theta_n(a_1, \dots, a_n)$.

Thus, if $\phi(\bar{X}, \bar{x})$ is any θ-formula, $M \subseteq_L N$ are models of T, and $\bar{a}, \bar{\alpha} \in M$, then $M \models \phi(\bar{a}, \bar{\alpha}) \iff N \models \phi(\bar{a}, \bar{\alpha})$ and since any formula is equivalent (modulo T) to a θ-formula we have $M \preccurlyeq N$. Therefore T is model complete. ■

Note that the proof of Proposition 9.3.4 shows that if T is the theory of vector spaces over algebraically closed fields with bilinear forms (so T does not specify the dimension of the vector space) then if $M \models T$ and N is any L-substructure of M then in fact N is an L_θ-substructure of M.

Corollary 9.3.5 *Assume the bilinear forms are either all symmetric or all alternating. Let T^{ACF} be the theory of vector spaces (of any dimension) over algebraically closed fields (of any characteristic) with non-degenerate bilinear forms. Then:*

1. *For each $m \in \mathbb{N} \cup \{\infty\}$ the theory T_m^{ACF} is model complete in the language L.*

2. *For any field K and for any language L' containing L such that $Th(K)$ is model complete in L', then T_∞^K is the model companion of T^K. In particular T_∞^{ACF} is the model companion of T^{ACF} .*

3. T_∞^{ACF} is the model companion of the theory T' of spaces (of any dimension) over an infinite field with a non-degenerate bilinear form.

Proof. (1) is clear, since if $M \subseteq N$ are models of T^{ACF} then M and N must be bilinear spaces over algebraically closed fields of the same characteristic and the result follows from Proposition 9.3.4.

(2) We prove the case when K is an algebraically closed field of some fixed characteristic. By Proposition 9.3.4 T_∞^K is model complete. Clearly any model of T_∞^K is a model of T^K. Conversely, any model of T^K can easily be embedded in a model of T_∞^K. Let $M = (V, K)$ be any model of T^K. If V is infinite dimensional then M is already a model of T_∞^K. Otherwise let $n = \dim(V)$. Let $(e_i : i \leq n)$ be an orthonormal basis for V. Let W be an infinite dimensional K-bilinear space with orthonormal basis $(e_i' : i \in I)$. Then (W, K) is a model of T_∞^K. The map $\varphi : V \to W$ given by $\varphi(e_i) = e_i'$ and extending by K-linearity is clearly an embedding of M into (W, K).

(3) By (1) T_∞^{ACF} is model complete. Any model of T_∞^K is a model of T'. Let $(V, K) \models T'$. Let F be the algebraic closure of the field K. Let $W = F \otimes_K V$. Then by the results of Section 5.5 there is a unique F-bilinear form on W extending the form on V. Hence $N = (W, F)$ is a model of T^{ACF} and $M = (V, K)$ embeds in N. Then by the proof of (2), N embeds in a model of T_∞^{ACF}. Hence M embeds in this model also and so T_∞^{ACF} is the model companion of T'. ■

9.4 Relative Categoricity

At this point it is useful to introduce the notion of (ω, ω)-categoricity defined in [32]. Recall from Proposition 9.1.5 that if $M_1 = (V_1, K_1)$ and $M_2 = (V_2, K_2)$ are bilinear spaces with $\dim(V_1/K_1) = \dim(V_2/K_2) \leq \aleph_0$ and either the forms on V_1 and V_2 are both alternating, or the forms are both symmetric and the fields K_1 and K_2 have square roots, then $K_1 \cong K_2 \Rightarrow M_1 \cong M_2$. Therefore, if T is a *good theory* as described in (6) of Definition 9.1.3, then, not only is T complete, but any countable model of T is determined up to isomorphism by its field part. This is clearly some kind of weakening of the notion of $\aleph_0$-categoricity.

Definition 9.4.1 *Let $L^- \subseteq L$ be first order languages with P a unary predicate in $L \backslash L^-$. Let T be a complete L-theory such that whenever $M \models T$ then $P(M)$ is a substructure of the reduct $M \restriction L^-$. Then T is said to be (ω, ω)-*categorical *(relative to P) if whenever M, N are countable models of T and $P(M)$ and $P(N)$ are isomorphic as L^--structures, then M and N are isomorphic as L-structures.*

Thus a theory T is (ω, ω)-categorical if its countable models can be classified up to isomorphism by classifying the structures $P(M)$, for M a countable model of T.

Remark 9.4.2 *The above is not quite the definition of (ω, ω)-categoricity given in [32]. The definition there says that T is (ω, ω)-categorical if whenever M and N are countable models of T with $P(M) = P(N)$ then M and N are isomorphic. However these two definitions clearly amount to the same thing.*

Theorem 9.4.3 *[32] Let L, L^-, P, T be as in Definition 9.4.1. Suppose that T is (ω, ω)-categorical. Then:*

1. *T is* ***reduced*** *over P: Given any formula $\phi(\bar{x}) \in L$, there is a formula $\phi^*(\bar{x}) \in L^-$ such that for any $M \models T$ and any $\bar{a} \in P(M)$, $M \models \phi(\bar{a}) \iff P(M) \models \phi^*(\bar{a})$. (The map $\phi \mapsto \phi^*$ is called a* reduction map.*)*

2. *T is* ***atomic*** *over P: Given any model $M \models T$, and any $\bar{b} \in M$, there is a formula $\psi(\bar{x}) \in L$ with parameters from $P(M)$ such that if $\phi(\bar{x})$ is any formula from L with parameters from $P(M)$ then $M \models \phi(\bar{b}) \iff M \models \forall \bar{x}(\psi(\bar{x}) \to \phi(\bar{x}))$. (i.e. $tp(\bar{b}/P(M))$ is isolated)*

3. *Let M be a countable model of T, and let $\bar{b}, \bar{c}$ be tuples from M such that $tp(\bar{b}/P(M)) = tp(\bar{c}/P(M))$. Then there is an automorphism of M, fixing $P(M)$ pointwise and taking $\bar{b}$ to $\bar{c}$.*

4. *Let M be a countable model of T, and let $X \subseteq P(M)$. Then $acl_{P(M)}(X) = P(M) \cap acl_M(X)$.*

Proof. See [32]. ■

In fact it is also proved in [32] that, for a theory T, conditions (1) and (2) of Theorem 9.4.3 are equivalent to T being (ω, ω)-categorical. That is T is (ω, ω)-categorical (relative to P) if and only if T is reduced (over P) and atomic (over P).

Suppose that T is a *good* theory of (two-sorted) vector spaces with bilinear forms in the language L as given in Definition 9.1.3. Take P to be a predicate picking out the K-sort elements. (P is not formally part of the language L, but we can consider L as a one-sorted language with predicates distinguishing the field elements and vector elements and all the results discussed so far apply to this one-sorted language.) Let L^- be the language of rings, which is clearly contained in L. Therefore, by Proposition 9.1.5, (since any countable model obviously has countable dimension) T is (ω, ω)-categorical. Then 9.4.3 says that if $M \models T$ the field sort elements interact relatively weakly with the rest of the structure. The following is just a restatement of Theorem 9.4.3 in the case that T is a good theory of vector spaces with bilinear forms.

Proposition 9.4.4 *Let T be a good theory of vector spaces with bilinear forms.*

1. *If $\phi(\bar{x})$ is any L-formula with field element variables then there is a formula $\phi^*(\bar{x})$ in the language of rings such that for any model $M = (V, K)$ of T, and any tuple $\bar{\alpha} \in K$, $M \models \phi(\bar{\alpha}) \iff K \models \phi^*(\bar{\alpha})$.*

2. *If $M = (V, K) \models T$ and $\bar{b} \in M$ then $tp(\bar{b}/K)$ is isolated.*

3. *If $M = (V, K)$ is a countable model of T and $\bar{b}, \bar{c} \in M$, then if $tp(\bar{b}/K) = tp(\bar{c}/K)$ there is an automorphism of M, fixing K pointwise and taking $\bar{b}$ to $\bar{c}$.*

4. *If $M = (V, K)$ is a countable model of T and $A \subseteq K$ then $acl_K(A) = K \cap acl_M(A)$.*

9.5 Algebraic and Definable Closure

Let T be the complete theory of a two sorted vector space with a bilinear form over an algebraically closed field and let M be some sufficiently saturated model of T. We aim to show that, in almost all circumstances, the definable and algebraic closures of a set $A \subset M$ are given by the sets $(\overline{A_V}, K_A)$ and $(\overline{A_V^{alg}}, K_A^{alg})$ respectively, as described in 9.2.4.

In the notation given in 9.2.4 the field K_A was defined as the field generated by certain sets definable from the set A. As the underlying field $\tilde{K}$, of M is algebraically closed, the model theoretic definable closure of K_A is exactly K_A if $Char(\tilde{K}) = 0$ and is $(K_A)_{ins}$ (the smallest perfect field containing K_A) if $Char(\tilde{K}) > 0$. Since the (algebraically closed) field $\tilde{K}$ has elimination of quantifiers, the elements of the field $(K_A)_{ins}$ are still quantifier free definable from A. From now on we redefine K_A to be the model theoretic definable closure of the previous definition of K_A. By the preceding remarks this is relatively harmless. (K_A is replaced by $(K_A)_{ins}$ in the finite characteristic case and otherwise remains the same.) We could also redefine K_A^{alg} to be the model theoretic algebraic closure of the previous definition of K_A^{alg} but this is unnecessary, since in an algebraically closed field $\tilde{K}$, the model theoretic and field theoretic algebraic closures of a subfield K_A of $\tilde{K}$ coincide and hence K_A^{alg} is equal to the model theoretic algebraic closure of (either of the definitions of) K_A. (See [55] for details of these points.)

Proposition 9.5.1 *Let T be the complete theory of a two sorted vector space (of dimension m) over an algebraically closed field, with a bilinear form, where $m \in \mathbb{N} \cup \{\infty\}$. Let $M = (V, \tilde{K})$ be some sufficiently saturated model of T. Let $A \subseteq M$. Then:*

1. *$dcl(A) = (\overline{A_V}, K_A)$, and*

2. *except in the case that m is finite and $\dim(A) = m - 1$, $acl(A) = (\overline{A_V^{alg}}, K_A^{alg})$.*

Proof. By Lemma 9.2.6 we have that $(\overline{A_V}, K_A) \subseteq (\theta qf\text{-})\, dcl(A)$ and $(\overline{A_V^{alg}}, K_A^{alg}) \subseteq (\theta qf\text{-})\, acl(A)$. Hence it is sufficient to prove that for $c \notin (\overline{A_V}, K_A)$ (respectively $c \notin (\overline{A_V^{alg}}, K_A^{alg})$) there are at least two (resp. infinitely many) distinct A-conjugates of c in M.

Case 1 $c \in \tilde{K}$. Suppose $c \in dcl(A)$. Then there is some formula $\phi(x) \in L_A$ with unique solution c. Then by (ω, ω)-categoricity there is a reduction map as in 9.4.3 (1) and hence a formula $\phi^*(x)$ in the language of rings (with parameters from A) with unique solution c. Thus $c \in dcl_K(K_A)$ i.e. $c \in K_A$. Similarly if $c \in \tilde{K}$ and $c \in acl(A)$ then $c \in K_A^{alg}$.

Case 2 c is a vector element and $c \in \langle A \rangle$. Let $c = \lambda_1 a_1 + \cdots + \lambda_n a_n$ where $a_1, \ldots, a_n \in A_V$ and are $\tilde{K}$-linearly independent. Suppose $c \notin \overline{A_V}$. Then for some $i = 1, \ldots, n$, $\lambda_i \notin K_A$. Without loss of generality $i = 1$ and $\lambda_i \in K_A$ for $i = 2, \ldots, n$. Since $\lambda_1 \notin dcl(K_A)$, we can choose $\lambda_1' \in \tilde{K}$ such that $tp_K(\lambda_1'/K_A) = tp_K(\lambda_1/K_A)$ and $\lambda_1' \neq \lambda_1$. Put $c' = \lambda_1' a_1 + \cdots + \lambda_n a_n$. By quantifier elimination $tp(c/A) = tp(c'/A)$ and hence $c \notin dcl(A)$. Similarly, if $c \in \langle A \rangle$ and $c \notin A_V^{alg}$ we can assume, without loss of generality, that $\lambda_1 \notin K_A^{alg}$ and hence there are infinitely many conjugates for λ_1 over K_A^{alg} and so we can find infinitely many A-conjugates for c. Thus $c \notin \overline{A_V^{alg}}$.

Case 3(a) $c \notin \langle A \rangle$ and V is infinite dimensional over $\tilde{K}$. We show that $c \notin acl(\langle A \rangle, \tilde{K})$. Fix a $\tilde{K}$-basis $\{a_i : i \in I\}$ for $\langle A \rangle$. We aim to find distinct $(c_j : j \in \omega)$ such that $c_j \notin \langle A \rangle$, $[c_j, a_i] = [c, a_i]$ for all $j \in \omega$ and $i \in I$ and

$[c_j, c_j] = [c, c]$ for all $j \in \omega$. The c_j's are chosen inductively with the required properties, such that $c_j \notin Span(A \cup \{c_1, \ldots, c_{j-1}\})$. This ensures that the c_j are distinct. If the bilinear form is alternating then $[c, c] = 0$ for all c, and we can easily find the required c_j by Lemma 5.2.4. If the form is symmetric the situation is slightly more complicated and is resolved by Lemma 9.5.2 below.

Case 3(b) $c \notin \langle A \rangle$ and V has finite dimension m over $\tilde{K}$. Suppose that $\dim(A) < m - 1$. Let $A = \langle A \rangle$. We aim to prove that $c \notin acl(A, \tilde{K})$. Let $A = A_0 \overset{\perp}{\oplus} A_1$ where $A_0 = A \cap A^{\perp}$, the radical of A. Let $r_1, \ldots, r_n$ be a fixed basis of A_0. Then, by Lemma 5.2.1, there exist mutually orthogonal vectors $y_1, \ldots, y_n \in A_1^{\perp}$ with $[r_i, y_j] = \delta_{ij}$ and hence

$$V = \langle r_1, y_1 \rangle \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} \langle r_n, y_n \rangle \overset{\perp}{\oplus} A_1 \overset{\perp}{\oplus} E$$

where E is some non-degenerate subspace of V. Assume that the form is a symmetric form. (The case of an alternating form is easier to deal with, as $[x, x] = 0$ for all $x \in V$.) Fix an orthonormal basis $(e_1, \ldots e_k)$ for E. (Note $m = \dim(V) = 2n + r + k$.) Fix an orthonormal basis $(a_1, \ldots, a_r)$ for A_1. Then A has basis $\{a_1, \ldots a_r, r_1, \ldots, r_n\}$.

Let $c = \sum_{i=1}^{n} \lambda_i r_i + \sum_{i=1}^{n} \mu_i y_i + \sum_{i=1}^{r} \nu_i a_i + \sum_{i=1}^{k} \eta_i e_i.$

and let $c' = \sum_{i=1}^{n} \lambda'_i r_i + \sum_{i=1}^{n} \mu'_i y_i + \sum_{i=1}^{r} \nu'_i a_i + \sum_{i=1}^{k} \eta'_i e_i$ be any element of V.

Then $[c, r_i] = \mu_i$, $[c, y_i] = \lambda_i$, $[c, a_i] = \nu_i$ and $[c, e_i] = \eta_i$. Similarly $[c', r_i] = \mu'_i$ etc. By quantifier elimination, $tp(c/A) = tp(c'/A)$ if and only if $c' \notin A$ and

$$tp_K([c', c'], [c', r_i]_{i=1}^{n}, [c', a_i]_{i=1}^{r}/K_A) = tp_K([c, c], [c, r_i]_{i=1}^{n}, [c, a_i]_{i=1}^{r}/K_A).$$

Thus $tp(c/A) = tp(c'/A)$ if and only if $c' \notin A$ and

$$tp_K(\gamma', \mu'_1, \ldots \mu'_n, \nu'_1, \ldots, \mu'_r/K_A) = tp_K(\gamma, \mu_1, \ldots \mu_n, \nu_1, \ldots, \mu_r/K_A),$$

where $\gamma = [c, c] = 2\sum_{i=1}^{n} \lambda_i \mu_i + \sum_{i=1}^{r} \nu_i^2 + \sum_{i=1}^{k} \eta_i^2$ and $\gamma' = [c', c'] = 2\sum_{i=1}^{n} \lambda'_i \mu'_i + \sum_{i=1}^{r} (\nu'_i)^2 + \sum_{i=1}^{k} (\eta'_i)^2$. As $\dim(A) < m - 1$, $n + k \geq 2$ and, by changing the values of the λ'_i and the η'_i (since there are at least two of them) and using the fact that $\tilde{K}$ is algebraically closed, it is easy to choose infinitely many distinct c' such that $tp(c/A) = tp(c'/A)$ thus proving that $c \notin acl(A)$. ■

In the case where $\dim(V) = m$ is finite and $\dim(A) = m-1$, we have that $\dim((acl(A))_V) = m$. Let $M = (V, \tilde{K})$ be a space with a symmetric bilinear form. Let $(e_1, \ldots e_m)$ be an orthonormal basis for V. Let $A = \{e_1, \ldots, e_{m-1}\}$. Consider the formula $\phi(x) := \big([x,x] = 1 \wedge \bigwedge_{i=1}^{m-1}[x, e_i] = 0\big)$. Then $\phi(x)$ is an L-formula with parameters from the set A, and the only elements of M satisfying $\phi(x)$ are e_m and $-e_m$. Thus $e_m \in acl(A)$ but $e_m \notin \langle A \rangle$. In the case of a symmetric form it is then easy to see that if A is a set of dimension $m-1$ then $acl(A)_V = \overline{A_V^{alg}} + B$ where $B = \{b \in \langle A_V \rangle^{\perp} : [b,b] \in K_A^{alg}\}$. Equivalently, $acl(A)_V = \overline{A_V^{alg}} + B'$ where $B' = \{b : [b,a] \in K_A^{alg}$ for all $a \in \overline{A_V^{alg}}$ and $[b,b] \in K_A^{alg}\}$. If the form is alternating then $acl(A)_V = \overline{A_V^{alg}} + B^*$ where $B^* = \{b : [b,a] \in K_A^{alg}$ for all $a \in \overline{A_V^{alg}}\}$. As the underlying field is algebraically closed, even in the case $\dim(A) = m-1$, we still have $dcl(A) = (\overline{A_V}, K_A)$.

Lemma 9.5.2 *Let K be any field with square roots for all its elements and let V be an infinite dimensional symmetric bilinear space over K. Let $a_1, \ldots, a_n$ be any finite set of linearly independent elements in V. Then given any $\alpha_1, \ldots, \alpha_n, \gamma \in K$ there exists some $c \in V$ such that $[c, a_i] = \alpha_i$ for $i = 1, \ldots, n$, $[c,c] = \gamma$ and $c \notin Span(a_1, \ldots, a_n)$.*

Proof. By Lemma 5.2.3 there is $d \in V$ such that $[d, a_i] = \alpha_i$ for $i = 1, \ldots, n$. Let Ad be any non-degenerate finite-dimensional subspace of V containing $(a_1, \ldots, a_n, d)$. (Such a subspace clearly exists.) Then $V = Ad \bigoplus (Ad)^{\perp}$ and hence we can choose some non-zero $c' \in (Ad)^{\perp}$ such that $[c', c'] = \beta \neq 0$. Now choose $\lambda \in K$ such that $\lambda^2 = (\gamma - [d,d])/\beta$ and put $c = \lambda c' + d$. Then $c \notin Span(a_1, \ldots, a_n)$, $[c, a_i] = [\lambda c' + d, a_i] = \lambda[c', a_i] + [d, a_i] = 0 + \alpha_i$, and $[c,c,] = [\lambda c' + d\,, \lambda c' + d] = \lambda^2[c', c'] + 2\lambda[c,d] + [d,d] = (\gamma - [d,d]) + 0 + [d,d] = \gamma$.
■

Corollary 9.5.3 *Let $T = {}_F T_\infty^K$ be any good theory of bilinear forms and suppose that $Th(K)$ has a countable saturated model (also called K). Suppose that $M = (V, K) \models T$. Then M is $\aleph_0$-saturated. In particular, if V is taken to be countable dimensional over K, then $M = (V, K)$ is a countable $\aleph_0$-saturated model and hence T is a small theory (as it possesses a countable saturated model).*

Proof. It suffices to prove that all 1-types over finite sets are realised in M. Let A be a finite set. Without loss of generality, $A = \{a_1, \ldots, a_n\} \cup K_A$ where $a_1, \ldots, a_n$ are linearly independent vector elements and K_A is a finitely generated field. Suppose that $p \in S(A)$ is the type of a vector sort element and choose some $c \in \tilde{M}$ realising p. Then, by quantifier elimination, $p = tp(c/A)$ is determined by the following data:

$$\begin{array}{ll} (1) & c \in \langle A \rangle \text{ or } c \notin \langle A \rangle \\ (2) & tp_K([c,c],[c,a_1],\ldots,[c,a_n]/K_A) \end{array}$$

By saturation of K and Lemma 9.5.2 there is $c' \in M$ satisfying p.

If p is the type of a field element then the result is immediate by saturation of K. ■

Chapter 10

Finite Dimensional Spaces with Bilinear Forms

Let L be the two sorted language given in 9.1.1 and fix (a complete theory of) an algebraically closed field K. Let T be the (two sorted) theory of vector spaces over K with symmetric bilinear forms. For $r \in \omega$, let T_r be the theory extending T expressing that the vector space has dimension r and let T_∞ be the theory extending T expressing that the vector space is infinite dimensional. Thus T_∞ and T_r for each $r \in \omega$ are complete theories. Recall that $T_\infty = \bigcup_{m<\omega} \left(\bigcap_{m \leq r < \omega} T_r\right)$.

Suppose that $M = (V, K)$ is a countable model of T_∞. Then by Corollary 5.4.3, V has an orthonormal basis $(e_i : i \in \omega)$. For $r \in \omega$, let $V_r = K$-span$(e_1, \dots e_r)$. Let N_r be the L-structure (V_r, K). Then $V = \bigcup_{r \in \omega} V_r$ and so, by identifying all the copies of K, we can consider the countable structure M as an ascending union of the structures N_r. Model theoretically, the structures N_r are much easier to study than the structure M. Each of the structures N_r is stable (in fact $\aleph_1$-categorical and hence ω-stable of finite Morley rank) whereas, by Proposition 7.4.1, the structure M does not even have a simple theory. Then, analogously to the case of smoothly approximable structures, where a countably infinite structure can be studied in terms of certain finite substructures, we can study the infinite dimensional structure M by looking at certain finite *dimensional* substructures $(N_r : r \in \omega)$. In fact the analogy with smoothly approximable structures is quite close. This idea is explored further in Chapter 12. In this chapter we study the theories T_r for $r \in \omega$: the finite dimensional spaces with bilinear forms.

10.1 The Structures N_r

Throughout this section T_0 is the complete theory of an algebraically closed field in the language of rings L_0. L is the two sorted language as in Definition 9.1.1 (so $L_0 \subseteq L$) and T_r is the L-theory of r-dimensional vector spaces over models of T_0 with bilinear (symmetric) forms.

Lemma 10.1.1 *For each $r \in \omega$ the theory T_r is $\aleph_1$-categorical.*

Proof. Let $M_1 = (V_1, K_1)$ and $M_2 = (V_2, K_2)$ be models of T_r of cardinality $\aleph_1$. Then, as $\dim(V_1/K_1) = \dim(V_2/K_2) = r$, we must have that $|K_1| = |K_2| = \aleph_1$. Since $K_1, K_2 \models T_0$ and T_0 is $\aleph_1$-categorical it follows that $K_1 \cong K_2$. Thus by Proposition 9.1.5, $M_1 \cong M_2$. ∎

Corollary 10.1.2 *The theory T_r is ω-stable of finite Morley rank.*

In some sense the theory T_r is really no more complicated than the theory T_0 (of the algebraically closed fields contained in models of T). Let $N = (V, K) \models T_r$ then the field K is obviously definable in the structure N. A partial converse to this result holds: the structure N is interpretable in the field K. However, unless we add new constants to the language for the elements of an orthonormal basis for V, the structures N and K are not bi-interpretable.

Lemma 10.1.3 *Let $N = (V, K) \models T_r$. Then the structure N interpretable in the field K (and, of course, K is interpretable in N). Let $\{e_1, \ldots, e_r\}$ be an orthonormal basis for V over K and let N' be the expansion of N obtained by* *adding $e_1, \ldots, e_r$ to the language as new constant symbols. Then the structure N' is bi-interpretable with K.*

Proof. Define a ($\emptyset$-definable) equivalence relation $\sim$ on K^{r+2} by:

$$(a_1, \ldots a_r, a_{r+1}, a_{r+2}) \sim (b_1, \ldots b_r, b_{r+1}, b_{r+2})$$
$$\iff \begin{cases} (a_1, \ldots a_r) = (b_1, \ldots b_r) \text{ and } a_{r+2} = b_{r+2} = 0 \\ \text{or } a_{r+1} = b_{r+1} \text{ and } a_{r+2}, b_{r+2} \neq 0 \end{cases}$$

Elements of the form $(a_1, \ldots a_r, a_{r+1}, 0)$ will be used to represent vector elements $(a_1, \ldots a_r)$ and elements of the form $(a_1, \ldots a_r, a_{r+1}, a_{r+2})$ with $a_{r+2} \neq 0$ will represent field elements a_{r+1}. Let $U = K^{r+2}/\sim$. We can then define L_0-definable functions and constants on U to correspond to the functions and constants on N. This is done by associating to each unnested atomic formula of L a corresponding unnested atomic formula in L_0 as described in the table below: (Recall the convention that upper case letters denote vector variable and lower case letters denote field sort variables.)

L-formula	Corresponding L_0-formula
$X = Y$	$\bigwedge_{i=1}^r (x_i = y_i)$
$X + Y = Z$	$\bigwedge_{i=1}^r (x_i + y_i = z_i)$
$xY = Z$	$\bigwedge_{i=1}^r (xy_i = z_i)$
$[X, Y] = z$	$(\sum_{i=1}^r x_i y_i) = z$
$X = 0$	$\bigwedge_{i=1}^r x_i = 0$
$x + y = z$	$x_{r+1} + y_{r+1} = z_{r+1}$
$xy = z$	$x_{r+1}\, y_{r+1} = z_{r+1}$

Formally, we should define the tuples of variables $(x_1, \ldots x_r, x_{r+1}, x_{r+2})$ corresponding to a vector variable X, to have component x_{r+2} equal to zero and a tuple $(x_1, \ldots x_r, x_{r+1}, x_{r+2})$ corresponding to a field variable to have component x_{r+2} non-zero. For example, the formula $X = Y$ should correspond to the L_0-formula $\bigwedge_{i=1}^r (x_i = y_i) \wedge x_{r+2} = 0 \wedge y_{r+2} = 0$ and $x = y$ should correspond to the formula $x_{r+1} = y_{r+1} \wedge x_{r+2} \neq 0 \wedge y_{r+2} \neq 0$.

Suppose we fix some orthonormal basis $\{e_1, \ldots, e_r\}$ for V over K. Define $f : N \to K^{r+2}/\sim$ to be the map given by:

$$f(a) = \begin{cases} ([a, e_1], \ldots, [a, e_r], 0, 0)/\sim & \text{if } a \in V \\ (0, \ldots, 0, a, \beta)/\sim & \text{for some } \beta \neq 0, \text{ if } a \in K \end{cases}$$

Clearly f is a bijection. Then, using the above correspondence between atomic L-formulas and L_0-formulas, we can see that the L-structure N and $K^{r+2}/\sim$ (with the induced structure described above) are isomorphic, and thus N is interpretable in K.

The field K is obviously interpretable (in fact definable) in the structure N via the map g taking elements of the field K to themselves considered as elements of $N = (V, K)$. However, although each of the structures N and K is interpretable in the other, they are not bi-interpretable. For bi-interpretability we need to prove that given the interpretations f of N in K and g of K in N, that the function $g \circ f$ is definable in N and $f \circ g$ is definable in K. Consider the function $g \circ f : N \to N$ where f is the interpretation of N in K as defined above. Then if $a \in V$, $g \circ f\,(a) = ([a, e_1], \ldots, [a, e_r])$ and hence if $g \circ f$ is definable in N then so is the orthonormal basis $\{e_1, \ldots, e_r\}$. Hence, without adding new constants, no fixed orthonormal basis is defined in N and so N and K are not bi-definable. However, if we expand the language L to a language L' containing constant symbols $(e_1, \ldots, e_r)$ and let N' be the expansion of N to L' with $e_1, \ldots, e_r$ interpreted as an orthonormal basis, then taking f and g to be the maps described above, the functions $g \circ f$ and $f \circ g$ are definable in N' and K respectively, and hence the structures N' and K are bi-interpretable. ■

The proof given above works for any good theory, T, of finite dimensional vector spaces with bilinear forms. If T is any good theory of symmetric forms, then any model of T has an orthonormal basis and the proof is exactly as above. If T is a theory of alternating forms then we just replace the orthonormal basis $\{e_1, \ldots e_r\}$ with a normal basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ and correspond to the

L-formula $[X,Y]=z$, the L_0-formula $(\sum_{i=1}^{n} x_i y_{n+i}) + (\sum_{i=1}^{n} x_{n+i} y_i) = z$. The proof then goes through as above.

10.2 Independence and Non-forking in N_r

Let $N = (V_r, K) \models T_r$ and let N' be an expansion of N to the language L' fixing an orthonormal basis $(e_1, \ldots, e_r)$ for V_r. (N' is clearly $\aleph_1$-categorical and hence stable.) Let $A, B, C \subseteq N'$. Then $C \downarrow_A B$ in the structure N' if and only if $C^* \downarrow_{A^*} B^*$ in K, where $A^* = A_K \cup \{[a, e_i] : a \in A_V\,,\ i = 1, \ldots r\}$ and similarly for B^* and C^*. However, if we are working in the model N where there is no fixed orthonormal basis, then it is not immediately clear how to characterise non-forking in N. Since N is a stable structure (in fact ω-stable of finite Morley rank) there is a unique notion of independence on models of N such that all types over models are stationary. Thus, by Theorem 2.3.11, if we can find some notion of independence with this property then this notion must be non-forking.

Definition 10.2.1 *Let $r \in \omega$ and let $N = (V, \tilde{K})$ be some sufficiently saturated model of T_r. Let $A \subseteq B \subset N$ and let $c \in N$. Define: $tp(c/B)$ does not Γ-fork ($dn\Gamma f$) over A if $K_{Ac} \downarrow_{K_A} K_B$ (i.e. K_{Ac} and K_B are independent over K_A in the algebraically closed field $\tilde{K}$) and one of the following four conditions hold:*

1. *$c \in \tilde{K}$ (i.e. c is a field sort element)*
2. *$c \in \langle A \rangle$*
3. *$c \notin \langle B \rangle$ and whenever $\{b_1, \ldots, b_n\}$ is a linearly independent set in $B_V \cap (V \backslash \langle A \rangle)$ then the set $\{[c, b_1], \ldots, [c, b_n]\}$ is algebraically independent over the field $K_B(K_{Ac})$.*

4. $c \in \langle B\rangle \backslash \langle A\rangle$ and $\langle B\rangle = V$ and whenever $\{b_1, \ldots, b_n\}$ is a linearly independent set in $B_V \cap (V\backslash\langle A\rangle)$ then the set $\{[c, b_1], \ldots, [c, b_n]\}$ is algebraically independent over the field $K_B(K_{Ac})$.

We use the notation '$[c, B]$ is Φ-independent *over A' to mean that whenever $\{b_1, \ldots, b_n\}$ is a linearly independent set in $B_V \cap (V\backslash\langle A\rangle)$ then the set $\{[c, b_1], \ldots, [c, b_n]\}$ is algebraically independent over the field $K_B(K_{Ac})$. For example condition (3) above reads: $c \notin \langle B\rangle$ and $[c, B]$ is* Φ-independent *over A.*

This only defines Γ-non-forking for 1-types, but we can extend the definition to n-types by inductively defining: $tp(c_1, \ldots, c_n, c_{n+1}/B)$ $dn\Gamma f$ over A if and only if $tp(c_1, \ldots, c_n/B)$ $dn\Gamma f$ over A and $tp(c_{n+1}/Bc_1, \ldots c_n)$ $dn\Gamma f$ over $Ac_1, \ldots, c_n$.

The notation is the same as in 9.2.4. By $K_B(K_{Ac})$ we mean the field generated over K_B by the elements of the field K_{Ac}. Note that $K_B(K_{Ac}) = K_{Ac}(K_B)$ but that the field K_{ABc} is in general larger, since if $c \notin \langle B\rangle$ and $b \in B_V$ then $[c, b] \in K_{ABc}$ but is not necessarily in $K_B(K_{Ac})$.

We now show that Γ-non-forking coincides with the usual definition of non-forking in the theories T_r by demonstrating that Γ-non-forking is a well behaved notion of independence on models of T_r and then invoking Theorem 2.3.11. This method may appear somewhat heavy-handed given that we already know the theories T_r to be stable (even $\aleph_1$-categorical). However almost all of the proof is still valid in the case of the theory T_∞ and this fact is exploited later in Theorem 12.2.2 to show that Γ-non-forking also defines a meaningful dependence relation on models of T_∞.

Proposition 10.2.2 *Let $N \models T_r$ and let $A \subseteq B \subset N$, and $\bar{c} \in N$. Then $tp(\bar{c}/B)$ does not fork over A (in the sense of the theory T) if and only if $tp(\bar{c}/B)$ does not Γ-fork over A.*

Proof. By 2.3.11 it is sufficient to prove that Γ-non-forking is automorphism invariant, satisfies symmetry, transitivity, finite character, local character, given any type $p(\bar{x}) \in S(A)$ and any $B \supseteq A$ there is some $q(\bar{x}) \in S(B)$ extending $p(\bar{x})$ such that q $dn\Gamma f$ over A, and that if $A \models T_r$ then the extension $q(\bar{x}) \in S(B)$ such that q $dn\Gamma f$ over A is unique.

Automorphism Invariance. Suppose $tp(c/B)$ $dn\Gamma f$ over A. Let $\sigma = (\sigma_V, \sigma_K) \in Aut(M)$. We must prove that $tp(\sigma(c)/\sigma(B))$ $dn\Gamma f$ over $\sigma(A)$. Suppose $c \in \tilde{K}$. Then $K_{Ac} \downarrow_{K_A} K_B$ (in the sense of the theory of algebraically closed field). Hence $\sigma_K(K_{Ac}) \downarrow_{\sigma_K(K_A)} \sigma_K(K_B)$ as independence in algebraically closed fields is automorphism is invariant. As $\sigma = (\sigma_V, \sigma_K) \in Aut(M)$ we clearly have that $\sigma_K(K_{Ac}) = K_{\sigma(Ac)}$ etc. Hence $K_{\sigma(Ac)} \downarrow_{K_{\sigma(A)}} K_{\sigma(B)}$. Thus, by definition, $tp(\sigma(c)/\sigma(B))$ $dn\Gamma f$ over $\sigma(A)$ as required. If c is a vector element the proof is similar. It is clear that if $\{b_1, \ldots, b_n\}$ is a linearly independent set in $B_V \cap (V \backslash \langle A \rangle)$ then $\{\sigma_V(b_1), \ldots, \sigma_V(b_n)\}$ is linearly independent and contained in $\sigma_V(B_V) \cap (V \backslash \langle \sigma_V(A) \rangle)$. Also the set of field-sort elements $\{[c, b_1], \ldots, [c, b_n]\}$ is algebraically independent over the field $K_B(K_{Ac})$ if and only if $\{[\sigma_V(c), \sigma_V(b_1)], \ldots, [\sigma_V(c), \sigma_V(b_n)]\} = \{\sigma_K([c, b_1]), \ldots, \sigma_K([c, b_n])\}$ is algebraically independent over the field $\sigma_K(K_B(K_{Ac}))$.

Symmetry. Suppose that $tp(c/Ab)$ $dn\Gamma f$ over A. We must prove that $tp(b/Ac)$ $dn\Gamma f$ over A. Note first that $K_{Ac} \downarrow_{K_A} K_{Ab}$ and hence as independence in algebraically closed fields is symmetric we have that $K_{Ab} \downarrow_{K_A} K_{Ac}$. Suppose that $b \in \tilde{K}$ or $b \in \langle A \rangle$. Then, by definition of Γ-non-forking, we have that $tp(b/Ac)$ $dn\Gamma f$ over A. So suppose $b \notin \langle A \rangle$. If $c \in \langle A \rangle$ then $[b, Ac]$ is Φ-independent over A, as $\langle A \rangle = \langle Ac \rangle$. If $c \in \langle Ab \rangle \backslash \langle A \rangle$ then $b \in \langle Ac \rangle \backslash \langle A \rangle$. Hence $tp(c/Ab)$ $dn\Gamma f$ over A by condition (4) and so $\langle Ab \rangle = V$ and $[c, Ab]$ is Φ-independent over A. Therefore $\langle Ac \rangle = V$ and $[b, c] \notin K_{Ac}(K_{Ab})^{alg}$ and so $tp(b/Ac)$ $dn\Gamma f$ over A (also by condition (4)). If $c \notin \langle Ab \rangle$ then (assuming $b \notin \langle A \rangle$) we have $b \notin \langle Ac \rangle$ and as in the previous case $[b, c] \notin K_{Ac}(K_{Ab})^{alg}$ and $tp(b/Ac)$ $dn\Gamma f$ over A. This exhausts all possible cases and shows that Γ-non-forking is symmetric is symmetric on pairs of single elements over a given set. We show full symmetry after proving transitivity.

Transitivity. We must prove that for any tuples $\bar{b}$, $\bar{c}$, $\bar{d}$ and any set A,

$$\bar{d} \downarrow^*_A A\bar{b}\bar{c} \iff \bar{d} \downarrow^*_{A\bar{b}} A\bar{b}\bar{c} \text{ and } \bar{d} \downarrow^*_A A\bar{b}.$$

We first claim that for any tuple $\bar{b}$ and for d and c single elements we have:

$$d \downarrow^*_A A\bar{b}c \iff d \downarrow^*_{A\bar{b}} A\bar{b}c \text{ and } d \downarrow^*_A A\bar{b}.$$

The proof of the claim is by induction on $\lg(\bar{b})$. The base case where $\lg(\bar{b}) = 0$ and so $\bar{b} = \emptyset$ is trivial. Suppose the claim holds for all $\bar{b}$ with $\lg(\bar{b}) \leq n$. Let $\bar{b}$ be any tuple with $\lg(\bar{b}) \leq n$ and let b', c and d be any single elements with A any set. We must prove:

$$d \downarrow^*_A A\bar{b}b'c \iff d \downarrow^*_{A\bar{b}b'} A\bar{b}b'c \text{ and } d \downarrow^*_A A\bar{b}b'.$$

Without loss of generality we may assume that d is a vector element such that $d \notin \langle A\bar{b}b'c\rangle$. Otherwise we need only consider the algebraically closed field part of the definition of independence which we already know to be transitive. (There is also the case to consider where $d \notin \langle A\rangle$, $d \in \langle A\bar{b}b'c\rangle$ and $\langle A\bar{b}b'c\rangle = \langle V\rangle$ but this is virtually identical to the case $d \notin \langle A\bar{b}b'c\rangle$.)

($\Rightarrow$) We must prove $d \downarrow^*_A A\bar{b}b'c \Rightarrow d \downarrow^*_{A\bar{b}b'} A\bar{b}b'c$ and $d \downarrow^*_A A\bar{b}b'$. Since $d \downarrow^*_A A\bar{b}b'c$ we have that $K_{Ad} \downarrow_{K_A} K_{A\bar{b}b'c}$ and whenever $b_1, \ldots b_r$ are linearly independent elements from $\bar{b}b'c$ not contained in $\langle A\rangle$ then $\{[d, b_1], \ldots, [d, b_r]\}$ is algebraically independent over the field $K_{A\bar{b}b'c}(K_{Ad})$.

To simplify notation let $B = A\bar{b}b'$.

In order to prove $d \downarrow^*_{A\bar{b}b'} A\bar{b}b'c$ (i.e. $d \downarrow^*_B Bc$) we must show that

(i) $K_{Bd} \downarrow_{K_B} K_{Bc}$ and

(ii) $[d, c] \notin K_{Bc}(K_{Bd})^{alg}$ if $c \notin \langle B\rangle$

and in order to prove $d \downarrow^*_A A\bar{b}b$ (i.e. $d \downarrow^*_A B$) we must show that

(iii) $K_{Ad} \downarrow_{K_A} K_B$ and

(iv) $\{[d,b_1],\ldots,[d,b_k],[d,b']\}$ is algebraically independent over $K_B(K_{Ad})$ whenever $b_1,\ldots,b_k,b'$ is a linearly independent set in $\bar{b}b'\backslash\langle A\rangle$. Here we allow that possibility that $[d,b']$ does not appear in the above mentioned set, e.g. because b' is not linearly independent from A or even b' is a field element.

Given that $d \downarrow^*_A A\bar{b}b'c$, statements (iii) and (iv) are immediate from the transitivity of independence in algebraically closed fields. If c is a field element or $c \in \langle B\rangle$ then (ii) is trivial. So suppose $c \notin \langle B\rangle$ and suppose for a contradiction that $[d,c] \in K_{Bc}(K_{Bd})^{alg}$. Let $b_1,\ldots,b_m,b'$ be a maximal linearly independent set from $\bar{b}b'\backslash\langle A\rangle$ (so b' is not necessarily included in the list). Then it is clear that $K_{Bc}(K_{Bd}) = K_{Bc}(K_{Ad}([d,b_1],\ldots,[d,b_m],[d,b']))$. Thus $[d,c] \in K_{Bc}(K_{Bd})^{alg}$ implies that the set $\{[d,c],[d,b_1],\ldots,[d,b_m],[d,b']\}$ is algebraically dependent over $K_{Bc}(K_{Ad})$. which contradicts the original hypothesis, since by assumption on c the set $\{c,b_1,\ldots,b_m,b'\}$ is linearly independent over A. Hence (ii) holds.

To prove (i) we rewrite K_{Bd} as $K_B(K_{Ad}([d,b_1],\ldots,[d,b_m],[d,b']))$ where $b_1,\ldots,b_m,b'$ is a maximal set from $\bar{b}b'$ linearly independent over A. We must prove that $K_B(K_{Ad}([d,b_1],\ldots,[d,b_m],[d,b'])) \downarrow_{K_B} K_{Bc}$. By hypothesis we have that $K_{Ad} \downarrow_{K_A} K_{Bc}$ and that $\{[d,b_1],\ldots,[d,b_m],[d,b'],[d,c]\}$ is algebraically independent over $K_{Bc}(K_{Ad})$. Thus, in particular, $\{[d,b_1],\ldots,[d,b_m],[d,b']\}$ is algebraically independent over $K_{Bc}(K_{Ad})$ and hence we have that $K_B(K_{Ad}([d,b_1],\ldots,[d,b_m],[d,b'])) \downarrow_{K_B} K_{Bc}$ as required.

($\Leftarrow$) By assumption, we have that:

(i) $K_{Bd} \downarrow_{K_B} K_{Bc}$

(ii) $[d,c] \notin K_{Bc}(K_{Bd})^{alg}$ if $c \notin \langle B\rangle$

(iii) $K_{Ad} \downarrow_{K_A} K_B$

(iv) $\{[d,b_1],\ldots,[d,b_k],[d,b']\}$ is algebraically independent over $K_B(K_{Ad})$ whenever $b_1,\ldots,b_k,b'$ is a linearly independent set in $\bar{b}b'\backslash\langle A\rangle$.

We immediately have $K_{Ad} \downarrow_{K_A} K_{Bc}$ from (i), (iii) and the transitivity of independence in an algebraically closed field. We must prove that whenever

$b_1, \dots b_r$ are linearly independent elements from $\bar{b}b'c$ not contained in $\langle A \rangle$ then the set $\{[d, b_1], \dots, [d, b_r]\}$ is algebraically independent over the field $K_{Bc}(K_{Ad})$. Let $b_1, \dots, b_k, b', c$ be a subset of $\bar{b}b'c$ linearly independent over A. (Note that b' and/or c or even any of the b_i need not necessarily be included in the set, e.g. if they are field elements.) By (ii) if $c \notin \langle B \rangle$ then $[d, c] \notin K_{Bc}(K_{Bd})^{alg}$, otherwise $c \in \langle B \rangle$ in which case we only need consider linearly independent sets of $\bar{b}b'$. Suppose $c \notin \langle B \rangle$. Then if $b_1, \dots, b_k, b' \in \bar{b}b' \setminus \langle A \rangle$ we have $[d, c] \notin K_{Bc}(K_{Ad}([d, b_1], \dots, [d, b_k], [d, b']))^{alg}$. Therefore if we can prove that the set $\{[d, b_1], \dots, [d, b_k], [d, b']\}$ is algebraically independent over $K_{Bc}(K_{Ad})$ then we are done. By (iv) we know that $\{[d, b_1], \dots, [d, b_k], [d, b']\}$ is algebraically independent over $K_B(K_{Ad})$ and by (i) we know that $K_B(K_{Ad}) \downarrow_{K_B} K_{Bc}$. Thus if $[d, b_1], \dots, [d, b_k], [d, b']$ satisfy some polynomial over $K_{Bc}(K_{Ad})$ they already satisfy a polynomial over $K_B(K_{Ad})$ contradicting (iv).

We have now proved the claim.

It is then clear (by an easy induction) that if $\bar{b}$ and $\bar{c}$ are tuples and d is a single element

$$d \downarrow^*_A A\bar{b}\bar{c} \iff d \downarrow^*_{A\bar{b}} A\bar{b}\bar{c} \text{ and } d \downarrow^*_A A\bar{b}.$$

We now prove full transitivity by induction on $\lg(\bar{d})$. Suppose transitivity holds for all tuples $\bar{d}$ with $\lg(\bar{d}) \leq n$. Recall that we have defined $\{d_1, d_2\} \downarrow^*_A B$ to hold if and only if $d_1 \downarrow^*_A B$ and $d_2 \downarrow^*_{Ad_1} Bd_1$. Let $\bar{d}$ be a tuple of length n. Suppose $\bar{d}d' \downarrow^*_A C$ with $A \subseteq B \subseteq C$. By definition of Γ-independence we have $\bar{d} \downarrow^*_A C$ and $d' \downarrow^*_{A\bar{d}} C\bar{d}$. Then by inductive hypothesis we have:

(i) $\bar{d} \downarrow^*_A B$

(ii) $\bar{d} \downarrow^*_B C$

(iii) $d' \downarrow^*_{A\bar{d}} B$

(iv) $d' \downarrow^*_{B\bar{d}} C$

By definition (i) and (iiii) say that $\bar{d}d' \downarrow^*_A B$. Similarly (ii) and (iv) say that $\bar{d}d' \downarrow^*_B C$. The converse is exactly the same argument in reverse: $\bar{d}d' \downarrow^*_A B$ and $\bar{d}d' \downarrow^*_B C$ implies that $\bar{d}d' \downarrow^*_A C$. Thus induction goes through and we have full transitivity of Γ-non-forking.

Full Symmetry. Suppose inductively that for all tuples $\bar{b}$ and $\bar{c}$ with $\lg(\bar{b})$, $\lg(\bar{c}) \leq n$, and for any set A, $\bar{b} \downarrow^*_A \bar{c}$ if and only if $\bar{c} \downarrow^*_A \bar{b}$. Let $\bar{b}$ and $\bar{c}$ be tuples of length $\leq n$ and let b' be any single element. Suppose that $\bar{c} \downarrow^*_A \bar{b}b'$. Then by transitivity $\bar{c} \downarrow^*_A \bar{b}$ and $\bar{c} \downarrow^*_{A\bar{b}} b'$. By inductive hypothesis (since $\lg(\bar{b})$, $\lg(\bar{c}) \leq n$) we have $\bar{b} \downarrow^*_A \bar{c}$ and $b' \downarrow^*_{A\bar{b}} \bar{c}$. By the definition of Γ-non-forking this means exactly that $\bar{b}b' \downarrow^*_A \bar{c}$. Thus Γ-non-forking is symmetric and we can write 'B and C are Γ-independent over A'(or $B \downarrow^*_A C$) to mean that $tp(C/AB)$ $dn\Gamma f$ over A (equivalently $tp(B/AC)$ $dn\Gamma f$ over A).

Finite Character. We must prove that $tp(c/B)$ $dn\Gamma f$ over A if and only if for each finite subset $B_0 \subseteq B$, $tp(c/AB_0)$ $dn\Gamma f$ over A. Left to right is obvious. Right to left is also clear, since if $tp(c/B)$ Γ-forks over A then either $K_{Ac} \not\downarrow_{K_A} K_B$ in which case (by finite character of independence in algebraically closed fields) there is a finite $B_0 \subseteq B$ such that $K_{Ac} \not\downarrow_{K_A} K_{B_0}$ and hence $tp(c/AB_0)$ Γ-forks over A; or there are $\{b_1, \ldots, b_n\}$ linearly independent set in $B_V \cap (V \backslash \langle A \rangle)$ such that the set $\{[c, b_1], \ldots, [c, b_n]\}$ is not algebraically independent over the field $K_B(K_{Ac})$ and hence not algebraically independent over $K_{B_0}(K_{Ac})$ for some finite $B_0 \subseteq B$ thus $tp(c/AB_0b_1, \ldots, b_n)$ Γ-forks over A.

Local Character. Consider a type $tp(c/B)$. If c is a field element then local character holds because it holds in the field K. Suppose that c is a vector element. As $\dim(V)$ is finite we can choose $a_1, \ldots, a_n$ linearly independent in V such that $\langle a_1, \ldots, a_n \rangle = \langle B \rangle$. Let $A \subseteq B$ be any set such that $\{a_1, \ldots, a_n\} \subseteq A$. Then, by definition, $tp(c/B)$ $dn\Gamma f$ over A if and only if $K_{Ac} \downarrow_{K_A} K_B$. But K_{Ac} is simply the field $K_A([c, c], [c, a_1], \ldots, [c, a_n])$. Therefore it suffices to choose $A = \{a_1, \ldots, a_n\} \cup A_K$ such that $tp_K([c, c], [c, a_1], \ldots, [c, a_n]/K_B)$ does not fork over A_K (in the sense of the theory K) and this is clearly possible by the local character axiom in K. (In fact, since K is superstable, we can choose A_K and hence A to be finite sets, yielding a stronger version of local character: for every type $tp(c/B)$ there is a *finite* set A such that $tp(c/B)$ $dn\Gamma f$ over A.)

Extension. Given sets A and B and some $c \in N$ we must find an element $c' \in N$ such that $tp(c'/A) = tp(c/A)$ and $tp(c'/B)$ $dn\Gamma f$ over A. This is easy using the quantifier elimination result. For example, if c is a vector element and $c \notin \langle A \rangle$ then $tp(c'/A) = tp(c/A)$ if and only if $c' \notin \langle A \rangle$ and for some $\tilde{K}$-basis $\{a_1, \ldots, a_n\}$ for A_V

$$tp_K([c', c'], [c', a_1], \ldots, [c', a_n]/K_A) = tp_K([c, c], [c, a_1], \ldots, [c, a_n]/K_A).$$

Thus, by using the extension axiom for non-forking in algebraically closed fields, it follows that extension holds for Γ-non-forking.

Types over models. Suppose $p \in S(N)$. Let $B \supseteq N$. Let c realise p in some elementary extension $\tilde{N}$ of N. Since N is a model $c \in \langle N \rangle$. Let $\{a_1, \ldots, a_r\}$ be a basis for N. Then, by quantifier elimination, if c and c' are vector elements in some elementary extension $\tilde{N}$ of N and $B \supseteq N$, then $tp(c/B) = tp(c'/B)$ if and only if

$$tp_K([c', a_1], \ldots, [c', a_r]/K_B) = tp_K([c, a_1], \ldots, [c, a_r]/K_B).$$

But, since $\langle N \rangle = \langle B \rangle$, $tp(c/B)$ $dn\Gamma f$ over N iff $tp_K([c, a_1], \ldots, [c, a_r]/K_B)$ $dn\Gamma f$ over K_N. Since K_N is a model of the theory K which is stable, the type $tp_K([c, a_1], \ldots, [c, a_r]/K_N)$ is stationary and thus has a unique non-forking extension to K_B. Thus $tp(c/N)$ is stationary. (i.e. has a unique Γ-non-forking extension.) This argument shows that any 1-type over a model is stationary. However if $p \in S_n(N)$ is realised by $\bar{c} = (c_1, \ldots, c_n)$ in some elementary extension of N then p is determined by the type

$$tp_K(([c_i, a_k] : i = 1, \ldots, n,\ k = 1, \ldots, r, [c_i, c_j] : i, j \leq n)/K_B)$$

and exactly the same argument holds. ∎

Suppose, that instead of starting with an algebraically closed field, we take K to be any complete theory of stable (respectively simple) fields with independence relation '$\downarrow$ ' given on triples of subsets of models of K.

Let T be a two sorted theory of K-vector spaces with bilinear forms. We can then define Γ-non-forking on models of T as in Definition 10.2.1, the only difference being that we define '$[c, B]$ is Φ-*independent* over A' to mean that whenever $\{b_1, \ldots, b_n\}$ is a linearly independent set in $B_V \cap (V \backslash \langle A \rangle)$ then the set $\{[c, b_1], \ldots, [c, b_n]\}$ is independent over the field $K_B(K_{Ac})$ in the sense of the theory K (i.e. for each $j \leq n$ we have that $[c, b_j] \downarrow_{K_B(K_{Ac})} K_B(K_{Ac}) \cup \{[c, b_1], \ldots, [c, b_{j-1}]\}$). The proof of Proposition 10.2.2 then shows that Γ-non-forking defines an automorphism invariant notion of independence on models of T (called Γ-independence). If K is a stable theory then it is clear that types over models are stationary and hence T is stable and Γ-non-forking is non-forking in T. If K is a simple theory then it is straightforward to show that Γ-non-forking satisfies the independence theorem over a model and hence T is a simple theory and Γ-non-forking coincides with non-forking in T. Thus we have the following:

Theorem 10.2.3 *Let K be a complete theory of fields and let T be a complete theory of finite dimensional K-vector spaces with bilinear forms, then:*

1. *T is simple (respectively stable) if and only if K is simple (resp. stable) and*

2. *if K is simple theory of fields, $N \models T$, $A \subseteq B \subset N$ and $c \in N$, then $tp(c/B)$ does not fork over A if and only if $K_{Ac} \downarrow_{K_A} K_B$ (in the sense of the theory K) and either*

   (a) $c \in \tilde{K}$;

   (b) $c \in \langle A \rangle$;

   (c) $c \notin \langle B \rangle$ and $[c, B]$ is Φ-independent *over A;*

   (d) $c \in \langle B \rangle \backslash \langle A \rangle$ and $\langle B \rangle = V$ and $[c, B]$ is Φ-independent *over A.*

10.3 Morley Rank in Bilinear Spaces

Definition 10.3.1 *Let X be a definable set. The Morley rank of X (denoted $MR(X)$) is defined by induction as follows:*

- $MR(X) \geq 0$ *if X is consistent;*
- $MR(X) \geq \alpha+1$ *if there are pairwise disjoint definable (with parameters) sets $(X_i : i \in \omega)$ such that $MR(X \cap X_i) \geq \alpha$ for each $i \in \omega$;*
- $MR(X) \geq \delta$ *for δ a limit ordinal if $MR(X) > \alpha$ for all $\alpha < \delta$;*
- $MR(X) = \alpha$ *if $MR(X) \geq \alpha$ and $MR(X) \not\geq \alpha + 1$.*

A countable theory T is ω-stable if and only if every formula has ordinal valued Morley rank. In this case Morley rank defines non-forking in T in the following sense:

Definition 10.3.2 *Let p be a (partial) type. $MR(p) = Inf\{MR(\phi) : p \vdash \phi\}$.*

Proposition 10.3.3 *Let T be an ω-stable theory. Let $A \subseteq B$ with $p \in S(A)$, $q \in S(B)$ and $p \subseteq q$. Then q is a non-forking extension of p if and only if $MR(q) = MR(p)$.*

Suppose X is a definable set of Morley rank α. Then there is an integer $d \in \omega$ such that for every model M (containing the parameters of X) there are exactly d sets $X_1, \ldots, X_d$ which are pairwise disjoint with $MR(X \cap X_i) = \alpha$ for $i = 1, \ldots, d$. This d is called the *Morley degree* of X denoted $\deg(X)$.

A set X has Morley rank 0 if and only if it defines a finite set. A set X is called *strongly minimal* if X is infinite and for any definable set Y either $X \cap Y$ or $X \cap \neg Y$ is finite. A set X is strongly minimal if and only if X has Morley rank 1 and Morley degree 1.

Lemma 10.3.4 *Let N be a vector space with a bilinear form. Let a be a vector element. Then the definable set $x \in Span(a)$ is strongly minimal.*

Proof. Suppose X is a definable subset of $Span(a)$. Let $Y = \{\lambda \in K : \lambda a \in X\}$. Then Y is a definable subset of K and from the reduction map from L-formulas to K-formulas given by (ω, ω)-categoricity in 9.4.4 it follows that Y is definable in the algebraically closed field K. But an algebraically closed field is strongly minimal and hence Y is finite or cofinite. Thus X is also finite or cofinite. Therefore $Span(a)$ is strongly minimal. ■

We return to working with finite dimensional spaces with bilinear forms. These structures are $\aleph_1$-categorical and therefore every definable set in such a structure has finite Morley rank. Another important property of $\aleph_1$-categorical theories is that they do not have the finite cover property.

Definition 10.3.5 *A formula $\phi(x, y)$ has the* finite cover property *if for arbitrarily large $n \in \omega$ there are elements $a_0, \ldots a_{n-1}$ such that $\models \neg\exists x \left(\bigwedge_{k<n} \phi(x, a_k)\right)$ but for every $l < n$, $\models \exists x \left(\bigwedge_{k<n, k\neq l} \phi(x, a_k)\right)$.*

A theory T has the finite cover property if some formula has the finite cover property.

Any unstable theory has the finite cover property. Any $\aleph_1$-categorical theory does not have the finite cover property. (See [31] Corollary 9.5.6.)

Lemma 10.3.6 *A theory T eliminates $\exists^\infty$ if and only if T does not have the finite cover property. Thus if T does not have the finite cover property, then for every formula $\phi(x, y)$ there is a formula $\psi(y)$ such that for all tuples b, $\phi(x, b)$ is infinite if and only if $\models \psi(b)$.*

In particular, any uncountably categorical theory eliminates $\exists^\infty$.

Let N_r be a (two-sorted) vector space over an algebraically closed field with a symmetric bilinear form. Suppose that X is a definable set of vector elements. We can then define $MR(X)$ algebraically as follows.

Definition 10.3.7 *Let X be a definable subset of a finite dimensional vector space over an algebraically closed field with a bilinear form. Define Dim(X) inductively as follows:*

- *Dim$(X) \geq 0$ if X is consistent;*
- *Dim$(X) \geq n+1$ if there is z and $(y_i : i \in \omega) \subseteq Span(z)$ such that for each $i \in \omega$ Dim$(X \cap (y_i + z^{\perp})) \geq n$;*
- *Dim$(X) = n$ if Dim$(X) \geq n$ and Dim$(X) \not\geq n+1$.*

Note that this dimension is definable in each N_r in the sense that if $\phi(x, y)$ is any formula (with x a single vector variable) then for each $n \in \omega$ there is a formula $\psi(y)$ such that for each $b \in N_r$ $\mathrm{Dim}(\phi(x, b)) = n \Leftrightarrow N_r \models \psi(b)$. This is an easy induction. 'Dim$(X) = 0$' is clearly definable and $\mathrm{Dim}(X) \geq n+1$ $\iff \exists z\, \exists^{\infty} y \left(y \in Span(z) \wedge \mathrm{Dim}(X \cap (y + z^{\perp})) \geq n\right)$ which is definable in N_r as N_r eliminates $\exists^{\infty}$. Note also that the inductive stage of the definition can be given equivalently as: $\mathrm{Dim}(X) \geq n+1$ if there is z such that for all but finitely many $y \in Span(z)$, $\mathrm{Dim}(X \cap (y + z^{\perp})) \geq n$. This is because for any z, the set $\{y \in Span(z) : \mathrm{Dim}(X \cap (y + z^{\perp})) \geq n\}$ is strongly minimal by Lemma 10.3.4, and hence if infinite, is cofinite (in Span(z)).

Proposition 10.3.8 *Let N_r be as above and let X be a definable set of vector elements in N_r. Then $MR(X) =$Dim(X).*

Proof. By definition, $MR(X) \geq 0 \iff \mathrm{Dim}(X) \geq 0$. Suppose, inductively, that $MR(X) \geq n \iff \mathrm{Dim}(X) \geq n$. Suppose $\mathrm{Dim}(X) \geq n+1$. Then we can choose z and $(y_i : i \in \omega) \in Span(z)$ such that $\mathrm{Dim}(X \cap (y_i + z^{\perp})) \geq n$ and hence by induction $MR(X \cap (y_i + z^{\perp})) \geq n$. Then, by definition of Morley rank, $MR(X) \geq n+1$.

Now suppose that $MR(X) \geq n+1$. Choose pairwise disjoint sets $(X_i : i \in \omega)$ such that $MR(X \cap X_i) \geq n$ and hence, by the inductive hypothesis, $\mathrm{Dim}(X \cap X_i) \geq n$. Therefore for each $i \in \omega$ there is z_i and

$(y_{ij} : j \in \omega) \subseteq Span(z_i)$ such that $\mathrm{Dim}(X_i \cap (y_{ij} + z_i^\perp)) \geq n - 1$. Since N_r is a finite dimensional space, there is $N \in \omega$ such that for all $i \geq N$, $z_i \in Span(z_1, \ldots, z_N)$. Without loss of generality $z_i \in \{z_1, \ldots, z_N\}$.(See below for an explanation.) Then, by the pigeonhole principle, there is $k \in \{1, \ldots N\}$ such that 'z_k works for infinitely many X_i' i.e. there are $(i_l : l \in \omega)$ such that there are $(y_{lj} : j \in \omega) \subseteq Span(z_k)$ such that $\mathrm{Dim}(X_{i_l} \cap (y_{lj} + z^\perp)) \geq n - 1$. Without loss of generality, there is z such that for each $i \in \omega$ $\{y \in Span(z) : \mathrm{Dim}(X_i \cap (y + z^\perp)) \geq n - 1\}$ is cofinite. Therefore, by compactness, there are infinitely many $y \in Span(z)$ such that $\mathrm{Dim}(X_i \cap (y + z^\perp)) \geq n - 1$ for all $i \in \omega$ (i.e. y works for all X_i). By definition of Morley rank (as the sets $X_i \cap (y + z^\perp)$ are pairwise disjoint) for each of the (infinitely many) $y \in Span(z)$ we have $MR(X \cap (y + z^\perp)) \geq n$. By induction hypothesis $\mathrm{Dim}(X \cap (y + z^\perp)) \geq n$ for each of the infinitely many $y \in Span(z)$ and therefore $\mathrm{Dim}(X) \geq n + 1$. ∎

Suppose we are working in the structure N_r and X is a definable subset. Then, in the definition of $\mathrm{Dim}(X)$ we need only consider the elements of an orthonormal basis $\{e_1, \ldots, e_r\}$ as candidates for the z_i. That is, if $\mathrm{Dim}(X) \geq n + 1$ then there is $i \leq r$ such that $\{y \in Span(e_i) : \mathrm{Dim}(X \cap (y + e_i^\perp)) \geq n\}$ is infinite. Suppose not, then $\{y \in Span(e_1) : \mathrm{Dim}(X \cap (y + e_1^\perp)) \geq n\}$ is finite. Then there is $y_1 \in \mathrm{Span}(e_1)$ such that $\mathrm{Dim}(X \cap (y_1 + e_1^\perp)) \geq n + 1$. Similarly there is $y_2 \in \mathrm{Span}(e_2)$ such that $\mathrm{Dim}(X \cap (y_1 + e_1^\perp) \cap (y_2 + e_2^\perp)) \geq n + 1$. Eventually we get $y_i \in \mathrm{Span}(e_i)$ for $i = 1, \ldots, r$ such that $\mathrm{Dim}\big(X \cap \big(\bigcap_{i=1}^r (y_i + e_i^\perp)\big)\big) \geq n + 1$. But this is a contradiction because $\bigcap_{i=1}^r (y_i + e_i^\perp)$ defines a single element in N_r which has $\mathrm{Dim}(0)$. (This explains the assumption in the proof of the above proposition that there is $N \in \omega$ such that for all $i \geq N$, $z_i \in Span\{z_1, \ldots, z_N\}$.)

Using the above comments, we can adapt the definition of Dim to work in alternating spaces. Suppose that N_r is a finite dimensional alternating space with normal basis $(e_1, \ldots, e_r, f_1, \ldots f_r)$, then Dim has the same definition as in the symmetric case but with the inductive part of the definition replaced by: $\mathrm{Dim}(X) \geq n+1$ if there is $i \leq r$ such that either $\{y \in Span(e_i) : \mathrm{Dim}(X \cap (y+f_i^{\perp})) \geq n\}$ is infinite, or, $\{y \in Span(f_i) : \mathrm{Dim}(X \cap (y+e_i^{\perp})) \geq n\}$ is infinite.

Note that in both the symmetric and alternating cases we have already defined another dimension on subsets of N_r by: for $A \subseteq N_r$, $\dim(A) =$ the dimension of the $\tilde{K}$-subspace $\langle A \rangle$. This dimension is not quite the same as the dimension $\mathrm{Dim}(A)$. However we clearly have that for any definable set of vectors X, $\mathrm{Dim}(X) \leq \dim(X) \leq r$.

Chapter 11

Two Sorted Theories of Vector Spaces

In this chapter we apply the quantifier elimination result 9.2.3 to theories of vector spaces in a two-sorted language. This language is much stronger than the usual (one-sorted) language used for discussing theories of vector spaces. In particular, in such a two sorted structure, both linear dependence and the full structure on the underlying field are definable. We also show that if it is possible to define a well behaved notion of independence on the underlying field (e.g. if the field is simple or stable) then this extends to a notion of independence on the whole two-sorted structure, and in particular if the underlying field is simple (respectively stable) then so is the theory of any two sorted vector space over that field.

11.1 Quantifier Elimination in Two Sorted Vector Spaces

Definition 11.1.1 *Let L be a many sorted language and T an L-theory. Let S be some sort in L. Then T has* elimination of quantifiers over variables of sort S, *if for every L-formula* ϕ *there is another L-formula* ψ *such that* $T \models \phi \leftrightarrow \psi$ *and* ψ *has no quantifiers over variables of sort* S.

Definition 11.1.2 *Let* L *be the two sorted language with sorts* V *(vectors), and* K *(scalars).*

L has the following functions:

$+_V : V \times V \to V$ *(vector addition)*

$+_K : K \times K \to K$ *(field addition)*

$\circ_K : K \times K \to K$ *(field multiplication)*

$\gamma : K \times V \to V$ *(scalar multiplication)*

and constants 0_V, 0_K *and* 1_K

Let T *be the theory expressing that the* K*-sort elements form a field and the* V*-sort elements form a* K*-vector space with the functions and constants in* L *having their obvious meanings.*

Definition 11.1.3 *For* $n \in \omega$ *let* $\theta_n(X_1, \ldots X_n)$ *be the following formula:*

$$\theta_n(X_1, \ldots X_n) := \forall y_1, \ldots, \forall y_n((y_1 X_1 + \cdots + y_n X_n = 0) \to (\bigwedge_{i=1}^{n} y_i = 0)).$$

Let L_θ *be the expansion of* L *resulting from adding each formula* θ_n *to the language* L. *Let* Θ_n *be the sentence* $\Theta_n := \exists X_1, \ldots, \exists X_n \, \theta_n(X_1, \ldots, X_n)$.

In the short paper of Kuzichev [42], the following result and corollaries are proved:

Proposition 11.1.4 *Let* T' *be an L-theory extending* T. *Then* T' *has elimination of variables of sort* V *if and only if for each* $n \in \omega$ *there is some formula* $\sigma_n \in L$ *such that* $T' \models \Theta_n \leftrightarrow \sigma_n$ *and* σ_n *contains no variables of sort* V.

Corollary 11.1.5 *Let $m \in \mathbb{N} \cup \{\infty\}$. (For $n \in \omega$, $T_n = T \cup \{\Theta_m \wedge \neg\Theta_{m+1}\}$ and $T_\infty = T \cup \{\Theta_m : m \in \omega\}$.) Then the theory T_m has elimination of quantifiers of sort V.*

Corollary 11.1.6 *Let T' be an L-theory extending T. Then T' is complete if and only if T' contains the complete theory of a field and for each $n \in \omega$ either $T' \vDash \Theta_n$ or $T' \vDash \neg\Theta_n$.*

Using Theorem 9.2.3 we can slightly improve on Proposition 11.1.4 in the case that T' is a complete theory extending T: if K is the (complete) theory of the field contained in T' and L_θ^K is an extension of the language L_θ containing an elimination set for the theory K in the language of fields, then T' has elimination of quantifiers in the language L_θ^K. That is, in the original language L, the only non-quantifier free formulas containing *any* vector variables we need consider are the formulas θ_n. Therefore every formula is equivalent modulo T' to a boolean combination of θ-formulas, quantifier-free formulas and K-formulas. (where a K-formula is a formula in the language of rings, considered as a sublanguage of L.) In particular, if T' is the (complete) theory of vector spaces of some fixed dimension over an algebraically closed field, then T' has elimination of quantifiers in the language L_θ.

Proposition 11.1.7 *Let T' be an L-theory extending T. Suppose that for each $n \in \omega$ there is some formula $\sigma_n \in L$ such that $T' \vDash \Theta_n \leftrightarrow \sigma_n$ and σ_n contains no variables of sort V. Then given any L-formula ϕ, ϕ is equivalent modulo T' to a boolean combination of θ-formulas, quantifier-free formulas and K-formulas.*

Proof. Essentially the same as the relevant parts of the proof of 9.2.3. Note that if a and b are vector elements not in the span of a subset $A \subseteq M$ then there is an A-automorphism of M taking a to b and hence $tp(a/A) = tp(b/A)$. ∎

Let T' be the theory of vector spaces of some fixed dimension over algebraically closed fields. Then, in contrast to the corresponding theory of vector spaces with bilinear forms, T' is *not* model complete in the language L (although, it is of course model complete in L_θ). This is because there is no bilinear form and hence no way of expressing the formulas θ_n as $\exists_1$-formulas. To illustrate this point we give an example of two models $M, N \models T$ with (L-) embeddings $\sigma : M \to N$ and $\tau : M \to N$ such that σ is an elementary embedding but τ is not.

Let K be the field of algebraic numbers $\mathbb{Q}^{alg}$ and let F be the field of complex numbers $\mathbb{C}$. Let $n \in \omega$ and let V be an n-dimensional vector space over K and let W be an n-dimensional vector space over F. Let $M = (V, K)$ and let $N = (W, F)$. Then by Corollary 11.1.6 $M \equiv N$. As F can be considered as an infinite dimensional vector space over K, we can choose $h_1, \ldots, h_n \in F$ which are linearly independent over K. Choose a basis $\{a_1, \ldots, a_n\}$ for V over K and a basis $\{b_1, \ldots, b_n\}$ for W over F. Let $\sigma = (\sigma_V, \sigma_K) : M \to N$ be the map given by specifying that $\sigma_K : K \to F$ is any (necessarily elementary) embedding of K into F and σ_V is defined on V by specifying that $\sigma_V(a_i) = b_i$ and then extending this by linearly by σ_K. Then by elimination of quantifiers, since σ preserves the formulas θ_n, σ is an elementary embedding.

Now let $\tau = (\tau_V, \tau_K) : M \to N$ be given by $\tau_K = \sigma_K$ and let τ_V be defined on V by specifying that $\tau_V(a_i) = h_i b_1$ for $i = 1, \ldots, n$. Clearly τ is an embedding of L-structures. However, $M \models \theta_n(a_1, \ldots, a_n)$ but $N \models \neg\,\theta_n(\tau(a_1), \ldots, \tau(a_n))$. Therefore τ is *not* an elementary embedding and hence T' is not model complete in L.

11.2 Independence in Two Sorted Theories of Vector Spaces

If T is the theory of the two-sorted vector space (V, K) and V is a K-vector space of finite dimension n, then the theory of T is essentially the same as the theory of the structure K^{n+1}, see e.g. [50]. As in the case of finite dimensional bilinear spaces, there is a translation procedure between K-formulas and (V, K)-formulas, and in particular, T is stable (respectively simple) if and only if the field K is stable (resp. simple).

In fact it can be easily seen that such a translation is an interpretation of the two sorted vector space (V, K) in the field K. This follows immediately form the relevant parts of the proof of 10.1.3. As in the case of bilinear spaces, if we fix some basis for the vector space V then the structures (V, K) and K are bi-interpretable. However if $M = (V, K)$ is an infinite dimensional K-vector space then M is clearly not interpretable in K.

If K is an algebraically closed field and $M = (V, K)$ is an infinite dimensional K-vector space, then we can prove that $T = Th(M)$ is at least superstable by the following crude counting models argument. Suppose $N = (W, F) \models T$ and $|N| = \aleph_1$. Then either $|F| = \aleph_1$ and $\dim(V) \leq \aleph_1$, or $|F| < \aleph_1$ and $\dim(V) = \aleph_1$. There is a unique (up to isomorphism) algebraically closed field of cardinality $\aleph_1$ and only countably many cardinals $\leq \aleph_1$. Therefore there are only countably many vector spaces of dimension $\leq \aleph_1$ over fields of cardinality $\aleph_1$. Also there are only countably many algebraically closed fields (up to isomorphism) of cardinality $\leq \aleph_1$ and hence only countably many $\aleph_1$-dimensional vector spaces over these fields. Thus T has only countably many models of cardinality $\aleph_1$ and so by the following theorem of Shelah, T is superstable.

Theorem 11.2.1 *([61] Ch. VIII, Cor. 3.4) Suppose that the countable theory T is unsuperstable. Then T has 2^λ non-isomorphic models in every uncountable* *cardinality* λ.

We now show, that if K is a field admitting some kind of notion of independence and if T is the theory of a two-sorted vector space (V, K), then the notion of independence extends to a notion of independence on all models of T. In the case where V is finite dimensional over K, this is trivial by the remarks above. However, the result seems to be of some interest in the infinite dimensional case. it also allows us to give an algebraic characterisation of non-forking in these structures. We first work in vector spaces over algebraically closed fields.

By the remarks above, the two sorted theory T, of a vector space (of any dimension) over an algebraically closed field is (super)stable and hence there is unique notion of independence on T, which is non-forking.

Notation 11.2.2 *Let T be a complete theory of a two sorted vector space. We work in a very saturated model $M \models T$, written $M = (V, \tilde{K})$. Given $A = (A_V, A_K) \subseteq M$ let K_A be the field generated by A as in the notation 9.2.4, (but obviously without the clause involving the bilinear form), let K_A^{alg}, A_V, A_K^{alg} and $\langle A \rangle$ be as in the notation 9.2.4.*

Lemma 11.2.3 *Let $M \models T$ as above and let $A = (A_V, A_K) \subseteq M$. Then $dcl(A) = (\overline{A_V}, K_A)$ and $acl(A) = (\overline{A_V^{alg}}, K_A^{alg})$.*

Proof. This is just a simplification of the proof of 9.5.1. Since there is no bilinear form, the case where $\dim(V) = m$ is finite and $\dim(A) = m - 1$ is not problematical here. ■

Definition 11.2.4 *Suppose that T is a two sorted theory of vector spaces over algebraically closed fields. Let $M = (V, \tilde{K})$ be a saturated model of T. Let $A \subseteq M$ and let $c \in V$. Define*

$$R_A(c) = \begin{cases} Inf\{tr.\deg(\gamma_1, \dots, \gamma_n / K_A), c = \gamma_1 a_1 + \cdots + \gamma_n a_n \\ \qquad\qquad\qquad a_1, \dots, a_n \in A_v\} \ \text{if } c \in \langle A_V \rangle \\ \dim(V/\tilde{K}) \quad \text{if } c \notin \langle A_V \rangle \end{cases}$$

Definition 11.2.5 *Suppose that T is the two sorted theory of vector spaces over algebraically closed fields. Let $M = (V, \tilde{K})$ be a saturated model of T. Let $A \subseteq B \subseteq M$ and let $c \in V$. We say $c \notin \langle B_V \rangle_A$ if the following conditions holds:*

- *$c \notin \langle A \rangle$ and whenever $b_1, \dots, b_n$ are linearly independent elements of B_V and $c = \lambda_1 b_1 + \cdots + \lambda_n b_n$ then $\lambda_1, \dots, \lambda_n$ are algebraically independent over K_B and $n = \dim(V/\tilde{K})$.*

So if $c \notin \langle B_V \rangle$ then $c \notin \langle B_V \rangle_A$ is vacuously true. If the underlying vector space V is infinite dimensional then $c \notin \langle B \rangle \iff c \notin \langle B_V \rangle_A$. However, in the finite dimensional case it is possible that $c \in \langle B \rangle$ but $c \notin \langle B_V \rangle_A$.

Theorem 11.2.6 *Suppose that T is a complete two sorted theory of vector spaces over algebraically closed fields. Let $M = (V, \tilde{K})$ be a saturated model of T. Let $A \subseteq B \subseteq M$ and let $c \in V$. Then, the following are equivalent:*

1. *$tp(c/B)$ does not fork over A.*

2. *$R_A(c) = R_B(c)$.*

3. *$c \notin \langle B_V \rangle_A$ or, $c \in \langle A \rangle$ and whenever $e_1, \dots e_n$ is a $\tilde{K}$-basis for $\langle A \rangle$ contained in A_V, $f_1, \dots, f_m$ is a $\tilde{K}$-basis for $\langle B \rangle$ contained in B_V and $c = \lambda_1 e_1 + \cdots + \lambda_n e_n = \mu_1 f_1 + \cdots + \mu_m f_m$ then $tr.\deg\{\lambda_1, \dots, \lambda_n / K_A\} = tr.\deg\{\mu_1, \dots, \mu_m / K_B\}$.*

4. *$c \notin \langle B_V \rangle_A$ or, $c \in \langle A \rangle$ and for some basis $e_1, \dots e_n$ for $\langle A \rangle$ contained in A_V, and some basis $f_1, \dots, f_m$ for $\langle B \rangle$ contained in B_V and some $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m$ such that $c = \lambda_1 e_1 + \cdots + \lambda_n e_n = \mu_1 f_1 + \cdots + \mu_m f_m$ then $tr.\deg\{\lambda_1, \dots, \lambda_n / K_A\} = tr.\deg\{\mu_1, \dots, \mu_m / K_B\}$.*

5. *$c \notin \langle B_V \rangle_A$ or, $c \in \langle A_V \rangle$ and $K_{Ac} \downarrow_{K_A} K_B$. (i.e. the fields K_{Ac} and K_B and independent over the field K_A.)*

In particular if $\dim(V)$ is infinite and c is a vector element then $tp(c/B)$ does not fork over A if and only if $\langle c\rangle \cap \langle A\rangle = \langle c\rangle \cap \langle B\rangle$ and $K_{Ac} \downarrow_{K_A} K_B$. In this case, it can be seen very clearly how non-forking in $M = (V, \tilde{K})$ is a combination of linear independence in V and algebraic independence in $\tilde{K}$.

The proof of Theorem 11.2.6 (1) $\iff$ (5) is contained in the proof of 10.2.2. The equivalence of the other three statements to (5) follows easily from the characterisation of independence in algebraically closed fields and properties of linear independence in vector spaces.

Suppose that K is any complete theory of a simple (respectively stable) field with independence (non-forking) relation $\downarrow$. Then we can define an independence relation $\downarrow^*$ on models of the theory of two sorted vector spaces over K as in clause (5) of Theorem 11.2.6 (i.e. $c \downarrow^*_A B$ iff $\langle c\rangle \cap \langle A_V\rangle = \langle c\rangle \cap \langle B_V\rangle$ and $K_{Ac} \downarrow_{K_A} K_B$). It then follows immediately that for $m \in \mathbb{N} \cup \{\infty\}$, if T_m is the theory of two sorted vector spaces of dimension m over models of $Th(K)$, then T_m is simple (respectively stable) if and only if $Th(K)$ is simple (resp. stable) and the independence relation $\downarrow^*$ is exactly non-forking on models of T_m.

Chapter 12

Infinite Dimensional Spaces with Bilinear Forms

We keep the notation of Chapter 9. In this chapter, the underlying field K is always some fixed algebraically closed field and the form is of fixed type (i.e. either symmetric or alternating). In most proofs we deal with the case where the form is symmetric, the alternating case usually follows identically after replacing an orthonormal basis $(e_i : i \in I)$ by a normal basis $(e_j, f_j : j \in J)$. Thus T_r and T_∞ denote (complete) theories of vector spaces with bilinear forms.

Suppose that $M = (V, K)$ is a countable model of T_∞. Then by Corollary 5.4.3, V has an orthonormal basis $(e_i : i \in \omega)$. For $r \in \omega$, let $V_r = K$-span$(e_1, \ldots e_r)$. Let N_r be the L-structure (V_r, K). Then $V = \bigcup_{r\in\omega} V_r$ and so, by identifying all the copies of K, we can consider the countable structure M as an ascending union of the structures N_r. By the results of Chapter 10 the structures N_r are all interpretable in the theory of the field K , which is $\aleph_1$-categorical. In particular all the structures N_r are ω-stable of finite Morley rank. (In fact each structure N_r is itself $\aleph_1$-categorical.) However, by proposition 7.4.1 the theory of the structure $M = \bigcup_{r\in\omega} N_r$ is not even simple.

12.1 Smooth Approximation Revisited

Recall the definition of smoothly approximated structures:

Definition 12.1.1 *A structure M is* smoothly approximated *(by finite substructures) if*

1. *M is countable and $\aleph_0$-categorical, and*

2. *There is a chain $N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$ of finite substructures of M such that $M = \bigcup_{r\in\omega} N_r$ and each N_r is a finite homogenous substructure of M.*

We give a generalisation of this definition to structures smoothly approximated by some theory (which does not quite include the structures smoothly approximable by finite substructures).

Definition 12.1.2 *Let L be a language and T an L-theory. Let $\mathcal{K}$ be some category of structures. We say that T is* smoothly approximated by $\mathcal{K}$ *if for each countable model M of T we can associate in some canonical way a (countable) structure $K \in \mathcal{K}$ such that:*

1. *there is a chain $N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$ of substructures of M each interpretable in the structure K such that $M = \bigcup_{r\in\omega} N_r$ and each N_r is a homogenous substructure of M.*

2. *If K_1 and K_2 are the structures in $\mathcal{K}$ associated to countable models M_1 and M_2 of T respectively, then $M_1 \cong M_2$ as L-structures if an only if $K_1 \cong K_2$ (in the category $\mathcal{K}$).*

The situation we have in mind is usually where $\mathcal{K}$ is the class of models of some L_0-theory T_0, and there is some L-formula ϕ such that for each (countable) model $M \models T$, $\phi(M)$ is a substructure of $M \restriction L_0$ isomorphic to a (countable) model of T_0. (i.e. the map from countable models of T to countable models of T_0 is just the map $M \mapsto \phi(M)$.) In this case (2) of the above definition says exactly that T is (ω, ω)-categorical (over the predicate ϕ). We now give a formal definition for structures smoothly approximated by some theory.

Definition 12.1.3 *Let L be a language and T an L-theory. Let $L_0 \subseteq L$ and let T_0 be an L_0-theory. We say that T is* smoothly approximated by T_0 *if there is some formula $\phi \in L$ such that if $M \models T$ then $\phi(M)$ is a substructure of the reduct $M \restriction L_0$ isomorphic to a model of T_0 such that:*

1. *Whenever M is a countable model of T and K is the (countable) L_0-structure $\phi(M)$ then there is a chain $N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots$ of L-substructures of M, each interpretable in the structure K, such that $M = \bigcup_{r \in \omega} N_r$ and each N_r is a homogenous substructure of M.*

2. *Whenever M_1 and M_2 are countable models of T then $M_1 \cong M_2$ (as L-structures) if and only if $\phi(M_1)$ and $\phi(M_2)$ are isomorphic as models of T_0. (i.e. T is (ω, ω)-categorical with respect to the predicate ϕ.)*

If, for example, the approximating theory T_0 is stable (or strongly minimal) we say that T is smoothly approximated by a stable (or strongly minimal) theory. Note that if T is smoothly approximated by an $\aleph_1$-categorical theory then (by the Baldwin-Lachlan Theorem) T can have only countably many models of cardinality $\aleph_0$.

Proposition 12.1.4 *Let $T = {}_F T^K_\infty$ be any good theory of bilinear forms (e.g. K is an algebraically closed field). Then T is smoothly approximated by $T_0 = Th(K)$.*

Proof. Let ϕ be the formula picking out the field sort elements of models of T (We can consider this formula to be in the language by the remarks following 9.4.3.) Then T is (ω, ω)-categorical relative to ϕ by Proposition 9.4.4. Let $M = (V, K)$ be a countable model of T. Then by Corollary 5.4.3 V has an orthonormal basis $(e_i : i \in \omega)$. Let $V_r = K\text{-span}(e_1, \ldots, e_r)$ and let $N_r = (V_r, K)$. Clearly $M = \bigcup_{r \in \omega} N_r$ and each structure N_r is interpretable in $T_0 = Th(K)$ by Lemma 10.1.3. It remains to prove that each N_r is a homogenous substructure of M in the sense that if a and b are tuples in N_r and $f \in Aut(M)$ takes a to b then there is some $f' \in Aut(M)$ taking a to b and fixing N_r setwise. So suppose $f \in Aut(M)$ and $f(a) = b$. Then $tp_M(a) = tp_M(b)$. Hence $tp_N(a) = tp_N(b)$ and so, without loss of generality, there is some $f_0 \in Aut(N_r)$ such that $f_0(a) = b$. Then, by Witt's Theorem, f_0 extends to an automorphism of M. ■

Note that if M is a structure smoothly approximated by finite substructures, then, for every $\sigma \in Th(M)$, there is a finite substructure $N \leq_{\text{hom}} M$ such that $N \models \sigma$. We have already proved the following (slightly stronger) result for models M of the theory T_∞ : for any sentence $\sigma \in Th(M)$, there is $R \in \omega$ such that $M \models \sigma \iff N_r \models \sigma$ for all $r \geq R$. (See Proposition 9.3.3.)

From now on, unless otherwise stated, the underlying field will always be some algebraically closed field K, of fixed characteristic. We write T for the theory ${}_FT^K_\infty$ (for $F \in \{S, A\}$) and T_r for the theory ${}_FT^K_r$. T_0 will denote the theory of the field K. Suppose $M \models T$ then an *approximating sequence* for M is defined to be a sequence of homogenous substructures $N_r \subseteq M$ with each $N_r \models T_r$ and each N_r having the same field-sort part as M, such that $M = \bigcup_{r \in \omega} N_r$. Such a sequence exists by Proposition 12.1.4 (but is clearly not unique).

Lemma 12.1.5 *Let M be a model of T and let $(N_r : r \in \omega)$ be some approximating sequence for M. Let ϕ be any L-formula (possibly with parameters from M). Then there is $R \in \omega$ such that for all r with $R \leq r < \omega$ we have $\phi(N_r) = \phi(M) \cap N_r$ and hence $\phi(M) = \bigcup_{R \leq r < \omega} \phi(N_r)$.*

Proof. By Proposition 9.3.3 there is $R \in \omega$ and a θ-formula ψ such that $N_r \vDash \phi \leftrightarrow \psi$ for all $r \geq R$ (and $M \vDash \phi \leftrightarrow \psi$). By the proof of Proposition 9.3.4 N is an L_θ-substructure of M. Hence, for $r \geq R$, and $a \in N_r$, $M \vDash \phi(a) \iff N_r \vDash \phi(a)$ and so $\phi(N_r) = \phi(M) \cap N_r$. The second part follows immediately form the fact that $M = \bigcup_{r\in\omega} N_r = \bigcup_{R\leq r<\omega} N_r$. ■

Lemma 12.1.5 says that we can understand the definable sets in M by understanding the definable sets in the N_r for arbitrarily large $r \in \omega$. This is useful since the structures N_r have finite Morley rank and so it is relatively easy to understand their definable sets.

12.2 Independence in T_∞

The theory $T = T_\infty$ (infinite dimensional vector spaces over algebraically closed fields with a bilinear form) is not a simple theory (Proposition 7.4.1). Therefore by 2.3.11 it is impossible to define, on models of T, a notion of independence satisfying all the axioms given in 2.3.10. However, each of the 'approximating theories' T_r is ω-stable and hence admits such a notion of independence. We aim to show that by using the characterisation of independence in the theories T_r given in Proposition 10.2.2 it is possible to define a notion of independence for models of T which satisfies all the axioms of a notion of independence for stable theories except for the local character axiom.

Definition 12.2.1 *Let $r \in \omega$ and let $M = (V, \tilde{K})$ be some sufficiently saturated model of T_∞. Let $A \subseteq B \subset M$ and let $c \in M$. Define: $tp(c/B)$ does not Γ-fork ($dn\Gamma f$) over A if $K_{Ac} \downarrow_{K_A} K_B$ (i.e. K_{Ac} and K_B are independent over K_A in the algebraically closed field $\tilde{K}$) and one of the following three conditions hold:*

1. *$c \in \tilde{K}$ (i.e. c is a field sort element)*
2. *$c \in \langle A \rangle$*

3. $c \notin \langle B\rangle$ *and* $[c, B]$ *is* Φ-independent *over* A

As in Definition 10.2.1 '$[c, B]$ *is* Φ-independent *over* A *' means that whenever* $\{b_1, \ldots , b_n\}$ *is a linearly independent set in* $B_V \cap (V \backslash \langle A\rangle)$ *then the set* $\{[c, b_1], \ldots , [c, b_n]\}$ *is algebraically independent over the field* $K_B(K_{Ac})$.

As in 10.2.1 we can extend the definition of Γ*-non-forking from* 1*-types to n-types by defining:* $tp(c_1, \ldots , c_n, c_{n+1}/B)$ $dn\Gamma f$ *over* A *if and only if* $tp(c_1, \ldots , c_n/B)$ $dn\Gamma f$ *over* A *and* $tp(c_{n+1}/Bc_1, \ldots c_n)$ $dn\Gamma f$ *over* $Ac_1, \ldots , c_n$.

Theorem 12.2.2 *Let* $M = (V, K) \vDash T_\infty$. *Then the relation on subsets of* M *given by* Γ*-non-forking is automorphism invariant, symmetric, transitive, satisfies the finite character and extension axioms and types over models are stationary. There are uncountable models* $M \vDash T_\infty$ *in which* Γ*-non-forking does not satisfy the local character axiom on* M *(i.e. there are pairs* (c, B) *in* M *such that for every countable subset* $A \subseteq B$, $tp(c/B)$ Γ*-forks over* A*).*

Proof. Exactly the same as the proof of 10.2.2 (in fact slightly more straightforward since case (4) of Definition 10.2.1 does not arise). The only place in the proof of 10.2.2 where the fact that $\dim(V)$ is finite is used is in verifying the local character axiom. In the case of T_∞ this may fail.

Let $M = (V, K)$ be a very saturated model of T with K uncountable and V of uncountable dimension over K. Let B_K be a subset of K. Let $\{b_i : i \in I\}$ be a set of linearly independent vectors in V where I is an uncountable set. Let $B = B_K \cup \{b_i : i \in I\}$. Let $c \in V$ be a vector element such that $c \notin \langle B\rangle$ and $[c, b_i] = 1$ for all $i \in I$. (Such an element exists by Lemma 5.2.3, compactness and saturation of M.) Let A be any countable subset of B. Then there is $b \in B_V \backslash \langle A\rangle$ such that $[c, b] = 1$. In particular $[c, b] \in K_{Ac}(K_B)^{alg}$ and so by condition (3) of the definition, $tp(c/B)$ Γ-forks over A. ■

The reason that the local character axiom fails in uncountable models of T is that in considering the type of a single vector element c over a set $A = \{a_i : i \in I\}$ of infinitely many linearly independent vector elements, we are forced, by the two-sorted language, into considering the infinitary type $tp_K([c, a_i]_{i \in I}/K_A)$. The version of local character in stable (and simple) theories for infinitary types is: if c_I is a tuple of length $|I|$ then for any set B there is some subset $A \subseteq B$ with $|A| \leq |I| + |T|$ such that $tp(c_I/B)$ does not fork over A. Thus in the theory T_∞, given a type $tp(c/B)$ the best bound we can expect for a local character type axiom is $\aleph_0 + \dim(B)$.

It is worth noting that if M is a countable model of T_∞, then the local character axiom holds trivially in M and hence, even though it is unstable (and not even simple) the theory T_∞ has the property that on any countable model $M \models T_\infty$ the independence relation given by Γ-non-forking on M satisfies all the independence axioms satisfied by non-forking in a stable theory.

Proposition 12.2.3 *Let $M \models T_\infty$, let $A \subseteq B \subseteq M$ and let $p(x) \in S(B)$. Let $(N_r : r \in \omega)$ be some approximating sequence for M. Then the following are equivalent:*

1. *$p(x)$ does not Γ-fork over A;*

2. *Given any formula $\phi = \phi(x, b) \in p(x)$ there is $R_\phi \in \omega$ such that $\phi(x, b)$ does not fork over $A \cap N_r$ in the structure N_r for all $r \geq R_\phi$;*

3. *For each finite $b \subseteq B$ there is $R_b \in \omega$ such that $p(x) \restriction_{N_r \cap Ab}$ does not fork over $A \cap N_r$ in N_r for all $r \geq R_b$.*

Proof. (2)$\Rightarrow$(1) Let $\phi(x, b) \in p(x)$ and choose R such that $\phi(x, b)$ does not fork over $A \cap N_r$ in N_r for all $r \geq R$. By the finite character of Γ-non-forking in M it is sufficient to find c realising $\phi(x, b)$ such that $tp_M(c/Ab)$ does not Γ-fork over A. By the characterisation of non-forking in each N_r, there is c (in some elementary extension of N_r) realising $\phi(x, b)$ such that $K_{Ac \cap N_r} \downarrow_{K_{A \cap N_r}} K_{Ab \cap N_r}$ and $[c, Ab]$ is Φ-independent over $A \cap N_r$. Since all these elementary extensions of the N_r are contained in some $\tilde{M} \models T_\infty$ we can assume, by compactness that there is some c witnessing the non-forking of

$\phi(x,b)$ over $A\cap N_r$ in each structure N_r as r varies. (The same c for all $r \geq R$.) But independence in algebraically closed fields is described by: $K_C \downarrow_{K_A} K_B$ if and only if $K_C^{alg} \cap K_A^{alg} = K_C^{alg} \cap K_B^{alg}$. Clearly $K_A^{alg} = \bigcup_{r\in\omega} K_{A\cap N_r}^{alg}$. Hence:

$$\begin{aligned} K_{Ac}^{alg} \cap K_A^{alg} = \left(\bigcup_{r\in\omega} K_{Ac\cap N_r}^{alg}\right) \cap \left(\bigcup_{r\in\omega} K_{A\cap N_r}^{alg}\right) &= \bigcup_{R\leq r<\omega} \left(K_{Ac\cap N_r}^{alg} \cap K_{A\cap N_r}^{alg}\right) \\ &= \bigcup_{R\leq r<\omega} \left(K_{Ac\cap N_r}^{alg} \cap K_{Ab\cap N_r}^{alg}\right) = K_{Ac}^{alg} \cap K_{Ab}^{alg}. \end{aligned}$$

If $c \in \langle Ab \rangle$ this is enough. Otherwise we must check that if $(b_1, \ldots b_n) \in (Ab_V)\backslash\langle A\rangle$ then $\{[c,b_1],\ldots,[c,b_n]\}$ are algebraically independent over the field $K_{Ab}(K_{Ac})$. But this follows immediately from the fact that $tp_{N_r}(c/Ab\cap N_r)$ does not fork over $A\cap N_r$ for each $r \geq R$ and by using the fact that b is a finite set and $K_{Ab}(K_{Ac}) = \bigcup_{R\leq r<\omega}(K_{Ab\cap N_r}(K_{Ac\cap N_r}))$. Hence, by definition of Γ-non-forking, $tp(c/Ab)$ $dn\Gamma f$ over A in M.

(1)$\Rightarrow$(3) Suppose that $p(x)$ does not Γ-fork over A and let $c \in \tilde{M}$ realise p. Without loss of generality we assume that $B = A \cup \{b\}$ for some finite set b, and aim to prove that for all sufficiently large $r \in \omega$, $p_r := p \restriction Ab\cap N_r$ does not fork over $A\cap N_r$ in N_r. We first show that (for large enough r) if $c \models p_r(x)$ then $K_{Ac\cap N_r} \downarrow_{K_{A\cap N_r}} K_{Ab\cap N_r}$. Suppose not, then for each $r \in \omega$, there is c_r realising p and $\lambda_r \in K_{Ac\cap N_r}^{alg} \cap K_{Ab\cap N_r}^{alg}$ such that $\lambda_r \notin K_{A\cap N_r}^{alg}$. Then, by compactness in M and the fact that the N_r are homogenous substructures of M, it follows that there is $c \in \tilde{M}$ realising p and $\lambda_r \in K_{Ac}^{alg} \cap K_{Ab}^{alg}$ with $\lambda_r \notin K_A^{alg}$ which contradicts that $p(x)$ does not Γ-fork over A. Secondly, as b is a finite set, we can choose a maximal linearly independent set $\{b_1,\ldots,b_n\} \subseteq b\backslash\langle A\rangle$. We claim that for all large enough $r \in \omega$, (i.e. $b \subseteq N_r$) and $c_r \models p_r$ that $\{[c_r,b_1],\ldots,[c_r,b_n]\}$ is algebraically independent over $K_{Ab\cap N_r}(K_{Ac\cap N_r})$. This is exactly the same argument as in the first case. Hence, by the characterisation of non-forking in the N_r, we have that p_r does not fork over $A\cap N_r$ in N_r for all sufficiently large r. Note that the second part of this argument does always work if the set b is infinite, since there may be infinitely many linearly independent elements in $b\backslash\langle A\rangle$.

(3)$\Rightarrow$(2) is obvious. ■

Note that the statement: 'There is $R \in \omega$ such that $p(x) \upharpoonright N_r$ does not fork over $A \cap N_r$ for all $r \geq R$.' is stronger than the assertion that $p(x)$ does not Γ-fork over A. We can find a type $p(x) \in S(B)$ and a set $A \subseteq B$ such that $p(x)$ does not Γ-fork over A but for each sufficiently large $r \in \omega$, $p(x) \upharpoonright_{B \cap N_r}$ forks over $A \cap N_r$ in N_r.

We can easily define the concept of a formula $\phi(x, b)$ not Γ-forking over a set A by: $\phi(x, b)$ $dn\Gamma f$ over A if there is some c realising $\phi(x, b)$ such that $tp(c/Ab)$ $dn\Gamma f$ over A. Note also that if there is no R such that $\phi(x, b)$ does not fork over $A \cap N_r$ for all $r \geq R$ then it follows that $\phi(x, b)$ forks (hence divides) over $A \cap N_r$ in N_r for all sufficiently large r. We can then use Proposition 12.2.3 to characterise Γ-forking of formulas in T_∞ by:

Corollary 12.2.4 *Let $\phi(x, b)$ be a formula and A a set of parameters. The following are equivalent:*

1. *$\phi(x, b)$ Γ-forks over A;*

2. *Given any approximating sequence $(N_r : r \in \omega)$ for M there exists $R \in \omega$ such that $\phi(x, b)$ forks (divides) over $A \cap N_r$ in N_r for all $r \geq R$.*

12.3 Forking and Dividing in T_∞

Recall the following definitions:

Definition 12.3.1 *Let T be any theory in a language L, let $\phi(x, b) \in L$ and let A be a set of parameters:*

1. *The formula $\phi(x, b)$* divides *over A* with respect to *k if there is an infinite A-indiscernible sequence $\{b_i : i \in \omega\}$ such that $\{\phi(x, b_i) : i \in \omega\}$ is k-inconsistent. $\phi(x, b)$* divides *over A if $\phi(x, b)$* divides *over A with respect to k for some $k \in \omega$. (Hence $\phi(x, b)$* divides *over A if and only if there*

is an infinite A-indiscernible sequence $\{b_i : i \in \omega\}$ such that $\{\phi(x, b_i) : i \in \omega\}$ is inconsistent.)

2. $\phi(x, b)$ forks *over A if there is $n \in \omega$ and formulas $\phi_i(x, b^i)$ for $i < n$ such that $\phi(x, b) \vdash \bigvee_{i<n} \phi_i(x, b^i)$ and for each $i < n$, $\phi_i(x, b^i)$ divides over A.*

If T is simple then the notions of forking and dividing coincide, and define, on all models of T, a notion of independence satisfying symmetry, transitivity and the other axioms of Definition 2.3.11. If the theory T is not simple, then forking is not necessarily equivalent to dividing and neither of them need induce, on models of T, a well behaved notion of independence. Note also that a theory T is simple if and only if for all sets B and all $p \in S(B)$ there is some set $B_0 \subseteq B$ with $|B_0| \leq |T|$ such that p does not fork over B_0 (equivalently there is some such $B_0 \subseteq B$ such that p does not divide over B_0). We can, in fact, prove that the theory T_∞ is not simple by showing directly from the definition of dividing, using quantifier elimination, that there is some type over an uncountable set which divides over every countable subset of its domain. The example of the type given in the proof of 12.2.2 to refute the local character axiom for Γ-non-forking in T_∞ provides an example in this case also.

In this section we investigate the properties of dividing and some related concepts on models of T_∞. The original idea had been to try to characterise dividing on (countable) models of T_∞ in terms of dividing in approximating sequences for these models. If such a characterisation were possible we show that this would imply the equivalence of forking and dividing in T_∞. For technical reasons it is much easier to restrict attention to the question of formulas dividing over countable sets of parameters. This allows us at least to consider all aspects of dividing in countable models of T_∞.

We write $\phi(x, b)$ divides over A (with respect to a theory T) to mean that $\phi(x, b)$ divides over A in a monster model $\tilde{M} \models T$ (i.e. there are $(b_i : i \in \omega) \subseteq \tilde{M}$ and indiscernible over A with $b_0 = b$ such that $\{\phi(x, b_i) : i \in \omega\}$ is inconsistent in $\tilde{M}$). Clearly if $\phi(x, b)$ divides over A in some $M \models T$ (containing

A and b) then by compactness $\phi(x,b)$ divides over A in $\tilde{M}$. It is also clear that if A is countable and $\phi(x,b)$ divides over A then there is some countable $M \models T$ such that $\phi(x,b)$ divides over A in the model M.

We first note that if $\phi(x,b)$ is a formula dividing over a countable set A in T_∞ then given any countable $M \models T_\infty$ containing $A \cup \{b\}$ such that $\phi(x,b)$ divides over A in M, and any approximating sequence $(N_r : r \in \omega)$ for M then for all sufficiently large $r \in \omega$, $\phi(x,b)$ divides over $A \cap N_r$ in the theory T_r $(= Th(N_r))$.

Lemma 12.3.2 *Suppose that $\phi(x,b)$ divides over A in the countable structure M. Then given any approximating sequence $(N_r : r \in \omega)$ for M, there exists $R \in \omega$ such that for all $r \geq R$, $\phi(x,b)$ divides over $A \cap N_r$ in the theory T_r (and hence $\phi(x,b)$ Γ-forks over A).*

Proof. Since $\phi(x,b)$ divides over A, there is an A-indiscernible sequence $\{b_i : i \in \omega\}$ such that $\{\phi(x,b_i) : i \in \omega\}$ is k-inconsistent for some k. Without loss of generality we can assume that the sequence $\{b_i : i \in \omega\}$ is $acl(A)$-indiscernible. Now choose any r such that $b_0, \ldots, b_{k-1} \in N_r$. Therefore, the set $\{\phi(x,b_0), \ldots, \phi(x,b_{k-1})\}$ is an inconsistent set of $acl(A \cap N_r)$-conjugates of $\phi(x,b)$ in N_r. Since N_r is a model of a stable theory this implies that $\phi(x,b)$ forks (and hence divides) over $A \cap N_r$ in the theory T_r. (e.g. by Theorem 2.4.10.) $\blacksquare$

It is important to note that the converse to Lemma 12.3.2 does *not* hold in general. That is, there are formulas $\phi(x,b)$ and sets A such that $\phi(x,b)$ divides over $A \cap N_r$ in T_r for all $r \in \omega$ but $\phi(x,b)$ does not divide over A in the structure $M = \bigcup_{r \in \omega} N_r$ (and does not divide over A in T_∞).

Example 12.3.3 *Let $\phi(x,b)$ be the formula $(x \perp b \wedge x \neq 0)$. Then:*

1. *$\phi(x,b)$ divides over $\emptyset$ in the structure N_r for each $r \in \omega$.*

2. *$\phi(x,b)$ does not divide over $\emptyset$ in the structure $M = \bigcup_{r\in\omega} N_r$ (and hence does not divide over $\emptyset$ in T_∞).*

Proof. (1) Let $r \in \omega$ and construct an indiscernible sequence $\{b_i : i \in \omega\}$ such that $b_0 = b$ and $\{b_0, \ldots, b_{r-1}\}$ is linearly independent in N_r. This is possible by quantifier elimination in N_r. In fact, by the characterisation of non-forking in the N_r, this sequence will be the Morley sequence of $tp(b/A)$ in N_r. Suppose $c \in N_r$ satisfies $\{\phi(x,b_i) : i \in \omega\}$, then $c \perp \{b_i : i \in \omega\}$. But $\{b_i : i \in \omega\}$ is a spanning set for the vector space part of N_r and hence by non-degeneracy of the bilinear form, we have $c = 0$ and this contradicts c satisfying $\phi(x,b)$.

(2) Let $\{b_i : i \in \omega\}$ be any indiscernible sequence in some model $M \vDash T_\infty$ with $b_0 = b$. Then $\{x \perp b_i \wedge x \neq 0 : i \in \omega\}$ is finitely satisfiable and hence consistent. Thus $\phi(x,b)$ does not divide over $\emptyset$ in M. Thus $\phi(x,b)$ does not divide over $\emptyset$ in T_∞. ■

Note that since $\phi(x,b)$ divides (equivalently forks) over $\emptyset$ in T_r for each $r \in \omega$, there is some integer k_r such that $\phi(x,b)$ divides over $\emptyset$ in N_r with respect to k_r. However, in the above case there is no bound on the integers k_r as r increases. That is, there is no integer k such that $\phi(x,b)$ divides over $\emptyset$ in T_r with respect to (the same) k for all $r \in \omega$. Proposition 12.3.5 below shows that this is the only obstacle preventing Γ-forking being equivalent to dividing in M. (Compare with Corollary 12.2.4.)

Definition 12.3.4 *Let $\phi(x,b)$ be a formula and let A be any (countable) set. Then $\phi(x,b)$* divides uniformly *over A (with respect to k) if there is some countable $M \vDash T_\infty$ (with $Ab \subseteq M$), some $k \in \omega$ and some approximating sequence $(N_r : r \in \omega)$ for M such that there is $R \in \omega$ such that for all $r \geq R$ the formula $\phi(x,b)$ divides over $A \cap N_r$ in T_r with respect to k.*

Proposition 12.3.5 *Let $\phi(x,b)$ be a formula and let A be any (countable) set. Then, if $\phi(x,b)$ divides uniformly over A with respect to k, then $\phi(x,b)$ divides over A (in some countable $M \models T_\infty$) with respect to k.*

Proof. Let $M_0 \models T_\infty$ be as in Definition 12.3.4. Let $(e_i : i \in \omega)$ be an orthonormal basis for the vector space part of M_0. Let $N_r = Span(e_1, \ldots, e_r)$. Then $(N_r : r \in \omega)$ is an approximating sequence for M_0. Then given any $r \geq R$ we can find (in N_r) an A-indiscernible sequence $(b_i^r : i \in \omega)$ of realisations of $tp_{N_r}(b/A \cap N_r)$ such that $\{\phi(x, b_i^r) : i \in \omega\}$ is k-inconsistent.

Let $\Gamma(y_i : i \in \omega)$ be the following set of formulas:

$$\begin{aligned}\{\chi(y_1, \ldots, y_n, a) \leftrightarrow \chi(y_{i_1}, \ldots, y_{i_n}, a) : n \in \omega,\ \chi(y_1, \ldots, y_n, z) \in L,\\ i_1 < \ldots < i_n,\ a \in A \cap N_r,\ r \geq R\}\\ \cup\{\psi(y_i, a) \leftrightarrow \psi(b, a) : \psi(y_i, a) \in tp_{N_r}(b/A \cap N_r)\,,\ i \in \omega\,,\ r \geq R\}\\ \cup\{\neg\exists x(x \in Span(e_1, \ldots e_r) \wedge (\phi(x, y_1) \wedge \ldots \wedge \phi(x, y_k))) : r < \omega\}\\ \cup\,\{y_i \in Span(e_1, \ldots e_i) : i \in \omega\}\end{aligned}$$

Then by assumption $\Gamma(y_i : i \in \omega)$ is finitely satisfiable in $M_0 = \bigcup_{r \in \omega} N_r$ and hence, by compactness, there is a sequence $(b_i : i \in \omega)$ realising $\Gamma(y_i : i \in \omega)$ in some elementary extension M' of M_0. Such a sequence must be A-indiscernible, with each of its elements realising $tp_{M_0}(b/A)$.

Let $B = Ab \cup \{b_i : i \in \omega\} \cup \{e_i : i \in \omega\}$ and let M be the substructure of M' generated by $\{e_i : i \in \omega\} \cup K_B$. (Thus M is countable and $Ab \cup \{b_i : i \in \omega\} \subseteq M$.) Then $M \models T_\infty$ and since T_∞ is model complete, we have $M \preccurlyeq M'$. As $(b_i : i \in \omega)$ satisfy $\Gamma(y_i : i \in \omega)$ it follows that $M \models \neg\exists x(\phi(x, b_{i_1}) \wedge \ldots \wedge \phi(x, b_{i_k}))$ for all $i_1 < \ldots < i_k$. Thus $\phi(x,b)$ divides over A in M. ∎

Thus, if $\phi(x,b)$ divides uniformly over A then $\phi(x,b)$ divides over A. This immediately raises the question of whether or not the converse holds. If $\phi(x,b)$ divides over A is it the case that $\phi(x,b)$ divides uniformly over A with respect to some/any countable $M \supseteq A$ and some/any approximating sequence

$(N_r : r \in \omega)$ for M? The following definition and proposition show that if it were possible to prove the equivalence of these concepts then it would follow that forking and dividing are equivalent in the (non-simple) theory T_∞.

Definition 12.3.6 *A formula* $\phi(x, b)$ forks uniformly *over a set* A *if there is* $n \in \omega$ *and formulas* $\phi_i(x, b^i)$ *for* $i < n$ *such that* $\phi(x, b) \vdash \bigvee_{j<n} \phi_j(x, b^j)$ *and for each* $j < n$, $\phi_j(x, b^j)$ *divides uniformly over* A.

Proposition 12.3.7 *Suppose that* $\phi(x, b)$ *forks uniformly over* A, *then* $\phi(x, b)$ *divides uniformly over* A. *(In particular* $\phi(x, b)$ *divides over* A.*)*

Proof. Let $\phi_i(x, b^i)$ for $i < n$ be formulas such that $\phi(x, b) \vdash \bigvee_{j<n} \phi_j(x, b^j)$ and each $\phi_j(x, b^j)$ divides uniformly over A with respect to k_j. By compactness we can easily find some countable $M \vDash T_\infty$ with an approximating sequence $(N_r : r \in \omega)$ and $R \in \omega$ such that for each $j < n$ and each $r \geq R$, $\phi_j(x, b^j)$ divides over $A \cap N_r$ in the theory T_r with respect to k_j. Then, as the structures N_r are stable and hence simple, by the proof of the equivalence of forking and dividing in simple theories (Theorem 2.3.2) we have that, for each $r \geq R$, $\phi(x, b)$ divides over $A \cap N_r$ in T_r with respect to $k = Max\{k_0, \dots, k_{n-1}\}$. Thus $\phi(x, b)$ divides uniformly over A with respect k and in particular $\phi(x, b)$ divides over A. ■

Suppose we could prove that for all formulas $\phi(x, b)$ and all (countable) sets A, that if $\phi(x, b)$ divides over A in T_∞ then $\phi(x, b)$ divides uniformly over A; then we could prove the equivalence of forking and dividing (over countable sets of parameters) as follows. Suppose that $\phi(x, b)$ forks over A. Then $\phi(x, b)$ is covered by finitely many formulas each of which divide (and hence divide uniformly) over A. Thus $\phi(x, b)$ forks uniformly over A and hence by Proposition 12.3.7 $\phi(x, b)$ divides uniformly over A. Then Proposition 12.3.4 implies that $\phi(x, b)$ divides over A.

Therefore, for a formula $\phi(x, b)$ and a (countable) set A the following would be equivalent:

1. $\phi(x,b)$ divides over A;

2. $\phi(x,b)$ forks over A;

3. $\phi(x,b)$ divides uniformly over A;

4. $\phi(x,b)$ forks uniformly over A.

We could then characterise non-forking in countable models of T_∞ by: $tp(c/B)$ does not fork over A if and only if for each formula $\phi(x,b) \in tp(c/B)$, $\phi(x,b)$ does not divide uniformly over A.

Suppose that this was the case and suppose that $\phi(x,b)$ forks over A (A countable). Then $\phi(x,b)$ divides over A and hence Γ-forks over A. Thus if $tp(c/Ab)$ does not Γ-fork over A we would have that $tp(c/Ab)$ does not fork over A. (i.e. Γ-non-forking is a refinement of the non-forking relation on T_∞.)

In practice it seems that the best strategy for attempting to prove the equivalence of dividing and dividing uniformly is to try to prove the equivalence of dividing with other apparently stronger concepts such as 'dividing finitely' and 'dividing via a Γ-Morley sequence' both of which imply dividing uniformly. These approaches are discussed below.

Definition 12.3.8 *A formula $\phi(x,b)$* divides finitely *over a set A if there is an A-indiscernible sequence $(b_i : i \in \omega)$ of realisations of $tp(b/A)$ such that $\{\phi(x,b_i) : i \in \omega\}$ is inconsistent and $\{b_i : i \in \omega\}$ is contained in some finite dimensional substructure $N \subseteq \tilde{M}$. (A substructure $N \subseteq \tilde{M}$ is finite dimensional (of dimension r) if $N \models \Theta_r \wedge \neg\Theta_{r+1}$.) If $\phi(x,b)$ divides finitely over A we write $\phi(x,b)$ divides$_{FIN}$ A. Clearly $\phi(x,b)$ divides$_{FIN}$ A implies $\phi(x,b)$ divides over A (in the theory T_∞).*

Lemma 12.3.9 *Suppose A is a countable set and $\phi(x,b)$ divides$_{FIN}$ A. Then $\phi(x,b)$ divides uniformly over A (with respect to any approximating sequence $(N_r : r \in \omega)$).*

Proof. Let $(b_i : i \in \omega)$ be a finite dimensional A-indiscernible sequence witnessing the dividing of $\phi(x, b)$ over A in $\tilde{M}$. Then for some $k \in \omega$, the set $\{\phi(x, b_i) : i \in \omega\}$ is k-inconsistent. Let M be a countable elementary substructure of $\tilde{M}$ containing $A \cup \{b_i : i \in \omega\}$. Let $(N_r : r \in \omega)$ be any approximating sequence for M and choose R large enough such that $\{b_i : i \in \omega\} \subseteq N_r$ for all $r \geq R$. Then $(b_i : i \in \omega)$ is an $A \cap N_r$-indiscernible sequence of realisations of $tp_{N_r}(b/A \cap N_r)$ and obviously $\{\phi(x, b_i) : i \in \omega\}$ is k-inconsistent in N_r. Thus $\phi(x, b)$ divides uniformly over A (with respect to k). ■

The advantage of the notion of dividing finitely over dividing uniformly is that dividing finitely makes sense over any set of parameters whereas dividing uniformly only really makes sense over a countable set of parameters.

Recall that a Morley sequence for a type $tp(b/A)$ is an A-indiscernible sequence $\{b_i : i \in \omega\}$ of realisations of $tp(b/A)$ such that for each $n \in \omega$, $tp(b_n/Ab_0, \ldots, b_{n-1})$ does not fork over A. This definition makes sense for any theory, however in stable theories the type of a Morley sequence (over an algebraically closed set) is unique. One of the fundamental steps in the proof that forking is equivalent to dividing in simple theories is showing that a formula $\phi(x, b)$ divides over a set A if and only if for some (equivalently every) Morley sequence $\{b_i : i \in \omega\}$ for $tp(b/A)$ the set $\{\phi(x, b_i) : i \in \omega\}$ is inconsistent.

Here, instead of using a Morley sequence for $tp(b/A)$ in T_∞, we use a 'Γ-Morley sequence'. This is simply an A-indiscernible set $(b_i : i \in \omega)$ with $b_0 = b$ such that the set $\{b_i : i \in \omega\}$ is Γ-independent over A (in the sense of Definition 12.2.1). The results of Section 12.2 then show that for any tuple b and any set A a Γ-Morley sequence of $tp(b/A)$ exists and, as long as A is assumed to be algebraically closed, then such a Γ-Morley sequence is unique. The important point is that if A is countable, M is countable model of T_∞ containing A and $(N_r : r \in \omega)$ is an approximating sequence for M, then the Γ-Morley sequence

(ΓMS) for $tp(b/A)$ in M is a limit of the Morley sequences for $tp_{N_r}(b/A \cap N_r)$ as r increases. This is because non-forking in the approximating structures N_r is exactly Γ-non-forking and therefore Morley sequences and Γ-Morley sequences in the structures N_r are the same thing.

Lemma 12.3.10 *Let A be countable (and algebraically closed). Let b be a finite tuple and let M be a countable model containing $A \cup \{b\}$. Let $(b_i : i \in \omega)$ be the Γ-Morley sequence for $tp(b/A)$. Let $(N_r : r \in \omega)$ be some approximating sequence for M. Then given any $n \in \omega$ there is $R \in \omega$ such that if $r \geq R$ and $(b_i^r : i \in \omega)$ is the Morley sequence of $tp_{N_r}(b/A \cap N_r)$ then $tp_M(b_0^r, \dots, b_{n-1}^r/A \cap N_r) = tp_M(b_0, \dots, b_{n-1}/A \cap N_r)$.*

Proof. Let $(b_i : i \in \omega)$ be the Γ-Morley sequence for $tp(b/A)$ and let $n \in \omega$. Choose $R \in \omega$ such that $(b_0, \dots, b_{n-1}) \subseteq N_R$. Let $r \geq R$ and let $(b_i^r : i \in \omega)$ be the Morley sequence of $tp_{N_r}(b/A \cap N_r)$. Then by the characterisation of non-forking in the N_r, $(b_0, \dots, b_{n-1})$ is an $A \cap N_r$-indiscernible, $A \cap N_r$-independent sequence in N_r. Any such sequence extends to a Morley sequence for $tp_{N_r}(b/A \cap N_r)$ and hence by the uniqueness of Morley sequences in stable theories $tp_M(b_0^r, \dots, b_{n-1}^r/A \cap N_r) = tp_M(b_0, \dots, b_{n-1}/A \cap N_r)$. But N_r is a homogenous substructure of M and this gives the required result. (Note that since A is assumed to be algebraically closed then by the characterisation of algebraic closure in T_∞ and in T_r, given by Proposition 9.5.1, we have that $A \cap N_r$ is algebraically closed.) ■

Definition 12.3.11 *Let $\phi(x, b)$ be a formula and let A be a set of parameters. We say that $\phi(x, b)$* divides over A with respect to the Γ-Morley sequence *(for $tp(b/A)$) or $\phi(x, b)$* divides-ΓMS *over A if for some (equivalently any) Γ-Morley sequence $(b_i : i \in \omega)$ for $tp(b/A)$ the set $\{\phi(x, b_i) : i \in \omega\}$ is inconsistent.*

Note that since a Γ-Morley sequence over A is A-indiscernible it follows that if $\phi(x, b)$ divides-ΓMS over A then $\phi(x, b)$ divides over A. In fact we can show that if $\phi(x,b)$ divides-ΓMS over a countable set A then $\phi(x,b)$ divides uniformly and finitely over A.

Lemma 12.3.12 *Suppose that $\phi(x,b)$ divides-ΓMS over a countable set A. Then $\phi(x,b)$ divides uniformly over A and $\phi(x,b)$ divides finitely over A.*

Proof. Let M be a countable model containing $A \cup \{b\}$ and a Γ-Morley sequence $(b_i : i \in \omega)$ for $tp(b/A)$ such that $\{\phi(x,b_i) : i \in \omega\}$ is k-inconsistent for some $k \in \omega$. Let $(N_r : r \in \omega)$ be an approximating sequence for M. By Lemma 12.3.10 we can choose $R \in \omega$ such that $b_0, \dots, b_{k-1} \in N_R$ and for any $r \geq R$, if $(b_i^r : i \in \omega)$ is the Morley sequence of $tp_{N_r}(b/A \cap N_r)$ then $tp_M(b_0^r, \dots, b_{n-1}^r/A \cap N_r) = tp_M(b_0, \dots, b_{n-1}/A \cap N_r)$. In particular, if $r \geq R$, since $(b_i^r : i \in \omega)$ is $A \cap N_r$-indiscernible, we have that $\{\phi(x, b_i^r) : i \in \omega\}$ is k-inconsistent. (i.e. $\phi(x,b)$ divides over $A \cap N_r$ with respect to k in T_r.) Therefore $\phi(x,b)$ divides uniformly over A (with respect to k).

The sequence $(b_0^R, \dots b_{k-1}^R)$ extends to an infinite A-indiscernible sequence in N_R (namely the Morley sequence). It is then clear that for each $r \geq R$, it is possible to extend the sequence $(b_0, \dots, b_{k-1})$ to an $A \cap N_r$-indiscernible sequence in N_r contained in N_R. Then, by compactness, $(b_0, \dots, b_{k-1})$ extends to an infinite A-indiscernible sequence $(b_i : i \in \omega)$ in some model $M \models T_\infty$ with $(b_i : i \in \omega)$ contained in a finite dimensional substructure generated by N_R. ∎

We were unable to prove the converse: $\phi(x,b)$ divides uniformly over A implies $\phi(x,b)$ divides-ΓMS over A. However, it is the case that if $\phi(x,b)$ divides uniformly over A with the property that in some approximating sequence $(N_r : r \in \omega)$ there is $k \in \omega$ such that for all sufficiently large r, whenever $(b_i^r : i \in \omega)$ is the Morley sequence of $tp_{N_r}(b/A \cap N_r)$ then $\{\phi(x, b_i^r) : i \in \omega\}$ is k-inconsistent, then $\phi(x,b)$ divides-ΓMS over A. This is because a Γ-Morley sequence is essentially a union of Morley sequences in the N_r. Therefore if we could prove that the ($\aleph_1$-categorical) theories T_r have the property that whenever $\phi(x,b)$ divides over a set A with respect to $k \in \omega$, it is then the case that

$\{\phi(x, b_i) : i \in \omega\}$ is k-inconsistent, where $(b_i : i \in \omega)$ is the Morley sequence of $tp_{N_r}(b/A)$; then we would have that dividing uniformly and dividing with respect to the Γ-Morley sequence are equivalent (and hence dividing finitely is also equivalent to these).

It would be useful to know if $(b_i : i \in \omega)$ is a Γ-Morley sequence is actually a Morley sequence. That is, do we have that $tp(b_j/Ab_0, \dots b_{j-1})$ does not fork over A for each $j \in \omega$? The following lemma gives a slightly weaker version of this.

Lemma 12.3.13 *Let $\{b_i : i \in \omega\}$ be a Γ-Morley sequence (ΓMS) for $tp(b/A)$. Then for each $j \in \omega$, $tp(b_j/Ab_0, \dots b_{j-1})$ does not divide over A.*

Proof. We must prove that for each $j \in \omega$, $tp(b_j/Ab_0, \dots b_{j-1})$ does not divide over A. Given $j \in \omega$, we can choose $R_j \in \omega$ such that $\{b_0, \dots, b_{j-1}, b_j\}$ is the first $j + 1$ terms of the Morley sequence of $tp(b/A)$ in N_r for all $r \geq R_j$. Hence $tp_{N_r}(b_j/A \cap N_r, b_0, \dots, b_{j-1})$ does not fork over $A \cap N_r$ in each N_r. Thus $tp_{N_r}(b_j/A \cap N_r, b_0, \dots, b_{j-1})$ does not divide over $A \cap N_r$ in each N_r. Then by Lemma 12.3.2 $tp(b_j/Ab_0, \dots b_{j-1})$ does not divide over A in M. ■

Remark 12.3.14 *Suppose we could prove that dividing and dividing uniformly are equivalent on T_∞. Then forking is equivalent to dividing and it would follow immediately from Lemma 12.3.13 that every Γ-Morley sequence is in fact a Morley sequence (but certainly not conversely).*

Note that in general the Γ-Morley sequence of $tp(b/A)$ is *not* the unique sequence such that $\{b_i : i \in \omega\}$ is A-indiscernible and such that for each $j \in \omega$, $tp(b_j/Ab_0, \dots b_{j-1})$ does not divide over A. Suppose that b is a single vector element. Then a sequence of conjugates of b, $\{b_i : i \in \omega\}$ with the field elements $[b_i, b_j]$ algebraically independent over A is a Γ-Morley sequence for $tp(b/\emptyset)$ but a sequence $\{b'_i : i \in \omega\}$ of conjugates of b such that $b_i \perp b_j$ for all $i \neq j$ can be chosen such that $\{b'_i : i \in \omega\}$ is A-indiscernible and with $tp(b'_j/Ab'_0, \dots b'_{j-1})$ not dividing over A for each $j \in \omega$.

Suppose that $\phi(x, b)$ divides over a set A in a countable model M. We wish to prove that $\phi(x, b)$ divides uniformly (or finitely) over A. Suppose that $\phi(x, b)$ divides over A via an indiscernible sequence $(b_i : i \in \omega)$ with respect to $k \in \omega$. For each $j \in \omega$, we can choose r large enough so that $\{b_0, \ldots, b_j\} \subseteq N_r$. Then, if $j \geq k$, $\phi(x, b)$ has a set of k conjugates in N_r which are inconsistent. (In particular, $\phi(x, b)$ divides/forks over $A \cap N_r$ in N_r.) Suppose that $b_0, \ldots, b_{k-1}$ are A (or $A \cap N_r$-) independent. Then since $b_0, \ldots, b_{k-1}$ are $A \cap N_r$-indiscernible they form the first k terms of the (unique) Morley sequence $(b_i^r : i \in \omega)$ of $tp_{N_r}(b/A \cap N_r)$ and therefore $\{\phi(x, b_i^r) : i \in \omega\}$ is k-inconsistent giving the desired result. Hence, it is sufficient to prove, that if $(b_i : i \in \omega)$ is an infinite dimensional sequence (i.e. not contained in any N_r) and indiscernible over A, such that $\{\phi(x, b_i) : i \in \omega\}$ is k-inconsistent, then we can find $\{b'_0, \ldots, b'_{k-1}\} \in N_r$ (for large enough r) such that $\{b'_0, \ldots, b'_{k-1}\}$ are $A \cap N_r$-independent and indiscernible in N_r and such that $\{\phi(x, b'_0), \ldots, \phi(x, b'_{k-1})\}$ are inconsistent.

If we can do this it follows that if $\{b_i : i \in \omega\}$ is a (the) Γ-Morley sequence for $tp(b/A)$ then $\phi(x, b)$ divides over A if and only if $\{\phi(x, b_i) : i \in \omega\}$ is inconsistent (i.e. $\phi(x, b)$ divides-ΓMS over A). In particular this shows that dividing, dividing uniformly, dividing ΓMS and dividing finitely are equivalent concepts. Hence forking is equivalent to dividing in T_∞.

A different strategy for trying to prove the equivalence of dividing and dividing uniformly (or dividing-ΓMS) is given by the results of Section 12.4 on dimensions of definable sets. If $\phi(x, b)$ is a formula with a single field-sort variable it is clear (e.g. by (ω, ω)-categoricity) that $\phi(x, b)$ divides over a set A if and only if it divides-ΓMS over A. If $\phi(x, b)$ is a formula with a single vector variable then $\phi(x, b)$ defines either a set of finite dimension or finite codimension in any model of T_∞. If $\phi(x, b)$ is of finite dimension then the proof of Proposition 2.3.1 used for characterising dividing in simple theories can be used to show that $\phi(x, b)$ divides over A if and only if $\phi(x, b)$ divides over A with respect to some (equivalently any) Γ-Morley sequence for $tp(b/A)$.

This just leaves the case of formulas $\phi(x, b)$ defining sets of finite codimension. Intuitively, it seems that if a formula $\phi(x, b)$ of finite codimension divides over A then $\phi(x, b)$ should divide finitely over A, but again we have been unable to prove this.

12.4 Dimensions on Definable Sets

In this section we use the methods of Section 10.3 to show that if X is a definable set (of vectors) in a model of T_∞ then in some sense X either has finite dimension or finite codimension. If X is a finite dimensional set then X also has finite Morley rank and the Morley rank is equal to the dimension. For the sake of clarity we consider only spaces with symmetric bilinear forms, although, by the note at the end of Section 10.3, all of the following can be easily adapted to alternating spaces by making use of a normal basis which exists in all such spaces of countable dimension.

Let M be a countable model of T_∞ and let $(N_r : r \in \omega)$ be some approximating sequence for M. Let X be a definable subset (of vector elements) of M. Then there is $R \in \omega$ such that X is defined over N_r for all $r \geq R$. In particular we have that if X is defined by $\phi(x, b)$ then $\phi(N_r, b) = N_r \cap \phi(M, b)$ and this set has finite Morley rank ($\leq r$) in N_r. By the results of Section 10.3, Morley rank on definable subsets X of N_r is equivalent to the dimension $\mathrm{Dim}(X)$ with inductive definition given by:

$$\begin{aligned}&\mathrm{Dim}(X) \geq n+1 \Leftrightarrow \\ &\qquad \exists z \text{ such that } \{y \in \mathrm{Span}(z) : \mathrm{Dim}(X \cap (y + z^\perp)) \geq n\} \text{ is infinite.}\end{aligned}$$

In order to study the definable set X in M we look at the definable sets $\phi(N_r, b) = X \cap N_r$ in the structures N_r. We can describe the 'rank' of X in M by means of the following function $f_X : \omega \to \omega$,

$$f_X(r) = \mathrm{MR}_{N_r}(X \cap N_r) \;\; (= \mathrm{Dim}_{N_r}(X \cap N_r)).$$

From the definition of $\mathrm{Dim}(X)$ in each N_r it is immediately clear that $f_X(r) \leq r$, and for each $r \in \omega$:

$$f_X(r) \leq f_X(r+1) \leq f_X(r) + 1.$$

This is simply because $\mathrm{Dim}_{N_r}(X \cap N_r) = \mathrm{Dim}_{N_{r+1}}(X \cap N_{r+1} \cap e_{r+1}^{\perp}) \leq \mathrm{Dim}_{N_{r+1}}(X \cap N_r) \leq \mathrm{Dim}_{N_{r+1}}(X \cap N_{r+1} \cap e_{r+1}^{\perp}) + 1 = \mathrm{Dim}_{N_r}(X \cap N_r) + 1$. In fact we claim that as r increases, the function $f_X(r) = MR_{N_r}(X)$ is either eventually constant or is strictly increasing after a certain point.

Proposition 12.4.1 *Let M be a countable model of T_∞ with approximating sequence $(N_r : r \in \omega)$ and let X be a definable subset of (singleton) vector elements. Then X has either finite dimension or finite codimension (with respect to the approximating sequence $(N_r : r \in \omega)$) in the following sense: either,*

1. *there is some integer $d \in \omega$ and some $R \in \omega$ such that for all $r \geq R$, $MR_{N_r}(X \cap N_r) = d$ (i.e. X has* finite dimension *d); or*

2. *there is some integer $d \in \omega$ and some $R \in \omega$ such that for all $r \geq R$, $MR_{N_r}(X \cap N_r) = r - d$ (i.e. X has* finite codimension *d).*

Proof. Choose $R \in \omega$ such that X is definable over N_R. Suppose that X does not have finite dimension. Then it is clear that there is some $a \in X \backslash N_R$. Since X is definable over N_R, X is fixed setwise by any $f \in Aut_{N_R}(M)$. Hence X contains every realisation of the type $tp_M(a/N_R)$. Suppose that $[a, e_i] = \alpha_i$ for each $i = 1, \ldots, R$, and let $\gamma = \sum_{i=1}^{R} \alpha_i^2$. Then, by quantifier elimination $tp_M(a/N_R) = \{a' \in M : [a', e_i] = \alpha_i,\ a \notin N_r \text{ and } [a', a'] = [a, a]\}$. Therefore, for any $r > R$, $X \cap N_r$ contains the set $\{\beta_1 e_1 + \cdots + \beta_R e_R + \beta_{R+1} e_{R+1} + \cdots + \beta_r e_r : \beta_i = \alpha_i \text{ for } i = 1, \ldots, R,\ \beta_j \neq 0 \text{ for some } j > R \text{ and } \sum_{i=R+1}^{r} \beta_i^2 = [a, a] - \gamma\}$. This set clearly has Dim / Morley rank $\geq r - (R+1)$ in N_r. Thus for all $r \geq R$, $\mathrm{MR}_{N_r}(X \cap N_r) \geq r - (R+1)$. Therefore X has codimension d for some $d \leq (R+1)$. ∎

Corollary 12.4.2 *The dimension or codimension as described is Proposition 12.4.1 is independent of the choice of approximating sequence $(N_r : r \in \omega)$ and countable model M.*

Proof. Let M and M' be countable models containing the parameters of X and suppose that $(N_r : r \in \omega)$ is an approximating sequence for M and $(N'_r : r \in \omega)$ is an approximating sequence for M'. Choose R large enough so that X is defined over both N_R and N'_R. Let $r \geq R$. Then since N_r and N'_r are both models of T_r containing the parameters of X it follows that $\mathrm{MR}_{N_r}(X \cap N_r) = \mathrm{MR}_{N'_r}(X \cap N'_r)$. ■

Definition 12.4.3 *Let $\phi(x, b)$ be any formula with a single free vector variable x. Let M be a countable model of T_∞ containing b and let $(N_r : r \in \omega)$ be an approximating sequence for M. Let $X = \phi(M, b)$. Then for $d \in \omega$,* $\mathrm{Dim}(X) = d$ *if there is $R \in \omega$ such that for all $r \geq R$, $MR_{N_r}(X \cap N_r) = d$ and* $\mathrm{Codim}(X) = d$ *if there is $R \in \omega$ such that for all $r \geq R$, $MR_{N_r}(X \cap N_r) = r - d$.*

By Proposition 12.4.1, it is the case that for every such set X either $Dim(X) < \omega$ or $Codim(X) < \omega$. By Corollary 12.4.2 Dim and Codim are well defined on the formula $\phi(x, b)$ (i.e. independent of the choice of countable model and approximating sequence).

The use of the notation 'Dim' may seem ambiguous due to the fact that on the finite-dimensional structures, N_r, it defines the rank with inductive definition given in 10.3.7 and on the infinite dimensional structures $M \models T_\infty$ it defines the dimension defined above. However it is in fact the case that the dimension 'Dim' defined above on infinite dimensional spaces is given by extending the inductive definition of 'Dim' from the finite dimensional spaces.

Let Dim* be the dimension on definable sets of vectors in models of T_∞ given by

$$\mathrm{Dim}^*(X) \geq n + 1 \Leftrightarrow$$

$$\exists z \text{ such that } \{y \in \mathrm{Span}(z) : \mathrm{Dim}^*(X \cap (y + z^\perp)) \geq n\} \text{ is infinite.}$$

(i.e. Dim* is the dimension on models of T_∞ with the same definition as Dim on models of T_r for $r \in \omega$). We claim that Dim and Dim* define the same rank on models of T_∞.

Recall that, in each N_r, the dimension $\mathrm{Dim}(X)$ is definable. That is, for each formula $\phi(x, y)$ (with x a single vector variable) and each $n \in \omega$, there is a formula $\psi(y)$ such that for each $b \in N_r$, $\mathrm{Dim}_{N_r}(\phi(x, b)) = n \Leftrightarrow N_r \models \psi(b)$. By the definition of Dim, the formula $\psi(y)$ is the same in all N_r (at least for $r \geq n$). Suppose that $\phi(x, y)$ is an L-formula and let $b \in M$. Suppose that $X = \phi(x, b)$ is a finite dimensional set in M in the sense described above. (There is $R \in \omega$ and $n \in \omega$ such that $\mathrm{Dim}_{N_r}(X \cap N_r) = n$ for all $r \geq R$.) Then for all $r \geq R$, $N_r \models \psi(b)$. Therefore, by 9.3.3, $M \models \psi(b)$ and hence $\mathrm{Dim}^*(\phi(x, b)) = n$. Conversely suppose that $\mathrm{Dim}^*(\phi(x, b)) = n$. The rank Dim* is definable on T_∞ with exactly the same definition as the rank Dim on the theories T_r (using the fact that for any vector element a the formula '$x \in \mathrm{Span}(a)$' is strongly minimal). Therefore for all sufficiently large N_r in some approximating sequence, $\mathrm{Dim}_{N_r}(\phi(x, b)) = n$ and thus $\mathrm{Dim}_M(\phi(x, b)) = n$. Hence Dim is indeed defined on models of T_∞ (with the same definition as on the N_r's) and therefore has the same inductive definition:
$\mathrm{Dim}(X) \geq n + 1 \Leftrightarrow \exists z \, \{y \in \mathrm{Span}(z) : \mathrm{Dim}(X \cap (y + z^\perp)) \geq n$.

The proof of Proposition 12.4.1 shows that if X is a finite dimensional set (in the sense that $\mathrm{Dim}(X)$ is bounded in the N_r as r increases) then there is $R \in \omega$ such that $X \subseteq N_R$. In fact we can take this N_R to be any N_r containing a set of parameters over which X is defined. Similarly if X has finite codimension d and X is definable over N_R then $d \leq R + 1$. This is directly from the proof above. The reason that the possibility that $d = R + 1$ exists in this case is essentially because the definable set $\{x \in N_r : [x, x] = \alpha\}$ for some $\alpha \in K$ is defined over $N_0 = K$ but has dimension $r - 1$ in N_r.

Suppose that $\phi(x, y)$ is a formula with x a single vector variable and y any tuple of variables. Let R be the number of vector variables in the tuple

y. Let $\phi(x, b)$ be an instance of $\phi(x, y)$. Then given any countable model M containing b it is possible to find an approximating sequence $(N_r : r \in \omega)$ for M such that $b \subseteq N_R$. Then, by the proof of Proposition 12.4.1, either $\mathrm{Dim}(\phi(x, b)) \leq R$ or $\mathrm{Codim}(\phi(x, b)) \leq R + 1$. Therefore given any formula $\phi(x, y)$, with x a single vector variable and y and set of parameter variables, there is an integer R such that any instance of $\phi(x, y)$ (or $\neg\phi(x, y)$) has finite dimension if and only if it has dimension $\leq R$ and has finite codimension if and only if it has codimension $\leq R + 1$. In fact R can be taken to be the number of vector variables in the parameter set y.

Also note that if X is a finite dimensional definable set, then X is defined over N_R if and only if $X \subseteq N_R$. (Left to right is by the remarks above. Right to left follows since if X is a definable set then X is definable over a set A if and only if X is fixed setwise by every elementary map fixing A pointwise, so if $X \subseteq N_R$ then obviously X is fixed (pointwise even) by every automorphism fixing N_R pointwise.)

We now show that dimension on M is equal to Morley rank on M.

Corollary 12.4.4 *Let M be a (countable) model of T_∞ and let X be a definable subset of M. Then X has finite Morley rank in M if and only if Dim(X) is finite. In this case $MR(X) =$ Dim(X).*

Proof. Suppose X is a finite dimensional set. By the above comments there is $R \in \omega$ such that $X \subseteq N_R$ and hence X is defined over N_R. By definition of Dim, $MR(X) \geq \mathrm{Dim}(X)$. Suppose inductively that $MR(X) \geq n \Rightarrow \mathrm{Dim}(X) \geq n$. Suppose $MR(X) \geq n + 1$. Then there are pairwise disjoint definable sets $(X_i : i \in \omega)$ such that $MR(X \cap X_i) \geq n$ for each $i \in \omega$. Therefore, by the inductive hypothesis, $\mathrm{Dim}(X \cap X_i) \geq n$ and since X and each $X \cap X_i$ is contained in N_R (and definable over N_R) it follows that $MR_{N_R}(X \cap N_R) \geq n+1$. But in N_R we know Morley rank is equal to Dim. Therefore $\mathrm{Dim}(X) \geq n + 1$ as required.

Suppose that X is a set of finite codimension. Suppose, for a contradiction, that $\mathrm{MR}(X) = \alpha < \infty$. Then, by the characterisation of codimension, it follows that $\mathrm{MR}(V) < \infty$ (in fact $\mathrm{MR}(V) = \alpha + \mathrm{Codim}(X)$), where V is the set of all vector variables in a model of T_∞. But since $\mathrm{MR}(K) = 1$ (where K is the set of field elements in model of T_∞) it then follows that $\mathrm{MR}(T_\infty) < \infty$, which is clearly a contradiction as T_∞ is not even a simple theory and so is certainly not ω-stable. ■

Thus, although $M \models T_\infty$ is certainly not a structure of finite Morley rank, Morley rank is definable in M in the following sense: for every formula $\phi(x, y)$ (with x a single vector variable) and every $n \in \omega$, there is a formula $\psi(y) \in L$ such that for all $b \in M$, $MR(\phi(x, b)) = n \Leftrightarrow \models \psi(b)$. In particular such a formula $\phi(x, y)$ has ordinal valued Morley rank if and only if it has finite Morley rank (if and only if it has Morley rank $\leq R$ where R is the number of vector variables in y).

We can also define 'codimension'. This is done as follows. Let $\phi(x, y)$ be a formula with x a single vector variable. Let R be the number of vector variables in the tuple y. Then if $\phi(x, b)$ is any instance of $\phi(x, y)$ we have that $\mathrm{Codim}(\phi(x, b)) \leq R + 1$. Choose $N_R \models T_R$ such that $b \subseteq N_R$. Then, for $d \leq R + 1$ $\mathrm{Codim}(\phi(x, b)) = d$ if and only if $\mathrm{Dim}_{N_R}(\phi(x, b)) = R - d$ and $\mathrm{Dim}_{N_{R+1}}(\phi(x, b)) = (R + 1) - d$. This is definable by expressing that $\mathrm{Codim}(\phi(x, y)) = d$ if and only if $\mathrm{Dim}_{N_R}(\phi(x, y)) = R - d$ and $\mathrm{Dim}_{N_{R+1}}(\phi(x, b)) = (R + 1) - d$ for every pair of substructures $N_R \subseteq N_{R+1}$ with $N_R \models T_R$, $N_{R+1} \models T_{R+1}$ and $y \subseteq N_R$. This is definable by the definability Dim in the structures N_r.

We summarise the results of this section with the following theorem.

Theorem 12.4.5 *It is possible to define on T_∞, ranks 'Dim' and 'Codim' with the following properties:*

1. *If X is any definable set of vector variables then Dim(X) and Codim(X) both take values in $\mathbb{N} \cup \{\infty\}$ and exactly one of Dim(X) or Codim(X) is finite.*

2. *Dim$(X) = d < \omega$ if and only if for some (equivalently any) countable model M of T_∞ containing the parameters of X and $(N_r : r \in \omega)$ some (equivalently any) approximating sequence for M, there is $R \in \omega$ such that $MR_{N_r}(X \cap N_r) = d$ for all $r \geq R$.*

3. *Codim$(X) = d < \omega$ if and only if for some (equivalently any) countable model M of T_∞ containing the parameters of X and $(N_r : r \in \omega)$ some (equivalently any) approximating sequence for M, there is $R \in \omega$ such that $MR_{N_r}(X \cap N_r) = r - d$ for all $r \geq R$.*

4. *If $\phi(x, y)$ is a formula with a single vector variable then for each $n \in \omega$ there are formulas $(\psi_n(y) : n \in \omega)$, $(\chi_n(y) : n \in \omega)$ such that Dim$(\phi(x, b)) = n \Leftrightarrow \models \psi_n(b)$ and Codim$(\phi(x, b)) = n \Leftrightarrow \models \chi_n(b)$. (Dim and Codim are definable.) Moreover, if R is the number of vector variables in the tuple y then for all $n > R$, $\models \neg \exists y \psi_n(y)$ and for all $n > R + 1$, $\models \neg \exists y \chi_n(y)$. (i.e. for all b either Dim$(\phi(x, b)) \leq R$ or Codim$(\phi(x, b)) \leq R + 1$.)*

5. *If X is a definable set of vector variables then X has (ordinal valued) Morley rank if and only if Dim$(X) < \omega$ in which case MR$(X) =$ Dim(X).*

12.5 Ideas for Further Work

In Section 12.3 we tried to characterise forking and dividing in the theory T_∞ of infinite dimensional vector spaces over algebraically closed fields with bilinear forms. Although we were unable to do this we left open the question of whether dividing in T_∞ was equivalent to the notion of dividing finitely (or dividing uniformly) and showed that is this was the case then forking and dividing in T_∞ would be equivalent.

Another method for attempting to characterise dividing in an arbitrary theory is given by Lemma 2.2.3, which says that a (partial) type $p(x, b)$ over a set Ab does not divide over A if and only if for every infinite A-indiscernible sequence I containing b, there is c realising $p(x, b)$ such that I is Ac-indiscernible. By using the quantifier elimination result for T_∞ it is possible to characterise dividing of formulas and partial types in some special cases. For instance, if there is $a \in acl(Ab)$ such that $[a, b] = \beta$ then the formula $[X, b] = \beta$ does not divide over A. Motivated by this and a number of other cases we conjecture that if c and b are single vector elements then $tp(c/Ab)$ does not divide over A if and only if $K_{Ac} \downarrow_{K_A} K_{Ab}$ and either $c \in \langle A \rangle$ or, $c \notin \langle Ab \rangle$ and $[c, b] \in acl\{[a, b] : a \in A_V\}$ or $[c, b] \notin K_{Ab}(K_{Ac})^{alg}$. We were unable to prove this in general, there are a seemingly prohibitive number of cases to consider, but it is true in many special cases. If this was dividing on T_∞ (it would be easy to extend the definition to sets of elements including field elements) it would show very clearly that dividing extends the notion of Γ-forking in T_∞, defined in Section 12.2. There is also the question of whether we could show via this proposed characterisation that dividing is equivalent to dividing finitely (or uniformly) hence showing that forking and dividing are equivalent. It would also be interesting to know if the non-forking relation on T_∞ was symmetric or transitive.

Suppose that $M \vDash T_\infty$ and $c \notin \langle M \rangle$. Then given $A \supseteq M$, $tp(c/M)$ has a unique Γ-non-forking extension to a type over A. This is simply the type extending $tp(c/M)$ expressing that $c \notin \langle A \rangle$ and for any linearly independent set of vector elements $\{a_i : i \in I\} \subseteq A \backslash M$, the set $\{[c, a_i] : i \in I\}$ is algebraically independent over the field $K_A(K_{Mc})$. However by the characterisation of dividing given by Lemma 2.2.3, if $c \notin \langle A \rangle$ and for all $a \in A \backslash \langle M \rangle$, $c \perp a$ then $tp(c/A)$ does not divide over A. Thus if our characterisation is correct we certainly have that types over models are not necessarily stationary. This then raises the question of whether or not a result like the independence theorem over a model holds for non-forking in T_∞. This actually seems quite likely since independent sets would be at least linearly independent and results such as Lemma 5.2.3 together with compactness allows us to find elements with arbitrary bilinear products with any linearly independent set.

Another approach to studying the theory T_∞ would be to ask what kinds of structures are definable in models of T_∞. For instance, what are the groups definable in T_∞? This is motivated by results such as the theorem saying that a group interpretable in an algebraically closed field is an algebraic group over an algebraically closed field (see e.g. [55]) and a group interpretable in an $\aleph_0$-categorical structure without the strict order property is nilpotent-by-finite (see [47]). In fact Cherlin and Hrushovski have proved that a group interpretable in a smoothly approximable structure is finite-by-abelian-by-finite. Evans and Wagner reach the same conclusion for $\aleph_0$-categorical, supersimple structures. Perhaps a group definable in the theory T_∞ will be finite rank-by-abelian-by-finite rank?

Note that if G is a finite-dimensional group in the sense that $\mathrm{dom}(G)$ is a set of vector elements of finite dimension (as defined in Section 12.4), then G is contained in and defined over some finite dimensional substructure N_r which is itself interpretable in the underlying algebraically closed field. Hence G is interpretable in the field and is therefore an algebraic group over an algebraically closed field.

If M is a structure smoothly approximated by finite substructures then

M does not interpret the random graph (see e.g. [15]). In particular an infinite dimensional vector space over a finite field with a bilinear form does not interpret the random graph. It would be interesting to know if this result can be extended to such spaces over algebraically closed fields.

A useful property of the infinite dimensional vector spaces over finite fields is that they admit weak elimination of imaginaries. It is therefore natural to ask if the corresponding spaces over algebraically closed fields also have weak elimination of imaginaries. This seems feasible since algebraically closed fields have (full) elimination of imaginaries. Unfortunately the proofs of weak elimination of imaginaries for lie geometries (e.g. in [33] and [19]) rely heavily on the fact the independence in such structures is modular, which is certainly not the case in algebraically closed field. Thus we would not expect independence to be modular in the theory T_∞. Therefore these proofs are unlikely to generalise to the theory T_∞ and we must look elsewhere for a method of proof. (Perhaps by showing that models of T_∞ have the small index property.)

It may also be interesting to look at the affine and projective versions of T_∞ analogously to the structures defined in Section 6.3 from vector spaces with bilinear forms over finite fields. Perhaps there is a method of gluing together these structures (similar to the method described in Section 6.3) to produce 'structures coordinatised by spaces over algebraically closed fields'. Maybe then, analogously to the result which says that structures smoothly approximated by finite substructures are lie coordinatised, we would have that a structure smoothly approximated by the theory of algebraically closed fields (as in Definition 12.1.3) is coordinatised by spaces over algebraically closed fields?

Finally, it would be interesting to study the general concept of a structure smoothly approximated by some theory (as in Definition 12.1.3) and attempt to draw parallels between this and the theory of structures smoothly approximated by finite substructures. For instance, we could look at structures smoothly approximated by strongly minimal theories (as in the case of the theory T_∞ smoothly approximated by the theory of algebraically closed fields), or more generally structures smoothly approximated by stable (or simple) theories. It would be interesting to know what properties of the approximating theory T_0 are preserved in the class of theories, T, smoothly approximated by T_0. For instance if T_0 does not have the strict order property does it follow that T does not have the strict order property; if T_0 admits some notion of dimension of independence is there a notion of dimension or independence on models of T (e.g. the dimension-codimension dichotomy in T_∞). Note however that a theory T smoothly approximated by a theory T_0 can be very much more complicated than T_0. For example in the case of the theory T_∞, the approximating theory T_0 (of algebraically closed fields) is strongly minimal but the theory T_∞ does not even have a simple theory.

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