Noncommutative Algebra 2018, Assessment

To be handed in to Mathematics Reception by 4.00 p.m. on Tuesday 6th November

A maximum of 20 marks may be obtained (a maximum of 80 marks may be obtained on the final examination).
Beyond providing marks, the purpose of this coursework is to help you engage with and test your understanding of the course material. If you already understand the course material very well then you might be able to write out neat solutions to Questions 1, 2, 3 and 5 in quite a short time but most students will have to spend some time reading and thinking about the material. Question 4 will require some experimental computations in order to see what the answer should be and then some more thought about how to prove it.

Q1. Define the map $\theta : \mathbb{Q}[x] \to \mathbb{Q}[x]$ by $\theta(f(x)) = f(x-1)$.

(i) Show that $\theta(x^2 + x + 1) = x^2 - x + 1$. Find a polynomial $g(x)$ such that $\theta(g) = x^2 + x + 1$.

(ii) Is θ an automorphism of the ring $\mathbb{Q}[x]$? Justify your answer.

Q2. Let I, J be left ideals of a ring R.

(i) Prove that

$$IJ = \{\sum_i a_i b_i : a_i \in I,\, b_i \in J \text{ and the sum is finite}\}$$

is a left ideal of R.

(ii) Consider the case $R = M_3(K)$, where K is any field, I is the left ideal with all entries in columns 2 and 3 being zero, and J is the left ideal with all entries in columns 1 and 3 being 0. Determine IJ.

Q3. Let R be any ring, I a left ideal of R and $s \in R$. Define

$$(I : s) = \{r \in R : rs \in I\}.$$

(i) Prove that $(I : s)$ is a left ideal of R.

(ii) For which $s \in R$ is it the case that $(I : s) = R$?

(iii) Compute the ideals $(I : x)$ and $(I : x + 1)$ in $\mathbb{Z}[x]$ where I is the ideal generated by the polynomial $2x^2$.

(iv) Let R be any ring and I any left ideal of R. Let $s \in R$. Consider the module R/I. Prove that $\mathrm{ann}_R(s + I) = (I : s)$. Deduce that there is an embedding (i.e. injective homomorphism) $R/(I : s) \to R/I$.

(v) Suppose that I is a maximal left ideal of R and that $s \in R \setminus I$. Is $(I : s)$ a maximal left ideal of R? Justify your answer.

Q4. Consider $\mathbb{C}[x]$ as a left module over the first Weyl algebra $A_1 = A_1(\mathbb{C})$ (with the natural action).

(i) Find $\mathrm{ann}_{A_1}(x^2)$, fully justifying your answer (compare Example 2.21 in the notes).

(ii) Write down $(A_1\partial : x^2)$, justifying your answer.

Q5. Let M be any nonzero R-module and $a \in M$, $a \neq 0$. Prove that there is at least one submodule N of M such that $a \notin N$ and N is maximal with respect to not containing a (that is, if $N' \supseteq N$ and $N' \neq N$ then $a \in N'$). Prove also that $(Ra + N)/N$ is a simple submodule of the factor module M/N.