3 hours

THE UNIVERSITY OF MANCHESTER

NONCOMMUTATIVE ALGEBRA

16 January 2017

14:00–17:00

Answer **ALL FOUR** questions in **Section A** (20 marks in total).
Answer **THREE** of the **FOUR** questions in **Section B** (60 marks in total).

If more than three questions from Section B are attempted then credit will be given for the best three answers.

Electronic calculators are not permitted.

SECTION A

Answer **ALL** 4 questions.

A1. Define what is meant by the following terms.

(a) The *nilradical* $N(R)$ of a ring R.

(b) The *annihilator* $ann_R(S)$ of a subset S of a left R-module M.

(c) The *first Weyl algebra* $A_1(\mathbb{C})$.

[4 marks]

A2. State the Artin-Wedderburn Theorem.

[3 marks]

A3.

(a) Write down (without proof) a $\mathbb{C}$-basis of $A_1(\mathbb{C})$.

(b) Prove that the centre $Z(A_1(\mathbb{C}))$ equals $\mathbb{C}$.
[You may assume Leibniz's rule without comment.]

[6 marks]

A4. Let R be a ring and let $End(R_R)$ denote the ring of all R-module endomorphisms of R regarded as a right R-module. Assume that, as usual, endomorphisms act from the left.
Prove that $R \cong End(R_R)$ as rings.

[7 marks]

SECTION B

Answer **3** of the 4 questions.

B5.

(a) State the main theorem on the structure of finitely generated modules over a commutative principal ideal domain.

(b) Let $K = \mathbb{Z}(6, -3, 9) + \mathbb{Z}(2, 2, 3) + \mathbb{Z}(4, -14, 6) \subseteq \mathbb{Z}^{(3)}$. Write $M = \mathbb{Z}^{(3)}/K$ as a direct sum of cyclic $\mathbb{Z}$-modules.

(c) Find a $\mathbb{Z}$-basis of K (equivalently, write K as a direct sum of cyclic modules).

(d) Write down a $\mathbb{Z}$-module that cannot be written as a direct sum of cyclic modules. You do not need to justify your answer.

[20 marks]

B6.

(a) Let M be a non-zero, finitely generated left module over a ring R. Prove that M has a maximal submodule.

(b) Let S be a non-empty subset of a left R-module M. Prove that $ann_R(S)$ is a left ideal of R. If S is a submodule of M, prove that $ann_R(S)$ is an ideal.

(c) Suppose that R is a k-algebra over a field k and that M is a non-zero, finite dimensional left R-module (thus M is finite dimensional as a k-vector space). By considering annihilators or otherwise, prove that R has a two-sided ideal $I \neq R$ such that R/I is finite dimensional as a k-vector space.

[20 marks]

B7.

(a) Suppose that I is a minimal left ideal of a ring R. Prove that *either* $I^2 = 0$ *or* $I = Re$ for some idempotent e.

(b) Let R be a left Artinian ring with $N(R) = 0$. Prove that every nonzero left ideal J of R can be written as $J = I_1 \oplus I_2 \oplus \cdots \oplus I_t$ for some simple left ideals I_j.

(c) Suppose that R is a left artinian ring with no idempotent elements (other than 0 and 1) and no nilpotent elements (other than 0). Using (a) or otherwise, prove that R is simple as a left R-module and hence is a division ring.

[Throughout this question, you may assume that a left module M is the internal direct sum of the submodules N_1 and N_2 if and only if $M = N_1 + N_2$ and $N_1 \cap N_2 = 0$.]

[20 marks]

B8.

(a) If R is an algebra over a field k, define the *left regular representation* of R.

For the rest of the question you may assume that the left regular representation is an injective ring homomorphism. You may also assume any facts you need related to the Artin-Wedderburn Theorem.

(b) State and prove Maschke's Theorem concerning the structure of the group ring $\mathbb{C}G$ of a finite group G over the field of complex numbers.

(c) Let G be a group with $|G| = 8$. By part (b) and the Artin-Wedderburn Theorem, the group ring $\mathbb{C}G$ can be written as a direct sum $\bigoplus M_{n_i}(D_i)$ of matrix rings over division rings. Determine the possible choices for such a decomposition, giving one example for each such possibility.

[If you need any further results from the course to answer part (c) of the question, they should be clearly stated, but need not be proved.]

[20 marks]

END OF EXAMINATION PAPER