Categories of imaginaries for additive structures

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Some notation

R is a ring with 1

Mod-R is the category of right R-modules

mod-R is the category of finitely presented right R-modules

1: Context: Model theory of modules

We use the model theory of modules (always meaning over a fixed ring R).
Language for (right) R-modules: $+$, 0, $(- \times r)_{r \in R}$.

Theorem

(pp-elimination of quantifiers; Baur, Monk,...) Let R be any ring.
(1) If σ is a sentence in the language of R-modules then there is a finite boolean combination, τ, of sentences of the form $\mathrm{card}(\phi/\psi) \geq m$, where ϕ, ψ are pp formulas and m is a positive integer, such that σ is equivalent to τ in the sense that for every R-module $M \models \sigma$ iff $M \models \tau$.
(2) If χ is any formula in the language of R-modules then there is a sentence, τ, and a finite boolean combination, η, of pp formulas such that for every module M and tuple $\bar{a}$ from M (matching the free variables of χ) we have $M \models \chi(\bar{a})$ iff both $M \models \tau$ and $M \models \eta(\bar{a})$. In particular, the solution set to χ in every module M is a finite boolean combination of pp-definable subgroups. If non-zero constants are allowed in χ then the solution set will be a finite boolean combination of cosets of pp-definable subgroups.

Positive primitive (pp) formulas are those equivalent to an existentially quantified system of linear equations/conjunction of atomic formulas.

The (right) **Ziegler spectrum** of R, Zg_R, is a topological space, the points of which are the isomorphism types of direct-sum indecomposable pure-injective (=algebraically compact = positively saturated) modules. A basis of open sets consists of the $(\phi/\psi) = \{N \in \mathrm{Zg}_R : \phi(N)/\psi(N) \neq 0\}$ as $\psi \leq \phi$ range over pp-pairs.

The closed subsets of Zg_R are in natural bijection with the **definable subclasses** of $\mathrm{Mod}\text{-}R$ meaning those which are elementary and closed under finite (hence arbitrary) direct sums as well as direct summands.

To each module M we associate its **support**,

$\mathrm{supp}(M) = \{N \in \mathrm{Zg}_R : N \text{ is a direct summand of some } M' \equiv M\}$.

This is a (typical) closed subset of Zg_R and the modules with support contained in $\mathrm{supp}(M)$ form the definable subcategory generated by M.

Some examples of definable subcategories:

• of $\mathbf{Ab} = \mathrm{Mod}\text{-}\mathbb{Z}$: torsionfree abelian groups, divisible abelian groups; abelian groups without p-torsion; modules over the localisation $\mathbb{Z}_{(p)}$; modules over the localisation $\mathbb{Z}[1/2]$.

• of other module categories $\mathrm{Mod}\text{-}R$: very much depending on the (kind of) ring R, the extent to which there is a classification depending on the extent to which the Ziegler spectrum has been described.

Two extremes: discrete topology on Zg_R which, because Zg_R is always (quasi)compact, must be finite - this includes, and perhaps equals, the case that R is a ring of finite representation type; indiscrete topology - eg $R = D$ a division ring but there are more interesting examples - $(\mathrm{End}(D^{(\aleph_0)}))$ (Tyukavkin) - and more exotic examples (Prest, Rothmaler and Ziegler).

2: Imaginaries: $\mathbb{L}_R^{\mathrm{eq}+}$

We build the category, $\mathbb{L}_R^{\mathrm{eq}+}$, of pp-imaginaries. We use this, rather than the category $\mathbb{L}_R^{\mathrm{eq}}$ of all imaginaries because, in this additive context, it is natural to require that all sorts inherit the additive structure and that maps between them should preserve that structure. This is equivalent to insisting that all definitions be by pp formulas.

To $\mathrm{Mod}\text{-}R$ we associate its **category** $\mathbb{L}_R^{\mathrm{eq}+}$ **of pp-imaginaries** (it is Morita-invariant, hence associated to the category of modules rather than to the ring so a better notation would be $\mathbb{L}^{\mathrm{eq}+}(\mathrm{Mod}\text{-}R)$):

- the objects are the pp-pairs ϕ/ψ;

- the morphisms from ϕ/ψ to ϕ'/ψ' are the pp-definable maps - the equivalence classes of pp formulas $\rho(\overline{x}, \overline{y})$ such that in $\mathrm{Mod}\text{-}R$, $\forall \overline{x} \left(\phi(\overline{x}) \to \exists \overline{y}\, \phi'(\overline{y}) \wedge \rho(\overline{x}, \overline{y})\right)$ and $\forall \overline{x}\, \overline{y} \left((\psi(\overline{x} \wedge \rho(\overline{x}, \overline{y})) \to \psi'(\overline{y})\right)$.

Let $\mathcal{L}(\mathcal{D})^{\mathrm{eq}+}$ denote the corresponding language. Each R-module M has a canonical extension to an $\mathcal{L}_R^{\mathrm{eq}+}$-structure $M^{\mathrm{eq}+}$.

Theorem

The category $\mathbb{L}_R^{\mathrm{eq}+}$ is abelian.

For example, the direct sum of two sorts $\phi(\overline{x})/\psi(\overline{x})$ and $\phi'(\overline{y})/\psi'(\overline{y})$ is $(\phi(\overline{x}) \wedge \phi'(\overline{y}))/(\psi(\overline{x}) \wedge \psi'(\overline{y}))$. Also, for instance, both the kernel and cokernel of a pp-definable map are pp-definable.

If $\theta \leq \psi \leq \phi$ are pp then we have a corresponding exact sequence $0 \to \psi/\theta \to \phi/\theta \to \phi/\psi \to 0$ - up to isomorphism, a typical exact sequence in $\mathbb{L}_R^{\mathrm{eq}+}$.

In the sense that $^{\mathrm{eq}}$ of the theory of an algebraically closed field contains a lot of algebraic geometry (in particular, every affine variety is a sort), $\mathbb{L}_R^{\mathrm{eq}+}$ contains a lot of homological algebra for R-modules.

For instance, if $A \in \mathrm{mod}\text{-}R$ then, for every module M the group $\mathrm{Hom}_R(A, -)$ is a sort of $M^{\mathrm{eq}+}$. Similarly, if A is FP_2 then $\mathrm{Ext}^1_R(A, M)$ lies in $M^{\mathrm{eq}+}$ for every module M. The general case, and its dual, are as follows, where we say that M is $\mathbf{FP}_n$ if M has a projective resolution the first $n+1$ terms of which are finitely generated (so FP_0= finitely generated, FP_1= finitely presented). The case $n = 0$ is due to Auslander, the general case to me.

Theorem

Let $A \in \mathrm{Mod}\text{-}R$, $L \in R\text{-}\mathrm{Mod}$.
The sort $\mathrm{Ext}^n(A, -)$ *is in* $\mathbb{L}_R^{\mathrm{eq}+}$ *if* A *is* FP_{n+1}.
The sort $\mathrm{Tor}_n(L, -)$ *is in* $\mathbb{L}_R^{\mathrm{eq}+}$ *if* L *is* FP_{n+1}.

3: Evaluation at modules

Given an R-module M, evaluation-at-M is an additive functor from $\mathbb{L}_R^{\mathrm{eq}+}$ to $\mathbf{Ab}$: $\mathrm{ev}_M : \mathbb{L}_R^{\mathrm{eq}+} \to \mathbf{Ab}$. In fact, this functor is **exact**, meaning that it takes exact sequences to exact sequences.

That is just the observation that, given the exact sequence of sorts $0 \to \psi/\theta \to \phi/\theta \to \phi/\psi \to 0$, the sequence of abelian groups $0 \to \psi(M)/\theta(M) \to \phi(M)/\theta(M) \to \phi(M)/\psi(M) \to 0$ is exact.

In fact, the R-modules are precisely the exact functors on $\mathbb{L}_R^{\mathrm{eq}+}$.

Theorem

(Herzog) $\mathrm{Mod}\text{-}R \simeq \mathrm{Ex}(\mathbb{L}_R^{\mathrm{eq}+}, \mathbf{Ab})$

If $\mathcal{A}$, $\mathcal{B}$ are preadditive categories (and $\mathcal{A}$ is skeletally small) then we write $(\mathcal{A}, \mathcal{B})$ for the category of additive functors from $\mathcal{A}$ to $\mathcal{B}$: the objects are the functors, the morphisms are the natural transformations. If $\mathcal{A}$ and $\mathcal{B}$ are abelian then it makes sense to consider the full subcategory, $\mathrm{Ex}(\mathcal{A}, \mathcal{B})$, on those functors which are exact.

Languages for R-modules

A ring is nothing other than a preadditive category with one object; then a left, respectively right, module is just a covariant, resp. contavariant, functor from the ring, so conceived, to $\mathbf{Ab}$ (and module homomorphisms are just the natural transformations). That is:

- $(R, \mathbf{Ab}) \simeq \mathrm{Mod}\text{-}R$.

This reflects the usual 1-sorted language for R-modules.

A richer language for R-modules is obtained *via* the restricted Yoneda embedding $\mathrm{Mod}\text{-}R \to ((\mathrm{mod}\text{-}R)^{\mathrm{op}}, \mathbf{Ab})$ which takes M to $\mathrm{Hom}(-, M) \restriction \mathrm{mod}\text{-}R$. This has a sort $\mathrm{Hom}(A, -)$ for each finitely presented module A and a function symbol for each R-linear map between finitely presented modules. This corresponds to the equivalence (due to Roos)

- $\mathrm{Lex}((\mathrm{mod}\text{-}R)^{\mathrm{op}}, \mathbf{Ab}) \simeq \mathrm{Mod}\text{-}R$.

(Actually that's a lie, made for purposes of neat presentation, unless R is right coherent.)

The richest, and canonical, language for the category of right R-modules is that based on $\mathbb{L}_R^{\mathrm{eq}+}$ and expressed in Herzog's result:

- $\mathrm{Mod}\text{-}R \simeq \mathrm{Ex}(\mathbb{L}_R^{\mathrm{eq}+}, \mathbf{Ab})$.

4: Definable categories

The full subcategory on a definable subclass of $\mathrm{Mod}\text{-}R$ is a **definable subcategory** of $\mathrm{Mod}\text{-}R$ and can alternatively be characterised as a full subcategory of $\mathrm{Mod}\text{-}R$ which is closed under direct products, direct limits and pure submodules (and, as always intended but seldom said, isomorphism). ($A \leq B$ is **pure in** B if for every pp formula $\phi(\overline{x})$, $\phi(A) = A^{l(\overline{x})} \cap \phi(B)$.)

If $\mathcal{D}$ is a definable subcategory of $\mathrm{Mod}\text{-}\mathcal{R}$ then we can define $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$ using the language of $\mathcal{R}$-modules: the objects of $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$ are the same as those of $\mathbb{L}_R^{\mathrm{eq}+}$ but the morphisms in $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$ from ϕ/ψ to ϕ'/ψ' are the pp-definable relations $\rho(\overline{x}, \overline{y})$ which, on every member of $\mathcal{D}$, define a function from the first to the second sort (thus there are more morphisms in $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$ than in $\mathbb{L}_R^{\mathrm{eq}+}$).

If $\mathcal{D}$ is a definable subcategory of $\mathrm{Mod}\text{-}\mathcal{R}$ then set $\mathcal{S}_{\mathcal{D}} = \{\phi/\psi : \phi(D) = \psi(D)\ \forall D \in \mathcal{D}\}$ to be the full subcategory of $\mathbb{L}_{\mathcal{R}}^{\mathrm{eq}+}$ on those pp-pairs which are closed on $\mathcal{D}$.

Theorem

If $\mathcal{D}$ is a definable subcategory of $\mathrm{Mod}\text{-}\mathcal{R}$ then $\mathcal{S}_{\mathcal{D}}$ is a Serre subcategory of $\mathbb{L}_{\mathcal{R}}^{\mathrm{eq}+}$ and the quotient category $\mathbb{L}_{\mathcal{R}}^{\mathrm{eq}+}/\mathcal{S}_{\mathcal{D}}$ is, in a natural way, naturally equivalent to $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$.

A **Serre subcategory** of an abelian category $\mathcal{A}$ is a subcategory $\mathcal{S}$ closed under subobjects, quotient objects and extensions (the last meaning that if $A \le B$ and if both A and B/A are in $\mathcal{S}$ then so is B). The quotient category $\mathcal{A}/\mathcal{S}$ is characterised by its being abelian and there being an exact functor $\mathcal{A} \to \mathcal{A}/\mathcal{S}$ which sends every object of $\mathcal{S}$ to zero and which is the "minimal" way of doing this.

Just as we had for $\mathrm{Mod}\text{-}R$, we have the following.

Theorem

(Herzog, Krause) If $\mathcal{D}$ is a definable subcategory of $\mathrm{Mod}\text{-}\mathcal{R}$ and $D \in \mathcal{D}$ then evaluation at D is an exact functor from $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$ to $\mathbf{Ab}$. In fact $\mathcal{D} \simeq \mathrm{Ex}(\mathbb{L}^{\mathrm{eq}+}(\mathcal{D}), \mathbf{Ab})$.

A **definable category** is a category which is equivalent to a definable subcategory of some module category $\mathrm{Mod}\text{-}\mathcal{R}$ ($\simeq (\mathcal{R}^{\mathrm{op}}, \mathbf{Ab})$) over some skeletally small preadditive category $\mathcal{R}$. Equivalently it is, up to equivalence, one of the form $\mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ where $\mathcal{A}$ is a skeletally small abelian category.

More examples of definable categories:

• module categories $\mathrm{Mod}\text{-}R$ and, more generally, functor categories $\mathrm{Mod}\text{-}\mathcal{R} = (\mathcal{R}^{\mathrm{op}}, \mathbf{Ab})$;

• the category of chains/ chain complexes/ exact complexes of modules over a ring R;

• the category of C-comodules where C is a coalgebra over a field;

• the category of $\mathcal{O}_X$-modules where $\mathcal{O}_X$ is a sheaf of rings over a space with a basis of compact open sets, and categories of quasicoherent sheaves over nice enough schemes;

• categories of abelian structures in the sense of Fisher;

• locally finitely presented additive categories (for instance the category of torsion abelian groups), more generally finitely accessible additive categories with products;

• any definable subcategory of a definable category.

(A category $\mathcal{C}$ is **finitely accessible** if it has direct limits, if the subcategory $\mathcal{C}^{\mathrm{fp}}$ of finitely presented objects is skeletally small and if every object of $\mathcal{C}$ is a direct limit of finitely presented objects. Such a category is **locally finitely presented** if it is also complete and cocomplete.)

5: The model theory of a definable category

The title makes sense because it turns out that the model theory of a definable subcategory $\mathcal{D}$ *is intrinsic* to that category.

That begs the question of what we should mean by "the model theory of $\mathcal{D}$". But it is enough if we can define, from a definable category $\mathcal{D}$, its category, $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$, of imaginaries, since then we know that $\mathcal{D}$ can be recovered as $\mathrm{Ex}(\mathbb{L}^{\mathrm{eq}+}(\mathcal{D}), \mathbf{Ab})$, equivalently as a definable subcategory of $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})\text{-}\mathrm{Mod}$ and then all the model theory is there in the "evaluation" pairing between pp-sorts and structures.

A theorem of Krause shows that $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$ can be recovered from $\mathcal{D}$, though a bit indirectly, but there is a simpler way to obtain it.

Theorem

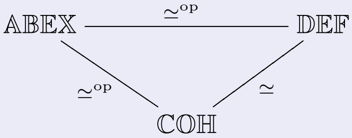
(Krause when $\mathcal{D}$ is finitely accessible, Prest in general) If $\mathcal{D}$ is a definable additive category then $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D}) \simeq (\mathcal{D}, \mathbf{Ab})^{\to\prod}$ - the category of functors from $\mathcal{D}$ to $\mathbf{Ab}$ which commute with direct limits and direct products.

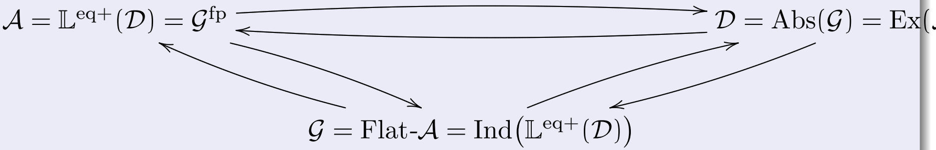
6: 3 2-categories

Let $\mathbb{ABEX}$ be the 2-category which has has for its objects the skeletally small abelian categories and for its morphisms the exact functors; $\mathbb{DEF}$ the category of definable additive categories and interpretation functors; $\mathbb{COH}$ the category of locally coherent Grothendieck categories and coherent morphisms. In each case the 2-arrows are just the natural transformations.

Theorem

(Prest and Rajani) There is a diagram of equivalences and anti-equivalences between $\mathbb{ABEX}$, $\mathbb{COH}$ *and* $\mathbb{DEF}$ *as follows.*





"Abs" and "Flat" refer to the full subcategories of absolutely pure and flat modules respectively; "Ind" denotes Ind-completion.

In particular every (skeletally) small abelian category is the category of imaginaries for some definable category of additive structures. We can ask, for instance, for which small abelian categories $\mathcal{A}$ is the corresponding theory [some algebraic or model-theoretic property]?

Morphisms

Let $\mathcal{C}$, $\mathcal{D}$ be definable categories; an interpretation of $\mathcal{D}$ in $\mathcal{C}$ is given by specifying:

• an axiomatisable subcategory $\mathcal{C}'$ of $\mathcal{C}$

• an interpretation $I : \mathcal{C}' \to \mathcal{D}$ in the usual sense, except that we insist on the additive structure being preserved, hence $\mathcal{C}'$ should be a definable subcategory and I should be an additive functor, and this forces everything to be given by pp formulas. In particular:

• to each sort of a chosen language $\mathcal{L}(\mathcal{D})$ for $\mathcal{D}$ there will correspond a pp pair in $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$

• to each basic function or relation symbol of $\mathcal{L}(\mathcal{D})$ there will correspond some pp formula of $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$ which, when applied to members of $\mathcal{C}'$, will define a corresponding relation or function

• *not* such that every object of $\mathcal{D}$ be thus obtained - easy examples show that's too much to ask - rather we ask that every object of $\mathcal{D}$ be pure in an object so obtained.

This is an **interpretation functor** from $\mathcal{C}'$ to $\mathcal{D}$.

Theorem

Let $I : \mathcal{C}' \to \mathcal{D}$ *be an additive functor between definable categories; then* I *is an interpretation functor iff* I *commutes with direct products and direct limits. There is a natural bijection (indeed, equivalence of categories) between interpretation functors from* $\mathcal{C}'$ *to* $\mathcal{D}$ *and exact functors from* $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$ *to* $\mathbb{L}(\mathcal{C}')^{\mathrm{eq}+}$*. (This is part of the equivalence between* $\mathbb{ABEX}$ *and* $\mathbb{DEF}$*.)*

So, for example, an interpretation functor from $\mathcal{D}$ to $\mathbf{Ab}$ is exactly a pp-sort.

We will say that the definable category $\mathcal{D}$ is **interpretable in** the definable category $\mathcal{C}$, writing $\mathcal{D} \prec \mathcal{C}$, if there is a definable subcategory $\mathcal{C}'$ of $\mathcal{C}$ and an interpretation functor I from $\mathcal{C}'$ to $\mathcal{D}$ such that the definable subcategory generated by $I\mathcal{C}$ is all of $\mathcal{D}$. If I preserves all induced structure then we will say that $\mathcal{D}$ is **strongly interpretable** in $\mathcal{C}$.

This gives us a preordering (indeed, two of them) on the collection of definable additive categories; what is its structure? Does it fit with notions of complexity like representation type which also are based on the existence of nice enough functors between categories?

7: Elimination of imaginaries

For module (or functor) categories $\mathrm{Mod}\text{-}\mathcal{R}$, both elimination of quantifiers and elimination of imaginaries can be formulated in terms of the pp-imaginaries category because each condition is equivalent to its pp version.

Elimination of quantifiers or of imaginaries is always with respect to some language $\mathcal{L}$. Moreover, for the latter we need also to specify a set, $\mathcal{H}$, of "home sorts" - that is, a set of objects of the category of (pp) sorts - and we may as well assume $\mathcal{H}$ to be closed under finite products. A theory T in $\mathcal{L}$ has **elimination of imaginaries to** $\mathcal{H}$ if every definable subset of every sort is in definable bijection with a definable subset of some sort in $\mathcal{H}$. In our context this is equivalent to the pp version ("pp-elimination of imaginaries").

Theorem

Suppose that $\mathcal{D}$ is a definable additive category and let $\mathcal{H}$ be an additive subcategory of $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$. Then (the theory of) $\mathcal{D}$ has elimination of imaginaries to $\mathcal{H}$ iff every object of $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$ is isomorphic to a subobject of some object of $\mathcal{H}$.

That, for module categories $\mathrm{Mod}\text{-}\mathcal{R}$, both elimination of quantifiers and elimination of imaginaries are equivalent to categorical properties of $\mathbb{L}_{\mathcal{R}}^{\mathrm{eq}+}$ is a consequence of the following.

Theorem

(Burke, where (1) is a reformulation of a result of Auslander) Let ϕ/ψ *be a pp-pair in* $\mathbb{L}_{\mathcal{R}}^{\mathrm{eq}+}$*. Then:*
(1) $\mathrm{pdim}(\phi/\psi) \leq 2$*;*
(2) $\mathrm{pdim}(\phi/\psi) \leq 1$ *iff* $\phi/\psi \simeq \phi'$ *(that is,* $\phi'/0$*) for some pp formula* ϕ'*);*
(3) $\mathrm{pdim}(\phi/\psi) = 0$ *iff* $\phi/\psi \simeq \theta$ *where* θ *is a system of linear equations.*

That is, with respect to the usual language for modules, the pp formulas are the objects of projective dimension ≤ 1 and the quantifier-free pp formulas are the projective objects.

Auslander proved that the global dimension - the sup of the projective dimensions of objects - of $\mathbb{L}_{\mathcal{R}}^{\mathrm{eq}+}$ (in one of its other incarnations) is either 2 or 0. This gives us the required algebraic criteria.

Theorem

For a ring (or, more generally, a skeletally small preadditive category) $\mathcal{R}$, the following are equivalent, where elimination of imaginaries refers to the usual home sorts:
(i) $\mathcal{R}$ is von Neumann regular;
(ii) the theory of $\mathcal{R}$-modules (right, equivalently left) has elimination of quantifiers;
(ii)$^+$ the theory of $\mathcal{R}$-modules (rt/l) has pp-elimination of quantifiers;
(iii) the theory of $\mathcal{R}$-modules (rt/l) has elimination of imaginaries.
(iii)$^+$ the theory of $\mathcal{R}$-modules (rt/l) has pp-elimination of imaginaries.

In the general case, with $\mathcal{D}$ in place of $\mathrm{Mod}\text{-}\mathcal{R}$, asking about either elimination make sense once a collection $\mathcal{H}$ of sorts has been chosen but, since $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$ may be any small abelian category, there would seem to be no collection of canonical "home" sorts in this generality.

Some fairly recent **references** giving accounts of parts of this:

M. Prest, Purity, Spectra and Localisation, Encyclopedia of Mathematics and its Applications, Vol. 121, Cambridge University Press, 2009.
M. Prest, Definable additive categories: purity and model theory, Mem. Amer. Math. Soc., Vol. 210/No. 987, 2011.
M. Prest and R. Rajani, Structure sheaves of definable additive categories, J. Pure Applied Algebra, 214 (2010), 1370-1383.
M. Prest, Model theory in additive categories, in "Models, Logics and Higher-dimensional Categories: a Tribute to the Work of Mihaly Makkai", CRM Proceedings and Lecture Notes, Vol. 53, Amer. Math. Soc., 2011.

The most comprehensive is/will be:
M. Prest, Abelian categories, definability and geometry [title likely to become a little less grandoise], to be submitted to arXiv around January 2012 (I intend/hope)