$2\frac{1}{2}$ hours

THE UNIVERSITY OF MANCHESTER

NONCOMMUTATIVE ALGEBRA

January 2015

Answer **ALL FOUR** questions in **Section A** (20 marks in total).
Answer **THREE** of the FOUR questions in **Section B** (60 marks in total).

If more than three questions from Section B are attempted
then credit will be given for the best three answers.

Electronic calculators are not permitted.

SECTION A

Answer **ALL** 4 questions

A1. Define what is meant by the following terms.

(a) A nil left ideal and a nilpotent left ideal.

(b) The nilradical $N(R)$ of a ring R.

(c) A left artinian ring.

[4 marks]

A2. State the Artin-Wedderburn theorem.

[4 marks]

A3. Let N_1 and N_2 be submodules of a left R-module M and assume that M/N_1 and M/N_2 are artinian. Prove that $M/(N_1 \cap N_2)$ is also artinian.

[Basic facts about artinian modules may be assumed.]

[4 marks]

A4. Let M be a left R-module.

(a) Define the annihilator $ann_R(S)$ for a subset S of M.

(b) Prove that $ann_R(M)$ is an ideal of R.

(c) Assume now that $M = R/I$ for some left ideal I of R. Prove that $ann_R(M)$ is the unique largest ideal K of R contained in I.

[8 marks]

SECTION B

Answer **3** of the 4 questions

B5.

(a) State the main theorem on the structure of finitely generated modules over a commutative principal ideal domain.

(b) Let $K = \mathbb{Z}(10,\,8,\,4) + \mathbb{Z}(5,\,5,\,3) + \mathbb{Z}(10,12,4) \subseteq \mathbb{Z}^{(3)}$. Write $\mathbb{Z}^{(3)}/K$ as a direct sum of cyclic $\mathbb{Z}$-modules.

(c) Find a $\mathbb{Z}$-module that cannot be written as a direct sum of cyclic modules and justify your answer.

[20 marks]

B6.

(a) Prove that every nil left ideal in a left artinian ring R is nilpotent.

(b) Prove that the nilradical of a left artinian ring R is nilpotent.

[You may assume that a sum of two nilpotent left ideals is nilpotent.]

(c) Let R be the ring $R = \begin{pmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} & \mathbb{C} \end{pmatrix}$. Find the nilradical $N(R)$ of R and identify $R/N(R)$ as a direct sum of simple artinian rings.

[20 marks]

B7.

(a) If R is an algebra over a field k, define the *regular representation* of R. You may assume that it is an injective ring homomorphism.

(b) State and prove Maschke's Theorem on the structure of the group ring kG of a finite group G over a field k of characteristic either zero or coprime to $|G|$.

(c) If G is the cyclic group of order 3, prove that $\mathbb{Q}G$ is a direct product of two fields.

[In Part (c), you may assume basic facts from the course without particular comment.]

[20 marks]

B8.

(a) Prove that any non-zero, finitely generated left module over a ring R has a simple factor module.

(b) Recall that Schur's Lemma states that the endomorphism ring $\mathrm{End}_R(M)$ of a simple left R-module M is a division ring. The aim of this part of the question is to show that converse to Schur's Lemma does not always hold. More precisely let

$$R = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix} \qquad \text{and} \qquad I = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}.$$

Then prove that I is a left R-module that is not simple, but satisfies $\mathrm{End}_R(I) \cong \mathbb{C}$.

To show this, argue as follows:

(i) If $\theta \in \mathrm{End}_R(I)$ show that θ is uniquely determined by $\theta(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})$.

(ii) If $\theta(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}$, show that $x = 0$.

(iii) Hence or otherwise prove that $\mathrm{End}_R(I) \cong \mathbb{C}$, but that I is not a simple left R-module.

(c) Prove that if M is a left module over a ring S such that $\mathrm{End}_S(M)$ is a division ring, then M cannot be written as a direct sum $M = M_1 \oplus M_2$ of S-submodules.

[20 marks]

END OF EXAMINATION PAPER