Three hours

THE UNIVERSITY OF MANCHESTER

NONCOMMUTATIVE ALGEBRA

Monday 21st January 2019

9:45-12:45

Answer **ALL** questions in Section A and THREE of the FOUR questions in Section B. If more than three questions are attempted from Section B, then credit will be given for the best three answers.

The use of electronic calculators is not permitted.

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SECTION A

Answer **ALL** four questions.

A1.

(a) Define what is meant by an element r of a ring R being nilpotent.

(b) Show that if r, s are nilpotent elements of a commutative ring R then $r+s$ is nilpotent.

(c) Give an example of a ring R and nilpotent elements $r, s \in R$ such that $r+s$ is not nilpotent.

[4 marks]

A2.

(a) State the Artin-Wedderburn Theorem.

(b) For each of $d = 2, 4, 16, 64$, write down an example of a simple artinian ring which is an $\mathbb{R}$-algebra of dimension d over $\mathbb{R}$. Given the fact that the only finite-dimensional $\mathbb{R}$-algebras which are division rings are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, is there a simple artinian $\mathbb{R}$-algebra of dimension 28? - give a reason for your answer.

[6 marks]

A3. Let N_1, N_2 be submodules of a module M such that $N_1 \cap N_2 = 0$. Suppose that each of M/N_1 and M/N_2 is artinian. Prove that M is artinian.

[Basic facts about artinian modules may be assumed.]

[4 marks]

A4. Prove Schur's Lemma - that if M is a simple module over a ring R then the endomorphism ring $\mathrm{End}_R(M)$ of M is a division ring.

Recall that the polynomial ring $\mathbb{C}[x]$ is a simple $A_1 = A_1(\mathbb{C})$-module under the natural action of A_1 as a ring of differential operators on $\mathbb{C}[x]$. Show that, if $\theta \in \mathrm{End}(\mathbb{C}[x])$ is an endomorphism of the A_1-module $\mathbb{C}[x]$, then the action of θ is determined by its value on $1 \in \mathbb{C}[x]$. Determine $\mathrm{End}(\mathbb{C}[x])$, justifying your answer.

[6 marks]

SECTION B

Answer **3** of the 4 questions.

B5.

(a) State the main theorem on the structure of finitely generated modules over a commutative principal ideal domain.

(b) Suppose that $M = \sum_{i=1}^{n} Ra_i$ is a finitely generated module over a commutative principal ideal domain R. Explain briefly how we can define homomorphisms $\theta : R^{(n)} \to M$ and $\psi : R^{(m)} \to R^{(n)}$ for some m, with θ surjective and with the action of ψ being given by right multiplication by some $m \times n$ matrix with entries from R (and from which we can compute a decomposition of M as a direct sum of cyclic modules).

(c) Let $K = \mathbb{Z}(6,\, 6,\, 15) \;+\; \mathbb{Z}(4,\, 6,\, 7) \;+\; \mathbb{Z}(2, 2, 4) \subseteq \mathbb{Z}^{(3)}$. Write $\mathbb{Z}^{(3)}/K$ as a direct sum of cyclic $\mathbb{Z}$-modules.

(d) Give a $\mathbb{Z}$-module that cannot be written as a direct sum of cyclic modules and briefly justify your answer.

[20 marks]

B6. Let R be a ring.

(a) Define what is meant by a left R-module M being completely reducible.

(b) Prove that if M is completely reducible and N is a submodule of M then N is completely reducible.

(c) Let M be an artinian left R-module. Prove that the following conditions are equivalent:

(i) M is a sum of simple submodules;

(ii) M is a direct sum of simple submodules;

(iii) M is completely reducible.

(d) Prove that $M = \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix}$ is not completely reducible as a left module over $R = \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} \end{pmatrix}$.

[20 marks]

B7.

(a) If R is an algebra over a field k, define the *(left) regular representation* of R. Prove that it is an injective ring homomorphism.

(b) State Maschke's Theorem on the structure of the group ring kG of a finite group G over a field k of characteristic either zero or coprime to $|G|$.

(c) If G is the cyclic group of order 2, prove that $\mathbb{Q}G$ is a direct product of two fields.

(d) If G is the cyclic group of order 2 and k is a field of characteristic 2, prove that the group ring kG is isomorphic to the ring $k[x]/(x^2)$, where (x^2) denotes the ideal of the polynomial ring $k[x]$ generated by x^2.

[20 marks]

B8. Let $a, b \in \mathbb{Q}$ and define $\mathbb{Q}(a, b)$ to be the $\mathbb{Q}$-algebra generated over $\mathbb{Q}$ by $\{1, i, j, k\}$ with multiplication determined by: $i^2 = -a$, $j^2 = -b$, $ij = k = -ji$.

(a) Show that $jk = bi = -kj$, that $ki = aj = -ik$ and that $k^2 = -ab$.

(b) For $x = \alpha + \beta i + \gamma j + \delta k \in \mathbb{Q}(a, b)$ (where α means $\alpha \cdot 1$) set $\overline{x} = \alpha - \beta i - \gamma j - \delta k$. Show that $x\overline{x} = \alpha^2 + \beta^2 a + \gamma^2 b + \delta^2 ab$. Deduce that, if $a, b > 0$, then $\mathbb{Q}(a, b)$ is a 4-dimensional $\mathbb{Q}$-algebra and that, furthermore, $\mathbb{Q}(a, b)$ is a division algebra.

(c) Now let $a = b = -1$ and consider the algebra $\mathbb{Q}(-1, -1)$. Show that $e_1 = \frac{1}{2}(1 + i)$ is an idempotent element of $\mathbb{Q}(-1, -1)$ and that the left ideal generated by e_1 is 2-dimensional over $\mathbb{Q}$. Explain why this shows that $\mathbb{Q}(-1, -1)$ is not a division ring.
Also show that $(j + k)^2 = 0$. Why does this show that $\mathbb{Q}(-1, -1)$ is not a product of division rings?

(d) Given the fact that $\mathbb{Q}(-1, -1)$ has nilradical $= 0$, deduce that $\mathbb{Q}(-1, -1)$ is a direct sum of matrix rings over division rings and identify it as a specific 4-dimensional $\mathbb{Q}$-algebra.

[20 marks]

END OF EXAMINATION PAPER