Model Theory of Modules

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1 Introduction

There are a number of introductions and surveys describing the model theory of modules; I give some references later. This is intended to be largely supplementary to those, in that my imagined reader has seen a little bit, perhaps some time in the past, about the model theory of modules. But let me give a little introduction to set the context.

In order to apply model theory to modules, we fix a ring (unital, with 1) and set up a language $\mathcal{L}_R$, described below, such that R-modules can be axiomatised within $\mathcal{L}_R$. The atomic formulas of $\mathcal{L}_R$ are the R-linear equations. A formula is said to be pp (for "positive primitive") if it is, or is equivalent modulo the common theory of R-modules to, an existentially quantified system of R-linear equations. A formula is pp precisely if its solution set in each module is a group under the addition inherited from the module. The pp formulas are also the key formulas in that there is a relative elimination of quantifiers result which reduces every formula to an equivalent one which is simply constructed from pp formulas. They are also the formulas whose solution sets are preserved by arbitrary homomorphisms between R-modules.

Thus model theory, algebra and category theory for R-modules are closely linked through the pp formulas and, as might then be expected, through associated ideas, especially that of purity.

The close links are preserved if we generalise beyond modules in the usual sense. If we allow rings with enough idempotents but perhaps not with a global 1, the corresponding modules include a wide range of types of additive structures; this is discussed below. By an additive structure we mean a possibly many-sorted structure each of whose sorts has an abelian group structure and such that any additional structure on and between sorts preserves this structure (which we write additively). Concretely - for any function symbol f in the language, we insist that it is additive (by an axiom like $f(\overline{a}+\overline{b}) = f(\overline{a}) + f(\overline{b})$) and for any relation symbol R of the language, we insist that its solution set be a subgroup $(R(\overline{a}) \wedge R(\overline{b}) \to R(\overline{a}-\overline{b}))$.

2 Modules

A module is an abelian group M with a specified set, which we may assume to be a ring R, of endomorphisms. This can be treated as a 2-sorted structure, with a sort for module elements and a sort for ring elements, but the resulting model theory becomes very complicated; it contains the model theory of modules in the sense that we will describe here but it also contains the highly complex model theory of rings. Much more tractable, and useful, is what we obtain if we fix the ring R by introducing a function symbol for each element of R, and then consider the (1-sorted) abelian groups M on which that ring acts. This, and its generalisations to many-sorted modules[1] (treated in Section 21), is *additive model theory*, a field with its own character, where elementary embeddings are replaced by pure embeddings and where saturated structures are replaced by the, far more numerous, pure-injective modules, where pp formulas are the important ones and where the space of types is replaced by the lattice of pp-types.

Let us, therefore, fix a ring R (associative, with 1) and set up a language for R-modules. This language, denoted $\mathcal{L}_R$, is the language for abelian groups (say with a binary function symbol for addition and a constant symbol for the 0 element) augmented by, for each element of $r \in R$, a unary function symbol μ_r used to represent multiplication by r.

To axiomatise R-modules we, of course, write down some axioms for abelian groups, but then have to decide whether we want to deal with left or right R-modules, these being distinguished by the axioms:

$\forall x\, (\mu_r(\mu_s(x)) = \mu_{rs}(x))$ in contrast to $\forall x\, (\mu_r(\mu_s(x)) = \mu_{sr}(x))$.

In practice we use the natural notation rx, respectively xr, instead of $\mu_r(x)$, depending on whether we are dealing with left or right modules. And we also use $\sum$ as a handy abbreviation in formulas.

In this paper, by R-module we will mean *right* R-module unless otherwise specified (because that fits with writing homomorphisms between R-modules on the left).

[1] That is, modules where there is no ring sort but there are many module sorts. For the reader who knows representations of quivers - these are naturally many-sorted, with one sort for each vertex of the quiver.

Let T_R denote the theory of (right) R-modules, or some suitable set of axioms for that. So the $\mathcal{L}_R$-structures which are models of T_R are the right R-modules. Denote by Mod-R the category whose objects are the right R-modules and whose arrows are the homomorphisms - R-linear maps - between R-modules. We also denote by mod-R the full subcategory of **finitely presented** (that is, finitely generated and finitely related) R-modules.

Already two features of additive model theory have just come into view. One is that it applies to the full algebraic category, not just the category which has the elementary embeddings as its arrows. The other is that, because Mod-R is a finitely accessible category, in particular every module is a directed colimit of finitely presented modules, the model theory of R-modules is strongly reflected in the category of finitely presented modules.

Throughout the paper, we deal with first-order, finitary model theory (though we mention infinitary languages in Section 28.9). In particular, formulas are finite strings.

I cite rather few original sources here and will mostly use [69], and to a lesser extent the older but more model-theory-directed [63], as a reference for proofs of results. There one can find references to the original papers (which, for some results with complex histories, are numerous) as well as a good deal of related material. There are a number of existing accounts, surveys and introductions, for instance: [43, Appx. 1], [44, Chpts. 6-8], [68], [94, Chpt. 15], as well the "Background" sections of many papers in the area.

3 Positive primitive (pp) formulas

A **positive primitive**, or just **pp**, formula is an existentially quantified conjunction of atomic formulas. These are also referred to as **regular** formulas. Modulo the theory T_R of R-modules, an atomic formula can be written in the form $\sum_{i=1}^{n} x_i r_i = 0$, where the x_i are variables and the r_i are (function symbols for multiplication by) elements of R. So a typical pp formula has the form

$$\exists y_1, \ldots, y_l \, \Big(\bigwedge_{j=1}^{m} \sum_{i=1}^{n} x_i r_{ij} + \sum_{k=1}^{l} y_k s_{kj} = 0 \Big).$$

It is convenient to use matrix notation for this, writing $\overline{x}$ for the $1 \times n$ matrix $(x_1 \ \ldots \ x_n)$ and similarly for $\overline{y}$, writing A for the $n \times m$ matrix $(r_{ij})_{ij}$ and B for the $l \times m$ matrix $(s_{kj})_{kj}$. Then the formula above becomes

$$\exists \overline{y} \ (\overline{x} \ \overline{y}) \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

or, writing H for the block-matrix $\begin{pmatrix} A \\ B \end{pmatrix}$,

$$\exists \overline{y} \ (\overline{x} \ \overline{y}) \, H = 0.$$

This makes it clear that a pp formula is simply an existentially quantified system of homogeneous R-linear equations. Since the solution set, in a module M, of a system of homogeneous R-linear equations is a subgroup of the relevant power of M (M^{n+l} in the formula above), the solution set of any existentially quantified version of this system is, being a projection to some coordinates, again a subgroup of the relevant power of M (M^n in the formula above). If we write $\phi(\overline{x})$ for the pp formula above, then

$$\phi(M) = \{\overline{a} \in M^n : \exists \overline{b} \in M^l \text{ such that } (\overline{a}\ \overline{b})H = 0\}$$

is the **solution set** of ϕ in M. It is referred to as a **pp-definable subgroup** of M (more accurately, as a subgroup of M^n pp-definable in M).[2] Easy examples show that, unless R is commutative or a division ring, such a solution set need not be a submodule of M^n. It is, however, an $\mathrm{End}(M)$-submodule of M^n, where the endomorphism ring, $\mathrm{End}(M)$, of M has the diagonal action on M^n.

More generally, pp-definable subgroups are preserved by homomorphisms.

Lemma 3.1. *(see [69, 1.1.7]) If $f : M \to N$ is a homomorphism between R-modules and ϕ is a pp-formula with n free variables, then the diagonal extension of f to a homomorphism $M^n \to N^n$ restricts to a homomorphism $\phi(M) \to \phi(N)$. In particular $f\phi(M)$ is a subgroup of $\phi(N)$.*

The diagonal extension of f is the homomorphism which takes $(a_1, \dots, a_n) \in M^n$ to $(fa_1, \dots, fa_n)$ and $f\phi(M)$ denotes the image of $\phi(M)$ under this map. So the above is the implication $M \models \phi(\overline{a}) \Rightarrow N \models \phi(f\overline{a})$.

The simplest examples of pp formulas in one free variable are annihilation $xr = 0$ and divisibility $r|x$, that is $\exists y\,(x = yr)$, formulas. Over $R = \mathbb{Z}$, pp formulas do not get much more complicated than this, the general form for 1-variable pp formulas being (see [63, §2.$\mathbb{Z}$]) a conjunction of those of the form $p^n|xp^m$ where p is a prime integer or 0, but, over general rings, one needs pp formulas with conjunctions of arbitrarily many equations in arbitrarily many quantified variables (see, e.g., [90], [69, §2.4.1] but also cf. [90] and [69, §2.4]).

Notice that if ϕ, ψ are pp formulas in the same free variables with $\psi \to \phi$ (modulo the theory of R-modules) then, in every (right R-)module M, $\psi(M)$ is a subgroup of $\phi(M)$. Therefore one may consider the quotient group $\phi(M)/\psi(M)$ and, given any integer t, there is a sentence σ in $\mathcal{L}_R$ such that $M \models \sigma$ iff $|\phi(M)/\psi(M)| \geq t$. That is, one can use sentences in the language of R-modules to express conditions on the cardinalities of such quotients of pp-definable subgroups. Of course a single sentence can only express a restriction involving a finite cardinality and even an infinite set of sentences can express that such a cardinality is infinite, but nothing about how infinite it is. Boolean combinations of such simple sentences as above, that is sentences expressing finite restrictions on cardinalities of finitely many quotients of pp-definable subgroups, are referred to as **invariants statements**.

4 Pp-elimination of quantifiers and the lattice of pp formulas

Theorem 4.1. *(see [63, §2.4]) Any formula in $\mathcal{L}_R$ is equivalent, modulo the theory of right R-modules, to the conjunction of an invariants statement and a finite boolean combination of pp formulas.*

[2] These have also, reflecting the multiple roots of this subject, been termed *sous-groupes de définition finie* - see [37] - and *matrizielle Untergruppen* (this latter term allowing infinitely-definable subgroups) - see [104].

This also applies to formulas with parameters, but note that a pp formula with parameters will be an existentially quantified, possibly inhomogeneous, system of R-linear equations, hence its solution set will be empty or a coset of the pp-definable subgroup which is the solution set of the corresponding existentially quantified *homogeneous* system of R-linear equations.

Therefore, in any module M, every definable set is a finite boolean combination of (cosets of) pp-definable subgroups and hence every type is determined by specifying which pp formulas are in it.

Corollary 4.2. *If M is a module and p is a type, possibly with parameters, modulo the theory of M, then p is equivalent, modulo that theory, to $p^+ \cup \neg p^-$, where p^+ denotes the set of pp formulas in p and $\neg p^- = \{\neg\psi : \psi$ is pp and $\neg\psi \in p\}$.*

A **pp-type**[3] is a set of pp formulas of the form p^+ where p is some type. For any tuple $\overline{a}$ from M, the set of pp-definable subgroups to which it belongs forms a filter (is upwards closed and closed under finite intersection) in the lattice, $\mathrm{pp}^n(M)$, of subgroups of M^n pp-definable in M.

Denote by pp^n_R the set of pp formulas[4], for right R-modules, in n free variables $x_1, \ldots, x_n$. These are ordered by implication (modulo the theory of R-modules) with $\overline{x} = \overline{0}$ at the bottom and $\overline{x} = \overline{x}$ at the top. Indeed this is a modular lattice - the **lattice of pp formulas** (in n free variables, for right R-modules), with meet being given by conjunction and join being given by

$$\psi(\overline{x}) + \phi(\overline{x}) = \exists \overline{x}' \, \overline{x}'' \, (\psi(\overline{x}') \wedge \phi(\overline{x}'') \wedge \overline{x} = \overline{x}' + \overline{x}'').$$

Given a module M and an n-tuple $\overline{a}$ from M, the **pp-type of $\overline{a}$ in M** is the set of pp formulas

$$\mathrm{pp}^M(\overline{a}) = \{\phi \in \mathrm{pp}^n_R : \overline{a} \in \phi(M)\}$$

and this is a filter in pp^n_R. It is the case that every filter of pp formulas occurs as a pp-type of some tuple in some module (see [69, 3.2.5]), so the pp-types are exactly the filters of pp formulas.

Note that there is a lattice homomorphism $\mathrm{pp}^n_R \to \mathrm{pp}^n(M)$, the homomorphism being simply evaluation (of a pp formula on M).

There is a very explicit criterion for implication between pp formulas.

Theorem 4.3. *([63, 8.10], see also [69, 1.1.13]) Let $\phi(\overline{x})$ be the pp formula $\exists \overline{y} \, (\overline{x} \, \overline{y}) H_\phi = 0$, and let $\psi(\overline{x})$ be the pp formula $\exists \overline{z} \, (\overline{x} \, \overline{z}) H_\psi = 0$. Then $\psi \leq \phi$ iff there are matrices $G = \begin{pmatrix} G' \\ G'' \end{pmatrix}$ and K such that*

$$\begin{pmatrix} I & G' \\ 0 & G'' \end{pmatrix} H_\phi = H_\psi K$$

where I is the $n \times n$ identity matrix, n being the length of $\overline{x}$, and where 0 is a zero matrix.

[3]Here we use the term for types without extra parameters but, of course, the notion makes sense if we add parameters to the language, in which case "subgroup" is replaced by "coset".

[4]More accurately, the set of equivalence classes (modulo the theory of right R-modules) of pp formulas, but we identify equivalent pp formulas without further comment.

Throughout, by a pp-type we will mean the set of pp formulas in some complete type without parameters, equivalently a filter in the lattice pp_R^n of pp-n-formulas for some n. We will also tend to use notation such as p for pp-types rather than for complete types (from now we will seldom have reason to refer to the latter).

5 Complete elimination of quantifiers

Complete elimination of quantifiers is a strong condition. The characterisation of those rings over which all modules have complete elimination of quantifiers was found independently by many people. A ring R is (**von Neumann**) **regular** if it satisfies the condition that, for every $r \in R$ there is $s \in R$ such that $r = rsr$, equivalently every embedding between R-modules is pure (for purity see Section 10). The conditions here are 2-sided: if they hold for, say, right modules then they hold for left modules. For the largest theory of R-modules, see Section 7.

Theorem 5.1. *(see [63, 16.16]) Every complete theory of R-modules has elimination of quantifiers iff the largest complete theory of R-modules has elimination of quantifiers iff R is von Neumann regular.*

In fact (von Neumann) regularity of R is also equivalent to the theory of R-modules having elimination of imaginaries ([73, §5]). Both elimination of quantifiers and elimination of imaginaries are completely language-dependent: essentially they are the question of what (definable or interpretable) sorts must be added to the language in order that every formula be equivalent to a quantifier-free one (elimination of quantifiers) or every interpretable sort be definably isomorphic to the solution set of a formula (elimination of imaginaries).

6 Stability and chain conditions on pp formulas

In order to show that every (complete theory of a) module is stable, we must count types, equivalently, by pp-elimination of quantifiers, we must count pp-types. A pp-type is a filter of cosets of pp-definable subgroups and each pp-definable subgroup can be represented at most once in such a filter (cosets of a given subgroup being equal or disjoint). Therefore, if $\kappa = |R| + \aleph_0$ is the cardinality of the set of formulas of $\mathcal{L}_R$, the number of pp-types over any set A of parameters is bounded above by $|A| \times 2^\kappa$. Thus we have the first statement below. The other parts are proved by making somewhat similar counts of types, equivalently pp-types.

Theorem 6.1. *(see [63, §3.1]) (i) Every module is stable.*
(ii) A module M is superstable iff, given any descending chain $\phi_1(M) \geq \phi_2(M) \geq \cdots \geq \phi_i(M) \geq \dots$ of pp-definable subgroups of M, there is j such that, for every $i \geq j$, the group $\phi_i(M)/\phi_{i+1}(M)$ is finite (it is enough to require this for pp formulas in one free variable).
(iii) A module M is totally transcendental iff M has the descending chain condition on pp-definable subgroups (again, it is enough to require this for pp formulas in one free variable).

For instance, $\mathbb{Z}$ is, as a module over itself, superstable but not totally transcendental, $\mathbb{Z}^{(\aleph_0)}$ is not superstable, and every Prüfer group $\mathbb{Z}_{p^\infty}$ (p prime) is totally transcendental, as is the direct sum $\mathbb{Q}/\mathbb{Z}$ of all the Prüfer modules.

It follows that, if M is superstable, then either M is totally transcendental or $M^{(\aleph_0)}$ ($\equiv M^{\aleph_0}$) is not superstable. In particular if there are no pairs of pp-definable subgroups with one of finite index in (and not equal to) the other, then there is no distinction between superstability and total transcendentality. For example, if R is an algebra over an infinite field, then every superstable R-module is totally transcendental, because every pp-definable subgroup is a vector space over that field. Forming the module $M^{(\aleph_0)}$ from M is an algebraically rather trivial operation, and this indicates that superstability *per se* is less meaningful algebraically than, for instance, total transcendentality (which is indeed equivalent to the algebraic condition of being Σ-pure-injective, for which see Section 12).

In this additive context, an algebraically more meaningful refinement of stability is the Krull-Gabriel dimension of (the complete theory of) a module. This is defined in terms of the lattice of pp formulas (as usual, pp formulas in one variable suffice). The original dimension of this kind - **elementary Krull dimension** [26] - defines the modules of elementary Krull dimension 0 to be the totally transcendental modules - those with the descending chain condition on pp-definable subgroups - and then proceeds inductively and transfinitely. Precisely (see [63, §10.5]), an interval in $\mathrm{pp}^1(M)$ is defined to have Krull dimension 0 if it has the descending chain condition. Then, inductively, an interval $[\phi(M), \psi(M)]$ in $\mathrm{pp}^1(M)$ has Krull dimension α (an ordinal) if it does not have Krull dimension $< \alpha$, but if, for every descending chain $\phi(M) \geq \phi_1(M) \geq \phi_2(M) \geq \cdots \geq \psi(M)$, there is i such that each subsequent subinterval $[\phi_{i+j}(M), \phi_{i+j+1}(M)]$ has Krull dimension $< \alpha$. We say that the elementary Krull dimension of M is α if that is the Krull dimension of the lattice $\mathrm{pp}^1(M)$. If there is no such α - this happens exactly if the lattice $\mathrm{pp}^1(M)$ has a densely ordered subchain - then we say that the elementary Krull dimension of M is ∞, or undefined.

For example, $\mathbb{Z}$, and equally $\mathbb{Z}^{\aleph_0}$, has elementary Krull dimension 1 and any Prüfer group $\mathbb{Z}_{p^\infty}$ has elementary Krull dimension 0.

A refinement of this dimension was introduced in [103]. This is referred to as **m-dimension** in [63] and it turned out to be equal to **Krull-Gabriel dimension** [30] which is an algebraic dimension defined in terms of localising an associated category of functors. This dimension is slower-growing than elementary Krull dimension, because the intervals of m-dimension 0 are defined to be those of finite length. An inductive definition of m-dimension can be given as for elementary Krull dimension, but it can alternatively be given in terms of forming the quotient lattice obtained from $\mathrm{pp}^1(M)$ by identifying all points in any interval of finite length, and continuing that process inductively and transfinitely (see [69, §§7.1, 7.2]). These dimensions coexist in the sense that, although they grow at different rates, if one is defined then so is the other.

For example, both $\mathbb{Z}$ and $\mathbb{Z}_{p^\infty}$ have Krull-Gabriel dimension 1. In fact, every abelian group has Krull-Gabriel dimension ≤ 2.

7 Theories of modules

It follows from the pp-elimination of quantifiers theorem that every complete theory of modules can be axiomatised by sentences of the form $|\phi(-)/\psi(-)| * t$ where t is a positive integer, where $*$ denotes $\leq$ or $\geq$ and where ϕ/ψ is a **pp-pair** - meaning that ϕ and ψ are pp formulas in the same free variables and $\psi \to \phi$ modulo the theory of R-modules.

If ϕ is a pp formula and $(M_i)_i$ are modules then $\phi(\bigoplus_i M_i) = \bigoplus_i \phi(M_i)$ and similarly for products, so the invariants $|\phi/\psi|$ of a direct sum (or product) of modules are the products of those of its components. In particular

$$|\phi(M \oplus N)/\psi(M \oplus N)| = |\phi(M)/\psi(M)| \times |\phi(N)/\psi(N)|.$$

So, if M is any module, then the invariants $|\phi(M^{(\aleph_0)})/\psi(M^{(\aleph_0)})|$ of the direct sum $M^{(\aleph_0)}$ of countably infinitely many copies of M are all either 1 (if ϕ/ψ is **closed** on M, meaning if $\phi(M) = \psi(M)$) or ∞ if ϕ/ψ is **open** (=not closed) on M. It follows that, if every invariant of M is either 1 or ∞, then $M \equiv M^{(\aleph_0)}$. Note also that the lattices of pp-definable subgroups of M and $M^{(\aleph_0)}$ are naturally isomorphic: $\mathrm{pp}^n(M) \simeq \mathrm{pp}^n(M^{(\aleph_0)})$, for every n. For such reasons, it is, for many purposes, enough to know which pp-pairs are closed (and which are open) on a module. That is, it is enough to work with theories satisfying the condition that every invariant is 1 or ∞ - this condition is denoted $T = T^{\aleph_0}$. For instance, if the ring R is an algebra over an infinite field, then every theory of modules satisfies this condition.

Complete theories T satisfying $T = T^{\aleph_0}$ are naturally ordered by $T' \leq T$ iff ϕ/ψ closed on T implies ϕ/ψ is closed on T'. The theory of the module 0 is the least element in this ordering and the top element (that theory in which every pp-pair open on some module is open, and the corresponding quotient is infinite) is referred to as the **largest** (**complete**) **theory of** (right R-)**modules**. This ordering is described algebraically by $\mathrm{Th}(M') \leq \mathrm{Th}(M)$ iff M' is a direct summand of some $M_1 \equiv M$, see [63, §2.6]. So every module is a direct summand of some model of the largest complete theory of modules. This ordering is also the same as the ordering on supports of modules (for these see Section 12): $\mathrm{Th}(M') \leq \mathrm{Th}(M)$ iff $\mathrm{supp}(M') \subseteq \mathrm{supp}(M)$.

If M is any module then $\mathrm{Add}(M)$ - the class of modules which are direct summands of direct sums of copies of M - is arguably the simplest class of modules constructed from M. We take account of forming direct sums (equivalently, direct products[5]) by concentrating on theories satisfying $T = T^{\aleph_0}$. We can take account of direct summands by concentrating on those theories which specify only which pp-pairs must be closed (in particular, which do not specify that any pp-pairs must be open). Of course, such theories are incomplete, but the classes of models of such theories - the definable subcategories (see Section 13) - are algebraically more natural than the class of models of a complete theory see, e.g. [18].

That is, to each complete theory T satisfying $T = T^{\aleph_0}$ we associate the incomplete theory which is (the deductive closure of) the set of all sentences of the form $\phi(-) = \psi(-)$ which are in T. Then the models of the latter theory are

[5]For any set $\{M_i : i \in I\}$ of modules, the modules $\bigoplus_{i \in I} M_i$ and $\prod_{i \in I} M_i$ are elementarily equivalent.

the direct summands of models of T. The inverse process is: given any set of pp-pairs and the incomplete theory which is (the deductive closure of) the set of sentences saying that all these pairs are closed, we take the largest theory (in the sense above) of a model of these sentences. This puts in bijection the complete theories with class of models closed under (finite, hence arbitrary) direct sums and the incomplete theories with class of models closed under direct sums and direct summands. Algebraically, and perhaps also from the viewpoint of regular logic[6], it is the latter which are more natural.

8 Classification of models

For totally transcendental modules there is a very simple classification theorem.

Theorem 8.1. *(see [63, §4.6]) Suppose that M is a totally transcendental module. Then M, and every model of* $\mathrm{Th}(M)$*, is a direct sum of indecomposable*[7] *modules, the factors being uniquely determined up to isomorphism and multiplicity. These indecomposables fall into four sets:*
1) those which occur (i.e. occur as direct summands in a direct-sum decomposition) a fixed and finite number of times in the decomposition of each model of $\mathrm{Th}(M)$*;*
2) those which must occur at least once in each model of $\mathrm{Th}(M)$ *but with no other restriction;*
3) those which must occur infinitely many times in each model of $\mathrm{Th}(M)$*;*
4) those which occur in some, but not every, model of $\mathrm{Th}(M)$*.*
The models of $\mathrm{Th}(M)$ *are the direct sums of indecomposables in these sets which conform with the requirements on each set.*

For an example, we may take the abelian group $\mathbb{Z}_{2^\infty}^3 \oplus \mathbb{Z}_3^{(\aleph_0)}$ where the factor $\mathbb{Z}_{2^\infty}$ is of the first type, the factor $\mathbb{Z}_3$ is of the third type and the module $\mathbb{Q}$ is of the fourth type. Thus the models of this theory have the form $\mathbb{Z}_{2^\infty}^3 \oplus \mathbb{Z}_3^{(\lambda)} \oplus \mathbb{Q}^{(\kappa)}$ where λ is any infinite cardinal and κ is any cardinal. And, over the polynomial ring $R = \mathbb{Q}[x]$, the module M which is the injective hull of the module R/xR is itself a factor of the second type (and the ring $\mathbb{Q}(x)$ of rational functions is a factor of the fourth type for models of $\mathrm{Th}(M)$).

For more general modules one can look for a classification result along these lines but will find one only on the pp-saturated = pure-injective modules (Section 10). That will be a generalisation since every model of a totally transcendental theory of modules is pure-injective, indeed this characterises that property. There is a general structure theorem for pure-injectives and, in the best cases, in particular if M has Krull-Gabriel dimension, there will be no superdecomposable[8] pure-injective direct summands of models, yielding a classification result for the pure-injective models of $\mathrm{Th}(M)$ somewhat analogous to the result above (see [63, 10.24], also Section 12 below).

As a consequence of these general structure theorems, most effort on classification is focussed on the Ziegler spectrum (Section 12): on describing its

[6] That is, the logic based on regular=pp formulas, see [14], [57].
[7] We say that a nonzero module M is **indecomposable** if $M = M' \oplus M''$ implies $M' = 0$ or $M'' = 0$.
[8] A nonzero module is **superdecomposable** if it has no indecomposable direct summand.

points (the indecomposable pure-injective modules) and its topology (the closure relations between sets of pp-pairs). Description of the points is typically a (highly non-trivial) extension, to infinite-dimensional modules, of existing algebraic classification projects, and description of the topology often fits well with questions about morphisms between modules. Some examples are [21], [36], [39], [56], [67], [78], [56], [86], [87], [93].

9 Finitely generated pp-types and finitely presented modules

A pp-type p is **finitely generated** if there is a pp formula ϕ such that $p = \{\psi \text{ pp: } \phi \to \psi\}$ where $\to$ means implication modulo the theory of R-modules. In this case one says that ϕ **generates** p. That is, a finitely generated pp-type is a principal filter in the lattice pp_R^n of pp formulas (for some n). The pp-part of any principal type will be finitely generated but the converse is far from being true: for instance the pp-type of $1 \in \mathbb{Z}$ is finitely generated but its complete type is non-principal.

Note that, if $\overline{a}$ is a tuple from some module M, then, for the pp-type of $\overline{a}$ in M, $\mathrm{pp}^M(\overline{a}) = \{\phi \text{ pp} : M \models \phi(\overline{a})\}$, to be finitely generated, it is not enough that it be finitely generated modulo $\mathrm{Th}(M)$, that is, it is not enough that it be a principal filter in $\mathrm{pp}^n(M)$. For example, if $R = \mathbb{Z}$, $M = \mathbb{Q}$ and $a = 1 \in \mathbb{Q}$, then $\mathrm{pp}^1(\mathbb{Q})$ contains just two elements (all nonzero elements have the same (pp-)type), so $\mathrm{pp}^{\mathbb{Q}}(1)$ is finitely generated modulo the theory of $\mathbb{Q}$, but it is not a finitely generated pp-type. One has to prove the latter but it follows, of course, because this pp-type contains, and essentially consists of, the infinitely many divisibility pp formulas $\exists y\, (yn = x)$ for $n \in \mathbb{Z}$, $n \neq 0$.

Proposition 9.1. *(see [69, 1.2.6]) If M is a finitely presented module and $\overline{a}$ is an n-tuple from M, then $\mathrm{pp}^M(\overline{a})$ is a finitely generated pp-type.*

The proof consists of taking any finite generating set for M, then choosing a finite generating set of linear relations between these generators and writing $\overline{a}$ as a linear combination of these generators. The resulting pp formula which, modulo the theory of R-modules, generates $\mathrm{pp}^n(\overline{a})$ has the form

$$\exists \overline{y}\, (\theta(\overline{y}) \wedge \overline{x} = \overline{y} \cdot \overline{r}),$$

where θ is a conjunction of R-linear equations, $\overline{r}$ is an n-tuple of elements of R and $\overline{y} \cdot \overline{r}$ means $\sum_{i=1}^n y_i r_i$.

Proposition 9.2. *(see [69, 1.2.14]) If p is a finitely generated pp-type for the theory of R-modules, then there is a finitely presented R-module M and a tuple $\overline{a}$ of elements of M such that $\mathrm{pp}^M(\overline{a}) = p$.*

That is, every finitely generated pp-type is precisely realised in some finitely presented module. A **free realisation** of a pp formula ϕ is $(C, \overline{c})$ where C is a finitely presented module and $\overline{c}$ is a tuple from C such that the pp-type of $\overline{c}$ in C is generated by ϕ. So the above result says that every pp formula has a free realisation. And the result before that says that, if $\overline{a}$ is a finite tuple in a finitely presented module M, then $(M, \overline{a})$ is a free realisation of any pp formula which generates $\mathrm{pp}^M(\overline{a})$. Free realisations are far from unique (though sometimes there is a minimal one) but they do have a nice universal property, which explains the term "free realisation".

Proposition 9.3. *(see [69, 1.2.17]) Suppose that* $(C, \overline{c})$ *is a free realisation of a pp formula* ϕ*. Then, given any module* M *and tuple* $\overline{a}$ *with* $M \models \phi(\overline{a})$*, there is a homomorphism* $f : C \to M$ *with* $f\overline{c} = \overline{a}$*.*

For instance, if R is any ring and $r \in R$, then (R, r) is a free realisation of the formula $r|x$. If M is any R-module and $a \in Mr$ then there is a homomorphism $R \to M$ taking r to a.

There is a somewhat analogous result, see Section 10, for arbitrary modules in place of C but, for the conclusion there, we need to assume a completeness/compactness property on M, namely that M is pure-injective.

10 Purity

Given the pp-elimination of quantifiers theorem, the following definition is a natural analogue of that of elementary embedding.

A submodule M of a module N is **pure** if, for every tuple $\overline{a}$ from M, and pp formula ϕ[9], we have $M \models \phi(\overline{a})$ iff $N \models \phi(\overline{a})$. Clearly the direction $\Rightarrow$ holds for any inclusion, indeed for any homomorphism, so pure embeddings are the morphisms that *reflect* pp-formulas. Note that an alternative formulation of the condition is that $\mathrm{pp}^M(\overline{a}) = \mathrm{pp}^N(\overline{a})$. Of course we extend the terminology to monomorphisms in the obvious way ($f : M \to N$ is a **pure embedding** if it is monic and fM is a pure submodule of N).

This concept has many alternative and equivalent definitions, see e.g. [69, §2.1]; it was defined for abelian groups by Prüfer [83] and over general rings by Cohn [16].

A module N is **pure-injective** if, given any pure embedding $f : M \to M'$ and any homomorphism $g : M \to N$, there is $g' : M' \to N$ such that $g'f = g$. A module N is **algebraically compact** if every pp-type with parameters in N has a solution in N.

Theorem 10.1. *(see [69, 4.3.11]) A module is pure-injective iff it is algebraically compact.*

We will use the first term.

For example, any injective module is pure-injective. If R is a K-algebra where K is a field, then any R-module which is finite-dimensional over K is algebraically compact, hence pure-injective. The indecomposable pure-injective abelian groups were determined by Kaplansky [45] and they are, as p ranges over positive primes and n over positive integers: the indecomposable finite abelian groups $\mathbb{Z}_{p^n} = \mathbb{Z}/p^n\mathbb{Z}$; the p-adic integers $\overline{\mathbb{Z}_{(p)}}$; the Prüfer groups $\mathbb{Z}_{p^\infty}$; the rationals $\mathbb{Q}$. The Pontryagin **dual** $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of any right R-module M is a pure-injective left R-module.

Proposition 10.2. *(see [69, 4.3.9]) Suppose that* M *is any module and* $\overline{a}$ *is a finite tuple from* M*. If* N *is a pure-injective module and* $\overline{b}$ *a tuple from* N *such that* $\mathrm{pp}^M(\overline{a}) \subseteq \mathrm{pp}^N(\overline{b})$*, then there is a homomorphism* $f : M \to N$ *with* $f\overline{a} = \overline{b}$*.*

[9] In fact, it is enough to require this for pp formulas in one free variable.

The pure-injective modules, being the pp-saturated ones, play a role in the model theory of modules very similar to that played by saturated structures in the general case. But there are many more pure-injective modules than saturated modules.

We say that a **pure-injective hull** (or **pure-injective envelope**) of a module M is a pure embedding into a pure-injective module $H(M)$ which is minimal in the sense that there is no proper direct summand of $H(M)$ containing M. The same term is used for the module $H(M)$.

Theorem 10.3. *(see [69, 4.3.18]) Every module M has a pure-injective hull $H(M)$. Moreover, any two such hulls are isomorphic over M: if $f : M \to N$ and $f' : M \to N'$ are pure-injective hulls of M, then there is an isomorphism $g : N \to N'$ such that $gf = f'$.*

For instance the pure-injective hull of the localisation $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at a prime p is its completion, $\overline{\mathbb{Z}_{(p)}}$, in the p-adic topology - the module of p-adic integers.

Theorem 10.4. *([95], see [63, 2.27]) Every module M is an elementary substructure of its pure-injective hull: $M \prec H(M)$.*

Thus every module has a minimal pp-saturated elementary extension.

11 Hulls of elements and pp-types

The hull of a pp-type is a minimal pure-injective module in which that pp-type is realised. This is a somewhat subtle concept, not least because, *a priori*, it is not clear that such a module will exist. The key algebraic result on which this depends is the following.

Theorem 11.1. *([24], [46], see [69, 4.3.33]) Suppose that N is a pure-injective module and that A is a submodule of N. Then there is a (necessarily pure-injective) direct summand $H(A)$ of N which is minimal such containing A. If N' is any other direct summand of N which contains A, then there is an endomorphism f of N which fixes A pointwise and such that $fH(A)$ is a direct summand of N'.*

In particular $H(A)$ is unique up to isomorphism over A.

We refer to $H(A)$ as the **hull of A in N**. This notation is compatible with the earlier use of $H(M)$ for the pure-injective hull of M since, if A is purely embedded in N, then $H(A)$ will be a copy of the pure-injective hull of A. But this notion of hull of A does depend on the 'context' - the embedding of A in N - not just on the isomorphism type of A. To obtain the pure-injective hull of A we take the 'context' N to be any pure-injective in which A is purely embedded (for instance the double dual A^{**} of A).

In fact it can be checked that $H(A)$ depends only on the pp-type, $\mathrm{pp}^N(A)$, of A in N, in the following sense.

Theorem 11.2. *(see [69, 4.3.35]) Suppose that N and N' are pure-injective modules (which we may assume to be equal), that A, A' are subsets (without loss of generality, submodules[10]) of N, N' respectively, and that there is a map*

[10] Since each element of the submodule generated by a set is definable over that set.

$f_0 : A \to A'$ such that $\mathrm{pp}^N(A) = \mathrm{pp}^{N'}(A')$ via this map (which must therefore be a bijection, indeed an isomorphism if A, A' are submodules).

Then there is an extension of f_0 to a morphism $f : N \to N'$. If we choose a copy of $H(A)$ which is a direct summand of N and a copy of $H(A')$ which is a direct summand of N', then there is such a morphism f which maps $H(A)$ isomorphically to $H(A')$.

Note that, in the above, we may assume that $N = N'$ since we can replace each by $N \oplus N'$. That does not change the pp-types of A and A'; of course, in general it changes their complete types but pp-types have less dependence on 'context' than complete types. This is a subtle point and perhaps contributes to hulls of types being under-appreciated[11].

Given the above, if p is any pp-type (possibly of an infinite tuple), we define the **hull** of p to be a pure-injective module $H(p)$ containing a **realisation** of p - a tuple $\overline{a}$ in $H(p)$ such that $\mathrm{pp}^{H(p)}(\overline{a}) = p$ - which is minimal such in the sense that no proper direct summand of $H(p)$ contains $\overline{a}$. The above results imply that this exists and is unique in that if $\overline{a}'$ in N is another hull of p then there is an isomorphism from $H(p)$ to N taking $\overline{a}$ to $\overline{a}'$. The extension of hulls to pp-imaginaries (see Section 23) is described in [51].

We say that a pp-type is **irreducible**[12] (or **indecomposable**) if its hull is an indecomposable module. Since every indecomposable pure-injective is the hull of at least one pp-type in 1 free variable, it follows that there is just a set of indecomposable pure-injective R-modules up to isomorphism.

12 Pure-injectives and the Ziegler spectrum

There is a structure theorem for pure-injective modules.

Theorem 12.1. *(see [69, 4.4.2]) If N is a pure-injective module then $N = H(\bigoplus_i N_i) \oplus N_c$ where each N_i is indecomposable pure-injective and N_c is 0 or a **superdecomposable** pure-injective - meaning that N_c has no indecomposable direct summands.*

Furthermore, this decomposition is essentially unique - the summands, with multiplicities, are determined up to isomorphism.

We need the "H" in the above because a direct sum of infinitely many pure-injectives need not be pure-injective. A pure-injective module N such that $N^{(\aleph_0)}$, and hence $N^{(\kappa)}$ for any cardinal κ, is pure-injective, is said to be **Σ-pure-injective**. These are, in fact, the totally transcendental modules and, as mentioned in Section 6, they are characterised as those having the descending chain condition on pp-definable subgroups (see [63, 2.11], [69, §4.4.2]). Finite direct sums of pure-injective modules are pure-injective, as are direct summands of pure-injective modules.

Over many rings (in particular over $\mathbb{Z}$) there are no superdecomposable pure-injectives. In any case, there are always indecomposable pure-injectives, indeed the indecomposables control the model theory.

[11] Judging by the strong results which have been proved using hulls of types compared with the small number of people who have used these hulls.

[12] This term is chosen because pp-types generalise right ideals so this is the corresponding generalisation of the notion of a (meet-)irreducible right ideal.

Theorem 12.2. *([103], see [63, 4.36]) Every module is elementarily equivalent to a direct sum of indecomposable pure-injectives.*

This leads to the definition of the **Ziegler spectrum** Zg_R for (right) R-modules. This is the topological space whose points are the isomorphism classes of indecomposable pure-injectives - let pinj_R denote the set (it was noted above that this is a set) of these - and whose open sets are the unions of sets of the form

$$(\phi/\psi) = \{N \in \mathrm{pinj}_R : \phi(N) > \psi(N)\}$$

as ϕ/ψ ranges over pp-pairs (recall that these are pairs ϕ, ψ of pp formulas in the same free variables with $\psi \to \phi$). In fact, pp-pairs in one free variable are enough to give a basis of open sets for the topology.

To each module M, we associate its **support**:

$$\mathrm{supp}(M) = \{N \in \mathrm{pinj}_R : N \text{ is a direct summand of some } M' \equiv M\}.$$

This is a closed subset of Zg_R and every closed subset of Zg_R has this form for some M ([103], see [63, §4.7], also [69, §5.1]). Of course $\mathrm{supp}(M)$ is an invariant of $\mathrm{Th}(M)$, so we may write $\mathrm{supp}(T)$ for this, where $T = \mathrm{Th}(M)$ is the complete theory of M. It is easy to show that $\mathrm{supp}(T^{\aleph_0}) = \mathrm{supp}(T)$, in fact $\mathrm{supp}(T) \subseteq \mathrm{supp}(T_1)$ iff every model of T is a direct summand of a model of $T_1^{\aleph_0}$ that is, if $\mathrm{Add}(\mathrm{Mod}(T)) \subseteq \mathrm{Add}(\mathrm{Mod}(T_1))$.

Since every closed subset of Zg_R is the support of some module, we have that the association of complete theories to closed subsets of the Ziegler spectrum is a bijection if we restrict to those complete theories T satisfying $T = T^{\aleph_0}$ (which, over many rings, for instance algebras over an infinite field, is all complete theories of their modules).

The closed subsets of Zg_R also parametrise certain incomplete theories which, as already discussed, are more algebraically natural than the class of models of a complete theory of modules, namely they parametrise the elementary classes of modules which are closed under direct sums and direct summands. These are the definable subcategories (of the category of modules), which we discuss next.

13 Definable subcategories

Given a set $\Phi = \{\phi_i \geq \psi_i : i \in I\}$ of pp-pairs we consider the class

$$\{M \in \text{Mod-}R : \phi_i(M) = \psi_i(M) \, \forall i \in I\}$$

of modules where each of these pp-pairs is **closed** (meaning that the two pp-formulas are equivalent on M, that is, the pair (ϕ_i, ψ_i) is in the kernel of the evaluation map $\mathrm{pp}_R^n \to \mathrm{pp}(M)$ for appropriate n). A class of modules arising in this way is referred to as a **definable class** of modules, and the full subcategory of Mod-R on that class is a **definable subcategory** of the category of R-modules. These classes of modules have various characterisations, including those below, where, recall, if $\mathcal{C}$ is a class of modules, then $\mathrm{Add}(\mathcal{C})$ denotes the closure of $\mathcal{C}$ under arbitrary direct sums and direct summands.

Theorem 13.1. *(see [69, 3.4.7]) The following are equivalent for a class $\mathcal{D}$ of R-modules which is closed under isomorphism.*

(i) $\mathcal{D}$ is definable;
(ii) $\mathcal{D}$ is an elementary class of modules satisfying $\mathcal{D} = \mathrm{Add}(\mathcal{D})$;
(iii) $\mathcal{D}$ is closed in Mod-R *under finite direct products, directed colimits and pure submodules;*
(iv) $\mathcal{D}$ is closed in Mod-R *under arbitrary direct products, directed colimits and pure submodules.*

We denote by $\langle M \rangle$ the definable subcategory of Mod-R generated by M (the full subcategory on the smallest definable subclass of Mod-R which contains M). Every definable subcategory is generated in this sense by some module, for instance by the direct sum of the indecomposable pure-injective modules in it.

The previously-mentioned natural bijection between the definable subclasses of Mod-R and the closed subsets of the Ziegler spectrum is given by

$$\mathcal{D} \mapsto \mathcal{D} \cap \mathrm{pinj}_R$$

and

$$C \subseteq \mathrm{pinj}_R \mapsto \{M \in \text{Mod-}R : \mathrm{supp}(M) \subseteq C\}.$$

This is order-preserving (both sets being ordered by inclusion). We define the **Ziegler spectrum of** $\mathcal{D}$ to be the corresponding closed subset of Zg_R, consisting of the indecomposable pure-injectives in $\mathcal{D}$, with the induced topology, denoting it by $\mathrm{Zg}(\mathcal{D})$. As remarked at the end of this section, this is an invariant of the category $\mathcal{D}$ and does not depend on the embedding of $\mathcal{D}$ as a definable subcategory of some module category.

Below are a few examples which hint at the breadth of this concept. It is these definable subcategories - the full subcategories of Mod-R with objects the modules in a definable subclass - which are algebraically more natural than the classes of models of a complete theory (because forming direct sums and taking direct summands of modules are useful and algebraically innocuous operations). If one allows the ring R to be replaced by a skeletally small preadditive category - equivalently a ring with many objects - that is, if we allow many-sorted R-modules (see Section 21), then the examples are even broader.

Examples 13.2. 1) If M is any R-module of finite length over its endomorphism ring then (see [69, 4.4.30]) the definable subcategory $\langle M \rangle$ of Mod-R that it generates is exactly $\mathrm{Add}(M)$ - the direct summands of direct sums of copies of M. Equivalently, if $M = N_1 \oplus \cdots \oplus N_t$ with each N_i indecomposable, then the modules in Mod-R are those of the form $N_1^{(\kappa_1)} \oplus \cdots \oplus N_t^{(\kappa_t)}$ for cardinals $\kappa_i \geq 0$. Indecomposable infinitely generated such modules ("generic modules") play an important role in the structure of module categories, see [17].

2) The torsionfree abelian groups form a definable subcategory of Mod-$\mathbb{Z}$ but the torsion abelian groups do not (since they are not closed under direct products).

3) If p is a prime then the category of modules over the localisation $\mathbb{Z}_{(p)}$ at p is a definable subcategory of Mod-$\mathbb{Z}$, but the category of modules over the completion $\overline{\mathbb{Z}_{(p)}}$ is not, since it is not closed under pure submodules.

4) The category of reduced abelian groups (those $\mathbb{Z}$-modules M satisfying the condition $\bigcap_{n\in\mathbb{Z}^+} Mn = 0$) is not a definable subcategory since it is not closed under directed colimits ($\mathbb{Q}$ is the colimit of a directed system of embeddings between copies of $\mathbb{Z}$).

5) Under suitable finiteness conditions on a ring, various classes of homologically-defined modules are definable. For instance, see [69, §3.4.3], the injective right R-modules form a definable class iff R is right noetherian, the larger class of absolutely pure = fp-injective modules[13] is definable iff R is right coherent and that is also exactly the condition for the class of flat *left* R-modules to be definable.

6) If L is a finitely presented left module then there is a pp-pair ϕ/ψ such that, for every right module M, we have $M \otimes_R L \simeq \phi(M)/\psi(M)$. Here the isomorphism is actually a natural equivalence between the functor $- \otimes_R L$ and the functor (see Section 23) from right R-modules to abelian groups which is given on objects by $M \mapsto \phi(M)/\psi(M)$. Similarly if A is a finitely presented right R-module then $M \mapsto \mathrm{Hom}_R(A, M)$ is given by a pp-pair. Each of these results generalises to the derived functors $\mathrm{Tor}_i^R(L, -)$ and $\mathrm{Ext}_R^i(A, -)$ provided L, A satisfy a suitable strengthening (FP_{i+1} to be precise) of the finitely presented condition (see [69, §4.4.6]). In particular the kernels of these homological functors are, if the modules A and L are suitably nice, definable subcategories. This is the basis (see [69, §18.2.3]) of applications in tilting (and related) theory.

In fact, but we do not detail this here, it is the definable categories which are the natural contexts for additive model theory. In particular the natural language and the theory of the modules in a definable subcategory $\mathcal{D}$ of Mod-R is intrinsic to $\mathcal{D}$, as therefore is its Ziegler spectrum $\mathrm{Zg}(\mathcal{D})$, in the sense that all this can be recovered purely from the category-theoretic structure of $\mathcal{D}$ (see [71, Chpt. 12]). From this point of view, the class of torsion abelian groups, being finitely accessible (see Section 22) and having products, is a definable category, though we have to use a multisorted language and realise it as a definable subcategory of the category of modules over a ring with many objects. But, of course, the embedding of the category of torsion abelian groups into **Ab** = Mod-$\mathbb{Z}$ is not an embedding as a definable subcategory.

14 The largest theory of modules

A module M is a model of the **largest theory of R-modules** (see Section 7 above, also [63, §2.6]) if, for every pp-pair $\phi \geq \psi$ with ϕ not equivalent (modulo the theory of R-modules) to ψ, we have $\phi(M) > \psi(M)$ and the quotient $\phi(M)/\psi(M)$ is infinite. Equivalently, $\mathrm{supp}(M) = \mathrm{pinj}_R$ and $M \equiv M^{(\aleph_0)}$.

Note that this largest theory of R-modules is indeed a complete theory by pp-elimination of quantifiers. It is largest also in the sense that, for any module M, there is a module M' such that $M \oplus M'$ is a model of the largest theory of R-modules.

If M is a model of the largest theory of modules, then M exhibits, in many senses, the full complexity of the model theory of R-modules. For instance the morphism $\mathrm{pp}_R^n \to \mathrm{pp}^n(M)$ is an isomorphism. Therefore, if M has Krull-Gabriel (or elementary Krull) dimension α then this is an upper bound on the values of this dimension among all R-modules. In particular, if M is totally

[13] A module is **absolutely pure** if it is a pure submodule of an injective module, equivalently if every embedding with it as domain is a pure embedding. As the alternative name suggests there is an equivalent homological definition.

transcendental, then so is every R-module[14] and M attains the maximum value of Morley rank for right R-modules.

The journey from the model theory of modules to more general additive model theory begins by replacing the definable category generated by a model of the largest theory of R-modules (that is, the category Mod-R) with the definable category generated by an arbitrary module. Once in that smaller category, one realises that all its model theory is intrinsic - there is no need to refer back to the surrounding module category (though of course it might be convenient to do so). The process of moving to the smaller category is *localisation*, figuratively but in some senses also literally, see Section 24. If one also moves to a more general starting point by allowing the ring R to be replaced by some skeletally small preadditive category, that is a 'ring with many objects', then localising gives the general notion of a definable category. One can argue that this is the "correct" notion in the additive context on the basis of a 2-category equivalence between, on the one hand, the 2-category of definable additive categories and interpretation functors between them and, on the other hand, the 2-category of skeletally small abelian categories and exact functors between these, see [72].

15 Interpretations and interpretation functors

In the context of modules and other additive structures, it is natural to require that the addition should be preserved when interpreting one structure within another. That condition forces the interpreting formulas to be pp formulas (for the precise result see [13, 2.1]). In particular if R and S are rings and we want to interpret (some) S-modules within (some) R-modules, then the data of an interpretation are as follows:
- a pair $\phi(\overline{x}) \geq \psi(\overline{x})$ of pp formulas in $\mathcal{L}_R$ in, say, n free variables;
- for each $s \in S$ a pp formula $\rho_s(\overline{x}, \overline{x}')$ with $\overline{x}$ and $\overline{x}'$ being tuples of length n.

Then, given $M \in$ Mod-R, we consider the abelian group $\phi(M)/\psi(M)$. For each $s \in S$ the condition that $\rho_s(\overline{x}, \overline{x}')$ well-define a total function (necessarily an additive homomorphism) from $\phi(M)/\psi(M)$ to itself is the condition that a certain pp-pair in $\mathcal{L}_R$ be closed. If we let $\mathcal{D}$ denote the definable subcategory of Mod-R given by closure of all these pp-pairs, then every $M \in \mathcal{D}$ interprets an S-module, namely that with underlying abelian group $\phi(M)/\psi(M)$ and with the action of $s \in S$ being given by ρ_s.

Thus the above data define an interpretation of certain S-modules within certain R-modules, namely each R-module $M \in \mathcal{D}$ interprets an S-module, as above (see, e.g., [71, Chpt. 25] or [66] for more detail).

In fact the assignment from M to the interpreted S-module is an additive functor from $\mathcal{D}$ to Mod-S. (The image of this functor need not be a definable subcategory of Mod-S but its closure under pure submodules will be definable, [72, 3.8].) It turns out that such **interpretation functors** have a purely algebraic characterisation; we state this is a somewhat general form (in the formulation below $\mathcal{C}$ is Mod-S or, if we prefer, we could take $\mathcal{C}$ to be the definable

[14] This is equivalent to the condition that every right R-module is pure-injective. It is an open question - the pure-semisimplicity conjecture - whether this implies that every left R-module is totally transcendental, equivalently whether it implies that every right R-module has finite Morley rank.

subcategory of Mod-S which is the closure of the image of the interpretation functor under pure submodules).

Theorem 15.1. *([71, 25.3]) Suppose that $\mathcal{C}$ and $\mathcal{D}$ are definable categories and that $I : \mathcal{D} \to \mathcal{C}$ is an additive functor. Then I is an interpretation functor (in the sense above) iff I commutes with direct products and directed colimits.*

This is actually part of the more general theory of regular categories and categories of models of regular theories (for which see, for instance, [57], [14]).

Given a pp-pair ϕ/ψ in the language for right R-modules, the functor $M \mapsto \phi(M)/\psi(M)$ is an interpretation functor from Mod-R to **Ab**. If R is commutative, then each pp-definable subgroup is naturally an R-module so, in this case, such a functor is actually an endofunctor of Mod-R.

Many, many interpretation functors occur in the representation theory of algebras; see, for instance, [35], [75, §8], [69, 18.2.4], [71, Chpt. 25], [77, §6] for some examples which are presented as such; there are many more, for instance representation embeddings, in the literature which are presented purely algebraically but which are clearly interpretation functors.

16 Rings of definable scalars

If M is an R-module, then the action of each element $r \in R$ is, of course, definable by a pp formula $\rho(x, y)$ (namely $y - xr = 0$) but there may be other pp-definable actions. For instance, if M is the Prüfer group $\mathbb{Z}_{p^\infty}$, and n is an integer coprime to p, then multiplication by n is invertible on M and clearly the inverse map is pp-definable. Indeed this Prüfer group has the natural structure of a module over the localisation $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at p and the action of each element of this ring is pp-definable on M. (There is also a natural action of the p-adic integers but those actions are definable using pp-types rather than pp formulas.)

The ring of definable scalars of an R-module M is the ring of pp-definable (in $\mathcal{L}_R$) actions on M; more formally, it is the set of equivalence-on-M classes of pp formulas which well-define a total function on M. The addition is pointwise addition and the multiplication is composition of functions. Note that this depends only on $\mathrm{Th}(M)$, indeed, each of these actions is pp-definable (by the same formula) on every module in the definable subcategory $\langle M \rangle$ generated by M. Thus, to every closed subset X of the Ziegler spectrum Zg_R, we have a ring R_X, indeed a ring homomorphism $R \to R_X$, such that every module M with support contained in X is naturally and definably (by formulas of $\mathcal{L}_R$, modulo the theory of right R-modules) a right R_X-module.

If this looks a little like localisation (of a ring), then that impression is correct. Indeed every classical localisation of a ring R occurs in this way, see [69, §6.1.3]. More generally every epimorphism of rings from R occurs in this way, see [69, 6.1.8]; but not every $R \to R_X$ need be an epimorphism of rings, see [69, 6.1.13]. To be clear, this means that if $R \to S$ is an epimorphism of rings[15] then the model theory of S-modules is contained in the model theory of R-modules. In particular universal localisations are ring epimorphisms and, for instance, Herzog showed in [41] that the ring of definable scalars associated to the finite-dimensional representations of the Lie algebra $\mathrm{sl}_2(k)$, for k an

[15] Which is much more general than a surjection of rings!

algebraically closed field of characteristic 0, is von Neumann regular and is a universal localisation of the enveloping algebra of $\mathrm{sl}_2(k)$.

Here is a general result relating rings of definable scalars to localisation.

Theorem 16.1. *(see [69, 4.3.44]) Suppose that N is an indecomposable pure-injective right module over the ring R. Let $Z(R)$ denote the centre of R. Then the set $P = \{r \in Z(R) : r$ acts non-invertibly on $N\}$ is a prime ideal in $Z(R)$ and N is naturally and pp-definably a right module over the central localisation*[16] *of R at P.*

In particular, if R is commutative, then every indecomposable pure-injective R-module is a module over a localisation of R at some prime ideal.

17 Elementary duality

If the ring R is commutative then the categories Mod-R and R-Mod of right and left modules are equivalent. If R is not commutative, they may be very different; for example a ring might be right artinian but not even left noetherian, so the structure of its right and left (even finitely generated) modules will be very different.

There are, however, various kinds of dualities that connect left and right modules. In particular, if we define, for any module M, its (character-, or Pontryagin-) dual module to be

$$M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$$

then, if M is a right, respectively left, R-module, M^* has a natural left, resp. right, R-module structure. Certainly not every module is a dual - in fact every module of the form M^* is pure-injective (see [69, 4.3.29]) - but the natural map $M \to M^{**}$ is a pure, indeed elementary, embedding, so M and M^{**} are elementarily equivalent.

For example $(\mathbb{Z}_{p^\infty})^* = \overline{\mathbb{Z}_{(p)}}$ and $(\mathbb{Z}_{(p)})^* = \mathbb{Z}_{p^\infty} \oplus \mathbb{Q}^{(2^{\aleph_0})}$.

Model-theoretic duality between left and right modules works at various levels. First, at the level of formulas.

Let $\phi(\overline{x})$ be a pp formula for right R-modules, say

$$\exists \overline{y}\, (\overline{x}\ \overline{y}) \begin{pmatrix} G \\ H \end{pmatrix} = 0.$$

We define the (**elementary**) **dual** of ϕ to be the pp formula $D\phi(\overline{x})$ which is

$$\exists \overline{z} \begin{pmatrix} I_n & G \\ 0 & H \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{z} \end{pmatrix} = 0$$

for left R-modules (where, of course, $\overline{x}$ now should be read as a column vector). In the other direction, from pp formulas for left modules to their dual pp formulas for right modules, we just use the transpose of this definition.

Theorem 17.1. *(see [69, §1.3]) For every pp formula ϕ, $DD\phi$ is equivalent to ϕ (modulo the theory of R-modules). Moreover, $\psi \to \phi$ iff $D\phi \to D\psi$, $D(\phi \wedge \psi) = D\phi + D\psi$ and $D(\phi + \psi) = D\phi \wedge D\psi$.*[17]

[16] That is, invert all elements of $Z(R) \setminus P$.

[17] Here, "=" means "are equivalent".

That is, for every n, elementary duality is a lattice anti-isomorphism between pp_R^n and $\mathrm{pp}_{R^{\mathrm{op}}}^n$ (the latter denotes the lattice of pp formulas in n free variables for left R-modules = right R^{op}-modules). Therefore, for example, the largest theory of right R-modules has Krull-Gabriel dimension α iff this is so for the largest theory of left R-modules (because the Krull-Gabriel dimension of a lattice, as opposed its elementary Krull dimension, is equal to that of the opposite lattice). Note that this gives us an alternative standard form for pp formulas: if $\phi(\overline{x})$ is

$$\exists \overline{y}\,(\overline{x}\ \overline{y})\begin{pmatrix} G \\ H \end{pmatrix} = 0$$

then its dual has the form

$$\exists \overline{z}\,(\overline{x} = G\overline{z} \wedge H\overline{z} = 0).$$

Since every pp formula is a dual pp formula, a typical pp formula can be written in the form

$$\exists \overline{y}\,(\overline{x} = \overline{y}G \wedge \overline{y}H = 0),$$

that is, every pp formula is a 'generalised divisibility formula'.

There is the following nice criterion ("Herzog's criterion"), in terms of pp formulas and their duals, for an element of the tensor product of a right and a left module to be 0.

Theorem 17.2. *([40], see [69, 1.3.7]) Let M be a right R-module, L a left R-module, $\overline{a} \in M^n$, $\overline{l} \in L^n$. Then $\overline{a} \otimes_R \overline{l} = 0$ (that is $\sum_{i=1}^n a_i \otimes l_i = 0$) in $M \otimes_R L$ iff there is a pp formula $\phi(\overline{x})$ such that $M \models \phi(\overline{a})$ and $L \models D\phi(\overline{l})$.*

Related to this is the next result which connects this model-theoretic duality with the algebraic duality seen before.

Theorem 17.3. *(see [69, 1.3.12]) Suppose that M is a right R-module and let M^* be its dual left R-module. Let ϕ be a pp formula for left R-modules, with n free variables. Then*

$$\phi(M^*) = \mathrm{ann}_{M^*}(D\phi(M))$$

where, for $X \subseteq M^n$,

$$\mathrm{ann}_{M^*}(X) = \{\overline{f} \in (M^*)^n : \sum_{i=1}^n f_i(a_i) = 0 \textit{ for every } \overline{a} \in X\}.$$

Elementary duality extends further.

18 Duality of theories and Ziegler spectra

The right and left Ziegler spectra of any ring R are homeomorphic at the level of topology, meaning that the complete Heyting algebra of open subsets of Zg_R is isomorphic to that of ${}_R\mathrm{Zg}$ ($= \mathrm{Zg}_{R^{\mathrm{op}}}$) ([40], see [69, §5.4]). The isomorphism takes the basic open subset (ϕ/ψ) of Zg_R to the basic open subset $(D\psi/D\phi)$ of ${}_R\mathrm{Zg}$. It is not known whether there is always a homeomorphism at the level of points. That is, can we, canonically or otherwise, associate to any indecomposable pure-injective right R-module N an indecomposable pure-injective left

R-module DN such that $N \in (\phi/\psi)$ iff $DN \in (D\psi/D\phi)$? For countable rings (see [69, 5.4.6]), for rings whose largest theory of modules has Krull-Gabriel dimension (see [69, 5.4.20]), and under various other conditions (see [69, 5.4.11]), this is so, but it is not known in general.

If R is commutative then this gives an endo-homeomorphism, at the level of topology (that is, on the lattice of open sets) and often at the level of points, on Zg_R. In the case $R = \mathbb{Z}$, this interchanges the p-Prüfer group and p-adic integers for each prime p and fixes every other point.

Since definable subcategories correspond bijectively to closed subsets of the Ziegler spectrum, this means that there is a natural, inclusion-preserving, bijection between definable subcategories of Mod-R and those of R-Mod. We denote the dual definable subcategory of $\mathcal{D}$ by $\mathcal{D}^{\mathrm{d}}$; we have $\mathcal{D}^{\mathrm{dd}} = \mathcal{D}$. Algebraically, $\mathcal{D}^{\mathrm{d}}$ is the class of pure submodules of modules of the form M^* with $M \in \mathcal{D}$, see [69, 3.4.17].

For instance, if R is right coherent, then the classes of right absolutely pure modules and left flat modules are dual definable subcategories (see [69, 3.2.24])

In fact, [40], this duality can be refined to give a natural bijection between complete theories of right R-modules and complete theories of left R-modules (supports of modules correspond as just described and, in fact, finite indices also correspond, the axiom $|\phi(-)/\psi(-)| \geq k$ corresponding to $|D\psi(-)/D\phi(-)| \geq k$.) Of course, we denote the dual of a (complete) theory T by DT.

As one might anticipate, the theories of the p-Prüfer group and the p-adic integers are dual.

19 Elementary cogenerators

An **elementary cogenerator** for a definable category $\mathcal{D}$ is a module N such that every module in $\mathcal{D}$ purely embeds in a power of N. Thus an elementary cogenerator for $\mathcal{D}$ generates $\mathcal{D}$ algebraically in a relatively simple way, and this can be useful when trying to understand the actions of functors between definable categories.

Theorem 19.1. *(see [63, §9.4], [69, 5.3.52, 5.2.54]) Every definable category $\mathcal{D}$ has an elementary cogenerator. In fact, if N is a pure-injective module which realises every irreducible 1-type (for the largest theory of a module supported on $\mathcal{D}$) then N is an elementary cogenerator of $\mathcal{D}$.*

If $\mathcal{D} = \langle N \rangle$ and N is a pure-injective module which is weakly saturated (or which just realises enough irreducible pp-types) then N is an elementary cogenerator of $\mathcal{D}$.

If $\mathcal{D} = \langle M \rangle$ with M totally transcendental, then M is an elementary cogenerator for $\mathcal{D}$.

20 Discrete and indiscrete rings

The Ziegler spectrum Zg_R of a ring is a compact space, that is, every open cover has a finite subcover, [103], see [69, 5.1.23], so, if discrete, it must be finite. Any ring R of **finite representation type**[18] has discrete Ziegler spectrum. The

[18]Meaning that every R-module is a direct sum of indecomposable modules and that there are, up to isomorphism, just finitely many indecomposable R-modules. The condition on, say,

converse is not known, though it is true if we make the, rather weak and perhaps empty, assumption that R satisfies the "isolation condition", see [69, 5.3.26].

The ring R is of finite representation type iff each pp-lattice pp_R^n is of finite length, equivalently if the theory of (right, equivalently left) R-modules has finite Morley rank.

At the other extreme are the **indiscrete** rings - those whose Ziegler spectrum has the indiscrete topology. By elementary duality this is a two-sided property of rings - the right Ziegler spectrum has the indiscrete topology iff this is true of the left Ziegler spectrum. Of course division rings have this property, having 1-point spectra. But there are more interesting examples of indiscrete rings, necessarily with many points in Zg_R ($2^{\aleph_0}$ if R is countable - [81, 2.2(b)], also see [69, 7.2.12-14]). Every simple von Neumann regular ring which is not a division ring is an example - a specific example is the endomorphism ring of a countably infinite-dimensional vector space V over a field, factored by its ideal consisting of the endomorphisms of finite rank. For a while it was open whether there were examples which are not von Neumann regular (and hence whose theory of modules does not have complete elimination of quantifiers). Such examples were constructed in [81].

A ring R is indiscrete iff, given any two right R-modules M and N, we have $M^{(\aleph_0)} \equiv N^{(\aleph_0)}$. Indeed, if R is indiscrete and not a finite field, then every two nonzero modules are elementarily equivalent.

21 Multisorted modules

The model theory of modules was largely developed for modules as usually defined: an abelian group with a ring acting on it as additive endomorphisms. But the model theory of multisorted modules hardly differs from that of 1-sorted modules, yet encompasses many more examples, so we give a quick introduction to these; for a fairly thorough exposition, see [76]. First we have to introduce multisorted rings = rings with many objects. These are actually the same thing as (skeletally small) preadditive categories, equivalently $\mathbb{Z}$-path algebras of (possibly infinite) quivers[19].

We will define a multisorted ring R to be given by:
• a set Sorts (which will index the sorts of R and of its modules);
• for every pair $(i, j) \in \mathrm{Sorts}^2$ an abelian group $R(i, j)$ (the union of these is the set of elements of R);
• for every triple $(i, j, k) \in \mathrm{Sorts}^3$, a bilinear map $R(j, k) \times R(i, j) \to R(i, k)$; this is the multiplication in R so we write gf for the image of $(g, f) \in R(j, k) \times R(i, j)$ in $R(i, k)$.

We require associativity for these bilinear maps: $h(gf) = (hg)f$ for all $f \in R(i, j), g \in R(j, k), h \in R(k, l)$ and also require, for each $i \in \mathrm{Sorts}$, that there is an identity $e_i \in R(i, i)$ at i: $e_j f = f = f e_i$ for all $f \in R(i, j)$.

Note that, for each $i \in \mathrm{Sorts}$, $R(i, i)$ is an abelian group with an associative ("multiplication") operation $R(i, i) \times R(i, i) \to R(i, i)$ for which there is an identity; that is, $R(i, i)$ is a ring in the usual, 1-sorted sense. In particular if

right modules implies the same condition on left modules.

[19] Under the preadditive categories view, a (normal) ring is a preadditive category with just one object (up to isomorphism); under the quivers view it is the path algebra of a quiver with just one vertex (but possibly many arrows).

Sorts is a singleton, then we have the usual notion of ring. What is extra in the general case is the multitude of sorts and also the ring elements that go between sorts.

Example 21.1. The definition above is equivalent to that of rings without identity but with enough idempotents. Let R be such a ring. Define Sorts to be the elements of a maximal direct-sum-independent set of idempotents, and set $R(e, f) = fRe$ for $e, f \in \text{Sorts}$.

The ring R is fixed by incorporating it into the language $\mathcal{L}_R$, which we define next.[20] The R-modules are then certain $\mathcal{L}_R$-structures.

So, given a multisorted ring R, we take Sorts to index the sorts of $\mathcal{L}_R$. For each sort σ of $\mathcal{L}_R$ we have a binary function symbol for addition in that sort, and a constant symbol for the zero element of that sort (of course we can introduce symbols for subtraction and negative if we wish). And for each element $r \in R(i, j)$ we introduce a unary function symbol with domain sort i and codomain sort j.

Then we axiomatise R-modules just as in the 1-sorted case. So a (multisorted) R-module consists of a collection $(M_i)_{i \in \text{Sorts}}$ of abelian groups with the interpretations of the various function symbols corresponding to the elements of R, all satisfying the obvious axioms. The next section gives some examples. The main point is that the model theory of these structures is essentially no different from that of 1-sorted modules. The only point to bear in mind is that any single formula can involve only finitely many sorts[21], so the Ziegler spectrum will not, in general, be compact, just locally compact.

22 Model theory in finitely accessible and presentable categories

A category $\mathcal{C}$ is **finitely accessible** (see [1]) if it has directed colimits, if there is just a set, up to isomorphism, of finitely presented objects[22] and if every object of $\mathcal{C}$ is a directed colimit of finitely presented objects. The categories of groups, of rings, of R-modules, ... all are finitely accessible, in fact these are **locally finitely presented** in that they also have all limits and colimits.

The categories we are interested in here are the additive finitely accessible categories $\mathcal{C}$ which have direct products. Such a category is a definable subcategory of a category of multisorted modules (see [71, 10.1]).[23] The multisorted ring is just the category $\mathcal{C}^{\text{fp}}$ of finitely presented objects of $\mathcal{C}$ (or, at least, some small version thereof). Then an object C of $\mathcal{C}$ becomes a (right) module over $\mathcal{C}^{\text{fp}}$, with its objects of sort $A \in \mathcal{C}^{\text{fp}}$ being the morphisms of $\mathcal{C}$ from A to C.[24]

[20] Languages which have the ring as a sort, so which allow it to vary, have been considered - see [96], [44, Chpt. 9] for instance. The model theory of such (ring,module)-structures is, however very different from the additive model theory that we consider here.

[21] To be clear, each variable of the language has a specified sort.

[22] In a general category an object X is said to be **finitely presented** if the representable functor $(X, -)$ commutes with direct colimits. This is equivalent to X being finitely generated and finitely related in categories where the latter terms make sense.

[23] The converse, however, is not true, definable categories need not be finitely accessible although, by downwards Löwenheim-Skolem, they will be κ-accessible for some κ. For instance the definable category of divisible abelian groups has no nonzero finitely presented objects.

[24] This generalises the observation that every module M over a ring (in the usual, 1-sorted, sense) R can be identified with the R-linear homomorphisms from the right module R to M,

Details of all this can be found, for instance in [71] or [69, Chpt. 16].

Examples of multisorted modules, and hence of structures to which the model theory of modules applies, include: chain complexes of modules; comodules over coalgebras [19]; sheaves over nice enough ringed spaces [79], [82]; quasicoherent sheaves over nice enough schemes [82]; categories of additive functors.

23 Pp formulas as functors

Given a ring R (which could be multisorted) and a pp formula $\phi = \phi(\overline{x})$ in $\mathcal{L}_R$, we have the (additive) functor $F_\phi : \text{Mod-}R \to \mathbf{Ab}$ which assigns to each right R-module M, the group $\phi(M)$. Every module M is a directed colimit $M = \varinjlim_\lambda M_\lambda$ of a directed system $\big((M_\lambda)_\lambda, (f_{\lambda\mu} : M_\lambda \to M_\mu)_{\lambda<\mu}\big)$ of finitely presented modules, and pp formulas commute with directed colimits: $\phi(M) = \varinjlim_\lambda \phi(M_\lambda)$. So the functor F_ϕ is determined by its restriction to the category mod-R of finitely presented modules (we will use the same notation for the functor and for its restriction). We remark that the functor category $(\text{mod-}R, \mathbf{Ab})$, being finitely accessible with products, indeed locally finitely presented, is itself a definable category.

Furthermore, the functor F_ϕ is a finitely presented object of the category $(\text{mod-}R, \mathbf{Ab})$ of all functors[25] from mod-R to $\mathbf{Ab}$. Therefore the functor $F_{\phi/\psi} = F_\phi/F_\psi$ associated to any pp-pair ϕ/ψ also is finitely presented. Indeed this gives all the finitely presented functors.

Theorem 23.1. *(see [69, 10.2.30]) The category* $(\text{mod-}R, \mathbf{Ab})^{\text{fp}}$ *of finitely presented functors on finitely presented right R-modules is equivalent to the category* $\mathbb{L}_R^{\text{eq}+}$ *of pp-pairs for right R-modules and pp-definable maps between these.*

The pp-definable maps from sort ϕ/ψ to sort ϕ'/ψ' are just those described by the name (generalising the definition of rings of definable scalars). Namely, those given by a pp formula $\rho(\overline{x}, \overline{x}')$ satisfying $\phi(\overline{x}) \wedge \rho(\overline{x}, \overline{x}') \to \phi'(\overline{x}')$ and $\psi(\overline{x}) \wedge \rho(\overline{x}, \overline{x}') \to \psi'(\overline{x}')$. This category $\mathbb{L}_R^{\text{eq}+}$ is referred to as the **category of pp-pairs** or **pp-imaginaries** for right R-modules. It is an abelian category, in fact the free abelian category on R^{op} (for which see [2], [25]), and its opposite is the category of pp-imaginaries for left R-modules.

Theorem 23.2. *(see [69, 10.2.30]) The category of pp-imaginaries for left modules is opposite to that for right modules:* $\mathbb{L}_{R^{\text{op}}}^{\text{eq}+} \simeq ((\mathbb{L}_R^{\text{eq}+}))^{\text{op}}$.

In this form, this appears in [40] and, in the equivalent functor category form in [4] and [38]. The categories were shown to be equivalent in [12].

The equivalence between pp-pairs and finitely presented functors in the first result above has been enormously useful and, as one might expect, the second result is part of elementary duality.

These results refer to the whole module category - to the largest theory of modules if one prefers to express it this way. Everything may be relativised to any definable subcategory - to any complete theory T of modules satisfying $T = T^{\aleph_0}$. The process involved is a kind of localisation.

via $f \in (R, M) \mapsto f(1) \in M$.

[25]Throughout this paper, "functor" will mean "additive functor".

24 Localisation

Localisation here is the process of moving from the model theory of a definable category $\mathcal{D}$ to the model theory of a definable subcategory $\mathcal{C}$. The process can be thought of as moving from the complete theory of a module which definably-generates $\mathcal{D}$, to that of a direct summand which definably-generates $\mathcal{C}$. We have already seen something of this in the surjective homomorphism from pp_R^n to $\mathrm{pp}^n(M)$ where $M \in \text{Mod-}R$; more generally we have such a homomorphism from $\mathrm{pp}^n(M)$ to $\mathrm{pp}^n(M')$ for any $M' \in \langle M \rangle$. This process is very broadly expressed through the corresponding categories of pp-imaginaries as follows.

Let $\mathcal{D}$ be a definable subcategory of Mod-R. We define its category $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$ of pp-imaginaries to have objects the pp-pairs ϕ/ψ (so just as in $\mathcal{L}_R$) and morphisms the pp-definable, *modulo* $\mathrm{Th}(\mathcal{D})$ maps - equivalence-modulo-$\mathcal{D}$ classes of pp formulas which, *on* $\mathcal{D}$, well-define a total map from ϕ/ψ to ϕ'/ψ'. So, for example, the ring of definable scalars of a module M is the endomorphism ring of the "home sort" $x = x/x = 0$ in $\mathbb{L}^{\mathrm{eq}+}(\langle M \rangle)$.[26]

If $\mathcal{C}$ is a definable subcategory of $\mathcal{D}$ then we may consider the set of pp-pairs - i.e. objects of $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$ - which are closed on, that is which are 0 when evaluated on, $\mathcal{C}$. These form a Serre subcategory[27] of $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$, so we may form the quotient abelian category $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})/\mathcal{S}$. Then $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})/\mathcal{S} \simeq \mathbb{L}^{\mathrm{eq}+}(\mathcal{C})$; thus the category of pp-imaginaries of a definable subcategory is obtained as a localisation, see, e.g., [69, §12.3].

There is much more that could be said here. For instance a definable category may be recovered as the category of exact functors on its category of pp-imaginaries; so an R-module is just an exact functor from the category of pp-imaginaries for R-modules to the category of abelian groups, see [48], [75]. That result leads to the previously mentioned 2-category equivalence between the 2-category of (skeletally small) abelian categories with exact functors between them and the 2-category of definable categories with interpretation functors between them, see [71], [72]. But this could take us a long way from classical model theory (though not from regular model theory), so we don't say more about it here.

25 Pp-types as functors

Here again we give just an indication of a large topic.

We saw that pp formulas can be seen as certain functors (indeed, when restricted to finitely presented modules, they are the functors of projective dimension ≤ 1, with the quantifier-free formulas being the projective functors, [12], see [69, 10.2.5, 10.2.6]). It is not so obvious how to conceive of pp-types in functor-category terms - they are filters of finitely presented functors but it would be better to associate pp-types with single algebraic objects rather than sets of such. Moving to the dual functor category gives a nice solution.

Let us consider, for simplicity, a pp-1-type p for right R-modules. Each pp formula $\phi \in p$ gives a finitely presented subfunctor F_ϕ of the forgetful functor

[26] We remarked earlier that the model theory of $\mathcal{D}$ is intrinsic so, in fact, we could define this without reference to an embedding of $\mathcal{D}$ as a definable subcategory of a module category, see [71, Chpt. 12].

[27] A **Serre subcategory** $\mathcal{S}$ of an abelian category $\mathcal{A}$ is one which is such that, for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, we have $B \in \mathcal{S}$ iff $A, C \in \mathcal{S}$.

$(R_R, -)$, that is $\mathrm{Hom}_R(R_R, -)$, in $(\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$, and these, for $\phi \in p$, form a filter in the lattice of finitely presented = finitely generated[28] subfunctors of $(R_R, -)$. If we apply elementary duality, then this filter maps to an ideal $\{F_{D\phi} : \phi \in p\}$ of finitely generated subfunctors of $({}_RR, -)$ - the forgetful functor for left R-modules. So we have the subfunctor, not necessarily finitely generated, of $({}_RR, -)$ which is the sum, we denote it F_{Dp}, of all these. Conversely, if G is any subfunctor of $({}_RR, -)$ then it has the form F_{Dp} for some pp-1-type p for right R-modules.

That is, stating the result for general n, pp-n-types for right R-modules are equivalent to subfunctors (not necessarily finitely generated) of the nth power of the forgetful functor for left R-modules, [69, 12.2.1].

This allows one to reconceive many results and techniques in the model theory of modules. For example, the hull $H(p)$ of a pp-n-type is characterised by the functor $H(p) \otimes_R (-)$ being the injective hull in $(R\text{-mod}, \mathbf{Ab})$ of $({}_RR^n, -)/F_{Dp}$ ([69, 12.2.6]). This also allows a flexibility of method in that one may easily mix model-theoretic and functor-category methods.

26 Other topologies

There are other natural topologies, which we just point to here, on the set pinj_R of isomorphism classes of indecomposable pure-injective (right) R-modules.

1) **Dual-Ziegler = rep-Zariski topology**: this is more or less the Hochster dual of the Ziegler topology. We take the compact open sets - the (ϕ/ψ) - of the Ziegler topology to be a generating family of closed sets in the new, dual-Ziegler, topology. Hochster's definition [42] was made for spectral spaces, which the Ziegler spectrum is not: even for $\mathbb{Z}$-modules it is T_0 but the intersection of two compact open sets need not be compact. Nevertheless, this leads us to an interesting space which can be seen as a noncommutative Zariski spectrum associated to the category of right R-modules [69, §14.1.2]. Over a commutative noetherian ring, the subset of pinj_R which consists of the indecomposable injective modules is a closed subset of the Ziegler spectrum which, if given the dual-Ziegler topology, is exactly the Zariski spectrum of the corresponding algebraic variety [69, §§14.1.1, 14.1.3]. Over rather more general rings it coincides [29] with the Thomason topology [100] which is used in various parametrisations connected with tilting in both abelian and triangulated categories. See [74, §6] for the relations between these various topologies and [69, Chpt. 14] for more on this topology. Also see [80] and [69, Chpt. 14] for the structure presheaf of rings of definable scalars over this space, generalising the structure sheaf of an affine variety.

2) **Full support topology**: This is defined, in [12], like the Ziegler topology but using pp-types in place of pp formulas, so it is finer than the Ziegler topology. It turns out that the closed sets in this topology on pinj_R are in natural bijection with the subcategories of Mod-R which are type-definable and closed under pure-injective hulls, equivalently which are closed under pure-injective hulls, products and pure submodules. See [69, §§5.3.7, 12.7].

[28]Every finitely generated subfunctor of a finitely presented functor is finitely presented because the category $(\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$ is locally coherent

27 Decidability of theories of modules

There is the question of whether there is a decision procedure, absolute or relative to some oracle, for the theory of modules. That is, is there a Turing machine (with oracle) which, given a sentence in the language $\mathcal{L}_R$ of R-modules, will determine whether or not it holds true in every R-module?

Undecidability of the theory of modules is typically much easier to establish than decidability. For (many) examples, see for instance [63, Chpt. 17] as well as more recent papers such as [77], [101].

To make sense of the question one should clarify what is assumed of the ring, since a significant part of the theory of the ring is encoded in the theory of its modules. See [34] for a discussion of this.

Given that, the decidability question reduces largely to questions about the topology of the Ziegler spectrum of the ring (though the first papers along these lines, [99], also [10], predate that topology): namely, whether a given basic open set is contained in the union of finitely many others. See [92] and particularly [33] for this. In particular, the algebraic project of describing the Ziegler spectrum (its points and, in particular its topology) feeds directly into the decidability question. Typically, for rings where the Ziegler spectrum has been described, one can deduce decidability of the theory of modules, though perhaps only with considerable effort, depending on how explicit is the description of the spectrum.

There has been a lot of work - [34] is a recent example - on decidability over valuation rings and rings related by localisation to these, since there one has a detailed knowledge of the relations between arithmetic in the ring, implications between pp-pairs and inclusions between the open subsets of the Ziegler spectrum that they define.

At least in the case where the ring is a (finite-dimensional) algebra R over a field, the main conjecture is that the theory of R-modules is decidable iff R has tame representation type[29]. The direction wild (=not tame) representation type implies undecidable theory of modules seems close to being established [35, §4]. A general proof that tame implies decidable is not yet in sight, rather it has been proved for particular classes of algebras, the strongest evidence to date being [33] which proves decidability for a class of tame but non-domestic algebras.

28 Some other topics

Here we just give some pointers and references for a few further topics.

28.1 The canonical language for R-modules

The language that we used for R-modules is not canonical. Recall that rings R and S are said to be Morita equivalent if their categories of modules, Mod-R and Mod-S are equivalent; for instance R and the ring of 2×2 matrices over R are Morita equivalent. So, if R and S are Morita equivalent, we could as well use a language based on S as one based on R in setting up a language for

[29] We don't define tame type here but, very roughly, it says that for every integer $d \geq 1$, the d-dimensional indecomposable R-modules sit in just finitely many 1-parameter (from the field) families.

their modules. We can enrich either of these languages by adding, as in Section 22, a sort Hom$(A, -)$ for every isomorphism class of finitely presented module A. The resulting language is a true invariant of the category of models (since "finitely presented module" can be defined purely in terms of the structure of that category). Or we could add all the pp-sorts and pp-definable maps between them, as in Section 23, and use the category of pp-pairs as the basis for the 'richest', so in that sense canonical, language for our category of modules. These choices of language, and the relations between them, are discussed in [70], also see [51].

28.2 Pure-injectives are injective

Not literally, but there is a full embedding of the category Mod-R of R-modules into the category (R-mod, **Ab**) of all additive functors on finitely presented left modules. It is given by sending a right module M to the functor $M \otimes_R -$ which takes a finitely presented left R-module L to the abelian group $M \otimes_R L$. Then M is pure-injective iff the functor $M \otimes_R -$ is an injective object in the functor category, and every injective in the functor category arises (up to isomorphism) in this way. See [44, Appx. B, Chpt. 7], [69, Chpt. 12].

This embedding was used very effectively by Gruson and Jensen (see, e.g., [37]) and subsequently by many others. It shows, for instance, that the early results [22] of Eklof and Sabbagh on absolutely pure and injective modules actually apply to arbitrary modules and pure-injective modules, with just a little translation.

28.3 Pure-projective modules

These are the modules which are defined dually to pure-injective modules - they are the modules which are projective over pure epimorphisms, equivalently they are the direct summands of direct sums of finitely presented modules. They have been less studied than the pure-injective modules, though see for instance [91]. Puninski developed, and made effective use of, their model theory in settling some questions about direct-sum decompositions of modules [88], [89].

28.4 Vaught's Conjecture for modules

Despite a considerable amount of attention, see e.g. [84], [85], this has not been proved over general rings.

28.5 Stability theory

There is a quite detailed working-through and interpretation of concepts from stability theory in Chapters 5-7 of [63] and in [52] (see also [61]).

28.6 Grothendieck rings

Here we mean Grothendieck rings in the sense of [47]. I conjectured that this ring should be nontrivial for any theory of nonzero modules; some initial results were obtain in [60]. The conjecture was proved by Kuber [49].

28.7 Model theory in triangulated categories

One may set up model theory in compactly generated triangulated categories very much as in finitely accessible categories (indeed, it can essentially be transported to that case). The triangulated structure does make a difference, for instance every pp formula is equivalent to a quantifier-free formula, hence also (by elementary duality) to a divisibility formula, [28]. Ziegler spectra of some triangulated categories have been computed [28], [3].

An approach *via* derivateurs has been initiated by Laking [55].

28.8 Abstract elementary classes of modules

There is some closeness between abstract elementary classes of modules and current investigations around deconstructibility in classes of modules (essentially considering how classes of modules can be built up as possibly transfinite extensions). See for instance [5], [98], [31, §10.3], as well as more model-theoretically-inspired work e.g. [53].

28.9 Infinitary languages

There was a little early work using infinitary languages, see [9], [97] for instance, and the languages $\mathcal{L}_{\alpha\omega}$ do fit well with not-necessarily-finitely accessible categories, see [1, Chpt. 5], and well-generated triangulated categories (for which see [59]) but, at least in the absence of a clear goal, there has not yet been very much development here yet.

28.10 Nonadditive additive model theory

By this I mean the very extensive parallel between categorical (in the topos theory or accessible categories sense) model theory and additive classical model theory. There has been a certain amount of transfer, e.g. [6], [50], [54], which is increasing, especially through the regular logic aspect.

28.11 Modules with additional structure

For example vector spaces with bilinear forms [32], and module (and more general representation) categories with a tensor product [7], [8].

28.12 Finally

In looking over this, I see many topics, in particular, regarding the model theory of modules over particular kinds of ring, which have been barely, or not at all, touched on here. The contents and bibliography of [69] must give some idea of why this is so - and much more has been done since that book was published. Beyond this, there are a number of contexts where modules and their model theory play a role but do not form the focus of attention; see [11], [15], [23], [62], [102] for a few quite recent examples.

This article has been mostly a summary of ideas and themes and then pointers to the wide and detailed understanding and the vast array of results that have been achieved in this area.

References

[1] J. Adámek and J. Rosický, Locally Presentable and Accessible Categories, London Math. Soc. Lecture Note Ser., Vol. 189, Cambridge University Press, 1994.

[2] M. Adelman, Abelian categories over additive ones, J. Pure Appl. Algebra, 3(2) (1973), 103-117.

[3] K. Arnesen, R. Laking, D. Pauksztello and M. Prest, The Ziegler spectrum for derived-discrete algebras, Adv. in Math., 319 (2017), 653-698.

[4] M. Auslander, Isolated singularities and existence of almost split sequences, (notes by Louise Unger), pp. 194-242 *in* Representation Theory II, Groups and Orders, Ottawa 1984, Lecture Notes in Mathematics, Vol. 1178, Springer-Verlag, 1986.

[5] J. T. Baldwin, P. C. Eklof and J. Trlifaj, ${}^{\perp}N$ as an abstract elementary class, Ann. Pure Appl. Logic, 149 (2007), 25-39.

[6] L. Barbieri-Viale, O. Caramello and L. Lafforgue, Syntactic categories for Nori motives, Selecta Math., 24(4), 3619-3648.

[7] L. Barbieri-Viale, A. Huber and M. Prest, Tensor structure for Nori motives, Pacific J. Math., *to appear.*

[8] L. Barbieri-Viale and M. Prest, Tensor product of motives via Künneth formula, J. Pure Applied Algebra, 224(6) (2020), DOI:10.1016/j.jpaa.2019.106267.

[9] J. Barwise and P. Eklof, Infinitary properties of abelian groups, Ann. Math. Logic, 2(1) (1970), 25-68.

[10] W. Baur, On the elementary theory of quadruples of vector spaces, Ann. Math. Logic, 19(3) (1980), 243-262.

[11] L. Bélair and F. Point, Separably closed fields and contractive Ore modules, J. Symbolic Logic, 80(4) (2015), 1315-1338.

[12] K. Burke, Some Model-Theoretic Properties of Functor Categories for Modules, Doctoral Thesis, University of Manchester, 1994, *available at* `https://personalpages.manchester.ac.uk/staff/mike.prest/publications.html`

[13] K. Burke and M. Prest, Rings of definable scalars and biendomorphism rings, pp. 188-201 *in* Model Theory of Groups and Automorphism Groups, London Math. Soc. Lecture Note Ser., Vol. 244, Cambridge University Press, 1997.

[14] C. Butz, Regular categories & regular logic, BRICS Lecture Series LS-98-2, 1998.

[15] A. Chernikov and M. Hils, Valued difference fields and NTP_2, Israel J. Math., 204 (2014), 299-327.

[16] P. M. Cohn, On the free product of associative rings, Math. Z., 71(1) (1959), 380-398.

[17] W. Crawley-Boevey, Modules of finite length over their endomorphism rings, pp. 127-184 *in* Representations of Algebras and Related Topics, London Math. Soc. Lecture Note Ser., Vol. 168, Cambridge University Press, 1992.

[18] W. Crawley-Boevey, Infinite-dimensional modules in the representation theory of finite-dimensional algebras, pp. 29-54 *in* I. Reiten, S. Smalø and Ø. Solberg (Eds.), Algebras and Modules I, Canadian Math. Soc. Conf. Proc., Vol. 23, Amer. Math. Soc, 1998.

[19] S. Crivei, M. Prest and G. Reynders, Model theory of comodules, J. Symbolic Logic, 69(1) (2004), 137-142.

[20] P. C. Eklof and E. Fisher, The elementary theory of abelian groups, Ann. Math. Logic, 4(2) (1972), 115-171.

[21] P. C. Eklof and I. Herzog, Model theory of modules over a serial ring, Ann. Pure Appl. Logic, 72(2) (1995), 145-176.

[22] P. Eklof and G. Sabbagh, Model-completions and modules, Ann. Math. Logic, 2(3) (1971), 251-295.

[23] G. Elek, Infinite dimensional representations of finite dimensional algebras and amenability, Math. Ann., 369 (2017), 307-439.

[24] E. Fisher, Abelian Structures, Yale University, 1974/5, partly published as Abelian Structures I, pp. 270-322 *in* Abelian Group Theory, Lecture Notes in Mathematics, Vol. 616, Springer-Verlag, 1977.

[25] P. Freyd, Representations in abelian categories, pp. 95-120 *in* Proceedings Conf. Categorical Algebra, La Jolla, 1965, Springer-Verlag, 1966.

[26] S. Garavaglia, Dimension and rank in the model theory of modules, *preprint*, University of Michigan, 1979, *revised* 1980.

[27] G. Garkusha and M. Prest, Injective objects in triangulated categories, J. Algebra Appl., 3(4) (2004), 367-389.

[28] G. Garkusha and M. Prest, Triangulated categories and the Ziegler spectrum, Algebras and Representation Theory, 8(4) (2005), 499-523.

[29] G. Garkusha and M. Prest, Classifying Serre subcategories of finitely presented modules, Proc. Amer. Math. Soc., 136(3) (2008), 761-770.

[30] W. Geigle, The Krull-Gabriel dimension of the representation theory of a tame hereditary artin algebra and applications to the structure of exact sequences, Manus. Math., 54(1-2) (1985), 83-106.

[31] R. Göbel and J. Trlifaj, Approximations and Endomorphism Algebras of Modules, *2nd revised and extended edition*, De Gruyter, 2012.

[32] N. Granger, Stability, Simplicity and the Model Theory of Bilinear Forms, Doctoral Thesis, University of Manchester, 1999, *available at* `https://personalpages.manchester.ac.uk/staff/mike.prest/publications.html`

[33] L. Gregory, Decidability of theories of modules over tubular algebras, arXiv:1603.03284.

[34] L. Gregory, S. L'Innocente and C. Toffalori, Decidability of the theory of modules over Prüfer domains with dense value groups, Ann. Pure Appl. Logic, 170(12) (2019), DOI:10.1016/j.apal.2019.102719.

[35] L. Gregory and M. Prest, Representation embeddings, interpretation functors and controlled wild algebras, J. London Math. Soc., 94(3) (2016), 747-766.

[36] L. Gregory and G. Puninski, Ziegler spectra of serial rings, Israel J. Math., 229(1) (2019), 415–459.

[37] L. Gruson and C. U. Jensen, Modules algébriquement compact et foncteurs $\varprojlim^{(i)}$, C. R. Acad. Sci. Paris, 276 (1973), 1651-1653.

[38] L. Gruson and C. U. Jensen, Dimensions cohomologiques reliées aux foncteurs $\varprojlim^{(i)}$, pp. 243-294 *in* Lecture Notes in Mathematics, Vol. 867, Springer-Verlag, 1981.

[39] R. Harland and M. Prest, Modules with irrational slope over tubular algebras, Proc. London Math. Soc., 110(3) (2015), 695-720.

[40] I. Herzog, Elementary duality of modules, Trans. Amer. Math. Soc., 340 (1993), 37-69.

[41] I. Herzog, The pseudo-finite dimensional representations of $sl(2, k)$, Selecta Mathematica, 7(2) (2001), 241-290.

[42] M. Hochster, Prime ideal structure in commutative rings, Trans. Amer. Math. Soc., 142 (1969), 43-60.

[43] W. Hodges, Model Theory, Encyclopedia of Mathematics, Vol. 42, Cambridge University Press, 1993.

[44] C. U. Jensen and H. Lenzing, Model Theoretic Algebra; with particular emphasis on Fields, Rings and Modules, Gordon and Breach, 1989.

[45] I. Kaplansky, Infinite Abelian Groups, Univ. of Michigan Press, Ann Arbor, 1954. *Also revised edition*, Ann Arbor, 1969.

[46] R. Kielpinski, On Γ-pure injective modules, Bull. Polon. Acad. Sci. Math., 15 (1967), 127-131.

[47] J. Krajicek and T. Scanlon, Combinatorics with definable sets: Euler characteristics and Grothendieck rings, Bull. Symbolic Logic, 6(3)(2000), 311-330.

[48] H. Krause, Exactly definable categories, J. Algebra, 201 (1998), 456-492.

[49] A. Kuber, Grothendieck rings of theories of modules, Ann. Pure Applied Logic, 166(3) (2015), 369-407.

[50] A. Kuber and J. Rosický, Definable categories. J. Pure Appl. Algebra, 222(5) (2018), 1006-1025.

[51] T. G. Kucera and M. Prest, Imaginary modules, J. Symbolic Logic, 57(2) (1992), 698-723.

[52] T. G. Kucera and M. Prest, Four concepts from "geometrical" stability theory in modules, J. Symbolic Logic, 57(2) (1992), 724-740.

[53] T. G. Kucera and M. Mazari-Armida, On universal modules with pure embeddings, arXiv:1903.00414.

[54] S. Lack and G. Tendas, Enriched regular theories, J. Pure Applied Algebra, 224(6) (2020), DOI:10.1016/j.jpaa.2019.106268.

[55] R. Laking, Purity in compactly generated derivators and t-structures with Grothendieck hearts, Math. Zeit., (2019), DOI:10.1007/s00209-019-02411-9.

[56] R. Laking, M. Prest and G. Puninksi, Krull-Gabriel dimension of domestic string algebras, Trans. Amer. Math. Soc., 370 (2018), 4813-4840.

[57] M. Makkai, A theorem on Barr-exact categories with an infinitary generalization, Ann. Pure Appl. Logic, 47 (1990), 225-268.

[58] B. Mitchell, Rings with several objects, Adv. in Math., 8 (1972), 1-161.

[59] A. Neeman, Triangulated Categories, Annals of Mathematics Studies, Vol. 148, Princeton University Press, 2001.

[60] S. Perera, Grothendieck Rings of Theories of Modules, Doctoral Thesis, University of Manchester, 2011, *available at* `https://personalpages.manchester.ac.uk/staff/mike.prest/publications.html`.

[61] A. Pillay and M. Prest, Forking and pushouts for modules, Proc. London Math. Soc., 46(2) (1983), 365-384.

[62] S. Posur, Methods of constructive category theory, arXiv:1908.04132.

[63] M. Prest, Model Theory and Modules, London Math. Soc. Lecture Note Ser., Vol. 130, Cambridge University Press, 1988.

[64] M. Prest, Wild representation type and undecidability, Comm. Algebra, 19(3) (1991), 919-929.

[65] M. Prest, Remarks on elementary duality, Ann. Pure Applied Logic, 62 (1993), 183-205.

[66] M. Prest, Interpreting modules in modules, Ann. Pure Applied Logic, 88(2-3) (1997), 193-215.

[67] M. Prest, Ziegler spectra of tame hereditary algebras, J. Algebra, 207(1) (1998), 146-164.

[68] M. Prest, Model theory and modules, pp. 227-253 *in* M. Hazewinkel (ed.), Handbook of Algebra, Vol. 3, Elsevier, 2003.

[69] M. Prest, Purity, Spectra and Localisation, Encyclopedia of Mathematics and its Applications, Vol. 121, Cambridge University Press, 2009.

[70] M. Prest: Model theory in additive categories, *in* Models, logics, and higher-dimensional categories, 231-244, CRM Proc. Lecture Notes, Vol. 53, Amer. Math. Soc., 2011.

[71] M. Prest, Definable additive categories: purity and model theory, Mem. Amer. Math. Soc., Vol. 210/No. 987, 2011.

[72] M. Prest, Abelian categories and definable additive categories, arXiv:1202.0426.

[73] M. Prest, Categories of imaginaries for definable additive categories, arXiv:1202.0427.

[74] M. Prest, Spectra of small abelian categories, arXiv:1202.0431.

[75] M. Prest, Modules as exact functors, pp. 37-65 *in* Surveys in Representation Theory of Algebras, Contemporary Mathematics, Vol. 716, Amer. Math. Soc., 2018.

[76] M. Prest, Multisorted modules and their model theory, pp. 115-151 *in* Model Theory of Modules, Algebras and Categories, Contemporary Mathematics, Vol. 730, Amer. Math. Soc., 2019.

[77] M. Prest and G. Puninski, Some model theory over hereditary noetherian domains, J. Algebra, 211(1) (1999), 268-297.

[78] G. Puninski and M. Prest, Ringel's conjecture for domestic string algebras, Math. Zeit., 282(1) (2016), 61-77.

[79] M. Prest, V. Puninskaya and A. Ralph, Some model theory of sheaves of modules, J. Symbolic Logic, 69(4) (2004), 1187-1199.

[80] M. Prest and R. Rajani, Structure sheaves of definable additive categories, J. Pure Applied Algebra, 214 (2010), 1370-1383.

[81] M. Prest, Ph. Rothmaler and M. Ziegler, Absolutely pure and flat modules and "indiscrete" rings, J. Algebra, 174(2) (1995), 349-372.

[82] M. Prest and A. Slavik, Purity in categories of sheaves, arXiv:1809.08981.

[83] H. Prüfer, Untersuchungen über die Zerlegbarkeit der abzählbaren primären abelschen Gruppen, Math. Z., 17(1) (1923), 35-61.

[84] V. Puninskaya, Vaught's Conjecture, J. Math. Sci., 109 (2002), 1649-1668 (*transl. from Russian original*).

[85] V. Puninskaya and C. Toffalori, Vaught's conjecture and group rings, Comm. Algebra, 33(11) (2005), 4267-4281.

[86] G. Puninski, Finite length and pure-injective modules over a ring of differential operators, J. Algebra, 231(2) (2000), 546-560.

[87] G. Puninski, Serial rings, Kluwer, Dordrecht, 2001.

[88] G. Puninski, Some model theory over an exceptional uniserial ring and decompositions of serial modules, J. London Math. Soc. (2), 64(2) (2001), 311-326.

[89] G. Puninski, Some model theory over a nearly simple uniserial domain and decompositions of serial modules, J. Pure Applied Algebra, 163(3) (2001), 319-337.

[90] G. Puninski, M. Prest and Ph. Rothmaler, Rings described by various purities, Comm. Algebra, 27(5) (1999), 2127-2162.

[91] G. Puninski and Ph. Rothmaler, Pure-projective modules, J. London Math. Soc., 71(2) (2005), 304-320.

[92] G. Puninski and C. Toffalori, Towards the decidability of the theory of modules over finite commutative rings, Ann. Pure Appl. Logic, 159(1-2) (2009), 49-70.

[93] C. M. Ringel, The Ziegler spectrum of a tame hereditary algebra, Colloq. Math., 76 (1998), 105-115.

[94] Ph. Rothmaler, Introduction to Model Theory, *transl./revised from German original* (1995), Algebra, Logic and Appl., Vol. 15., Gordon and Breach, 2000.

[95] G. Sabbagh, Aspects logiques de la pureté dans les modules, C. R. Acad. Sci. Paris, 271 (1970), 909-912.

[96] G. Sabbagh, Catégoricité et stabilité: quelques exemples parmi les groupes et anneaux, C. R. Acad. Sci. Paris, 280 (1975), 531-533.

[97] G. Sabbagh and P. Eklof, Definability problems for rings and modules, J. Symbolic Logic, 36(4) (1971), 623-649.

[98] J. Šaroch and J. Trlifaj, Kaplansky classes, finite character and $\aleph_1$-projectivity, Forum Math., 24(5) (2012), 1091-1109.

[99] W. Szmielew, Elementary properties of abelian groups, Fund. Math., 41 (1954), 203-271.

[100] R. W. Thomason, The classification of triangulated subcategories, Compos. Math., 105(1) (1997), 1-27.

[101] C. Toffalori, Wildness implies undecidability for lattices over group rings, J. Symbolic Logic, 62(4) (1997), 1429-1447.

[102] S. Walsh, Definability aspects of the Denjoy integral, Fund. Math., 237 (2017), 1-29.

[103] M. Ziegler, Model theory of modules, Ann. Pure Appl. Logic, 26(2) (1984), 149-213.

[104] W. Zimmermann, Rein injektive direkte Summen von Moduln, Comm. Algebra, 5(10) (1977), 1083-1117.