3 hours

THE UNIVERSITY OF MANCHESTER

NONCOMMUTATIVE ALGEBRA

27 January 2016

14:00–17:00

Answer **ALL FOUR** questions in **Section A** (20 marks in total).
Answer **THREE** of the **FOUR** questions in **Section B** (60 marks in total).

If more than three questions from Section B are attempted
then credit will be given for the best three answers.

Electronic calculators are not permitted.

SECTION A

Answer **ALL** 4 questions.

A1. Define what is meant by the following terms.

(a) A nil left ideal and a nilpotent left ideal of a ring R.

(b) The nilradical $N(R)$ of a ring R.

[3 marks]

A2.

(a) State the Artin-Wedderburn Theorem.

(b) Classify the left artinian rings R that have no non-zero nilpotent elements.

[7 marks]

A3.

(a) Let R be a ring. State what is meant for a left R-module M to be the internal direct sum of two submodules N_1 and N_2.

(b) Prove that a left R-module M is the internal direct sum of submodules N_1 and N_2 if and only if $M = N_1 + N_2$ and $N_1 \cap N_2 = 0$.

[6 marks]

A4. In the polynomial extension $\mathbb{H}[x]$ of the quaternions, find a maximal left ideal that is not a two-sided ideal. You should briefly justify your answer.

[4 marks]

SECTION B

Answer **3** of the 4 questions.

B5.

(a) Let $K = \mathbb{C}[x](x+1, 1, 2) + \mathbb{C}[x](1, x+1, -2) + \mathbb{C}[x](-1, 1, x+4) \subseteq \mathbb{C}[x]^{(3)}$. Write $\mathbb{C}[x]^{(3)}/K$ as a direct sum of cyclic $\mathbb{C}[x]$-modules.

(b) Hence or otherwise find the Jordan canonical form of the matrix $\begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & -1 \\ -2 & 2 & -4 \end{pmatrix}$.

You do not need to justify your answer.

(c) Find a module over the ring $\mathbb{C}[x]$ that cannot be written as a direct sum of cyclic modules. You should briefly explain why your example has the desired properties.

[20 marks]

B6. Let R be a ring.

(a) State what is meant for a left R-module M to be completely reducible.

(b) Let M be an artinian left R-module. Prove that the following conditions are equivalent:

(i) M is a sum of simple submodules;
(ii) M is a direct sum of simple submodules;
(iii) M is completely reducible.

(c) Prove that $M = \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix}$ is not completely reducible as a left module over $R = \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} \end{pmatrix}$.

[20 marks]

B7.

(a) Let S be a left noetherian ring. Prove that every nil one-sided ideal of S is nilpotent and that $N(S)$ is nilpotent.

(b) Prove that every left artinian ring R is left noetherian.

[In Part (b) of this question you may assume standard results from the course without particular comment; in particular you may assume that the nilradical $N(R)$ of a left artinian ring R is nilpotent and that any module over a semisimple left artinian ring is a direct sum of simple modules.]

[20 marks]

B8.

(a) Define the Weyl algebra $A_1 = A_1(\mathbb{C})$ of differential operators on $\mathbb{C}[x]$ and state Leibniz's rule. [You may assume that A_1 is a ring and you do not need to prove Leibniz's rule.]

(b) Prove that A_1 is a domain with basis $\{x^i\partial^j : 0 \leq i, j < \infty\}$, where $\partial = \frac{d}{dx}$.

(c) Consider $\mathbb{C}[x, x^{-1}] \supset \mathbb{C}[x]$ as left A_1-modules. Prove that $\mathbb{C}[x]$ is the unique simple A_1-submodule of $\mathbb{C}[x, x^{-1}]$ and that $M = \mathbb{C}[x, x^{-1}]/\mathbb{C}[x]$ is also simple as an A_1-module.

[20 marks]

END OF EXAMINATION PAPER