How to Define, Generalize, and Dualize the Notion of Torsion

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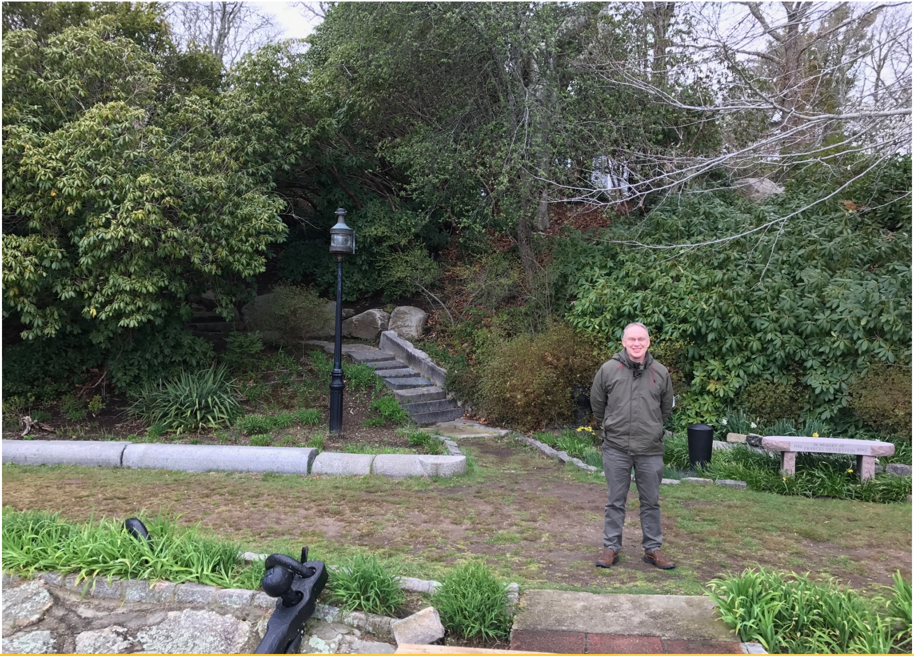
GOALS OF TODAY'S TALK

- For any module over any ring, define the torsion submodule, extending classical torsion over commutative domains.
- For any module over any ring, define the cotorsion quotient module.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PLAN OF THE TALK

Part 1. Statement of the problem

Part 2. Definition and properties of the new torsion

Part 3. A functorial interlude

Part 4. Definition and properties of cotorsion

Part 5. Duality and the exchange formula

REFERENCES

A. Martsinkovsky and J. Russell, *Injective stabilization of additive functors. I. Preliminaries*, arXiv:1701.00150, 2017

A. Martsinkovsky and J. Russell, *Injective stabilization of additive functors. II. (Co)torsion and the Auslander-Gruson-Jensen functor*, arXiv:1701.00151, 2017

Part 1. Statement of the problem

THE TERM "TORSION"

torsion | ˈtôrSHən |

noun

the action of twisting or the state of being twisted, especially of one end of an object relative to the other.

- *Mathematics* the extent to which a curve departs from being planar.
- *Zoology* (in a gastropod mollusk) the spontaneous twisting of the visceral hump through 180° during larval development.

DERIVATIVES

torsional | ˈtôrSH(ə)n(ə)l | adjective

torsionally | -SHənl-ē | adverb

torsionless adjective

ORIGIN

late Middle English (as a medical term denoting colic or in the sense 'twisting' (especially of a loop of the intestine)): via Old French from late Latin ***torsio(n-***), variant of ***tortio(n-***) **'twisting, torture'**, from Latin ***torquere*** **'to twist'**.

FIGURE: From New Oxford Dictionary

Torsion intérieure, POINCARÉ (1900)

CLASSICAL TORSION

- R a commutative domain
- A an R-module

DEFINITION

The torsion submodule of A is

$$T(A) := \{a \in A \mid \exists r \in R - \{0\},\ ra = 0\}$$

REMARK

- $T(A)$ is a submodule of A;
- $T(A)$ is defined by an annihilation condition

(PROVISIONAL) STATEMENT OF THE PROBLEM

Extend the definition of classical torsion to arbitrary rings and modules.

This can be done in more than one way, but we want to impose some restrictions.

A POSSIBLE CANDIDATE: 1-TORSION

Recall that the 1-torsion $\mathfrak{t}(A)$ of a module A is the kernel of the canonical evaluation map

$$e_A : A \longrightarrow A^{**} : a \mapsto (F_a : f \mapsto f(a))$$

Thus $\mathfrak{t}(A)$ is determined by the exact sequence

$$0 \longrightarrow \mathfrak{t}(A) \longrightarrow A \longrightarrow A^{**}$$

It is defined for any module over any ring and consists of those elements of A on which every linear form on A vanishes. Moreover:

LEMMA

If R is a commutative domain and A is finitely generated, then

$$T(A) = \mathfrak{t}(A)$$

1-Torsion is not a solution

Unfortunately, 1-torsion need not coincide with classical torsion for infinite modules:

Example

Let $R := \mathbb{Z}$ and $A := \mathbb{Q}$. Then

$$T(\mathbb{Q}) = \{0\} \quad \text{but} \quad \mathfrak{t}(\mathbb{Q}) = \mathbb{Q}$$

1-TORSION

However, 1-torsion is an important concept. It shows up in a variety of contexts:

- Stable module theory (Auslander - Bridger, 1969);
- PDE and constructive aspects of linear control systems (Oberst, et al. 1990, ...);
- Linkage of algebraic varieties (M - Strooker, 2004);
- Algebraic aspects of a question of Reiffen - Vetter (M, 2010).

Precise statement of the problem

All of the above applications deal with finitely generated modules over noetherian rings. This leads to a precise statement of the problem.

Problem Find a common generalization of:

- classical torsion for arbitrary modules over commutative domains,

  and

- 1-torsion for finitely presented modules over arbitrary rings,

applicable to arbitrary modules over arbitrary rings.

Part 2. Definition and properties of torsion

POSSIBLE APPROACHES

Q: Why not replace nonzero elements of the ring by its regular elements?

A: The resulting construct need not be a submodule.

POSSIBLE APPROACHES

Q: Why not use the axiomatic formalism of torsion theories?

A: The torsion class of any torsion theory is idempotent. On the other hand, we have

THEOREM (M, 2010)

Let R be a commutative artinian local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated Λ-module. Then $\mathfrak{t}(M) \subset \mathfrak{m}M$.

Now choose R and M as above with $\mathfrak{t}(M) \neq \{0\}$. Then

$$\mathfrak{t}(\mathfrak{t}(M)) \subseteq \mathfrak{m}\mathfrak{t}(M) \subsetneq \mathfrak{t}(M),$$

showing that $\mathfrak{t}$ is not idempotent.

Classical torsion via localization

Let K be the field of fractions of the domain R and

$$0 \longrightarrow R \longrightarrow K$$

the canonical embedding.

Tensoring a module A with this map, we have the localization map

$$\ell_A : A \cong A \otimes R \to A \otimes_R K$$

Lemma

$T(A) = \operatorname{Ker} \ell_A$.

Observation: In this definition K is the injective envelope of the ring.

The definition

Let Λ be a ring, A a **right** Λ-module, and

$$0 \longrightarrow \Lambda \longrightarrow I$$

the injective envelope of Λ viewed as a **left** module over itself.

Definition

The torsion of A is defined by the exact sequence

$$0 \longrightarrow \mathfrak{s}(A) \longrightarrow A \otimes \Lambda \longrightarrow A \otimes I$$

FIRST PROPERTIES OF $\mathfrak{s}$

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{s}(A) \arrow[r] & A \otimes \Lambda \arrow[r] \arrow[d, equal] & A \otimes I \\
 & & A &
\end{tikzcd}$$

- Because I is injective, the right multiplication on Λ extends to I, making $\mathfrak{s}(A)$ a submodule of A. It will be called the *torsion submodule* of A.

- $\mathfrak{s}$ is a subfunctor of the identity endofunctor on $\mathrm{Mod}\text{-}\Lambda$. In particular, $\mathfrak{s}$ preserves monomorphisms.

- If Λ is a commutative domain, then $\mathfrak{s}$ coincides with classical torsion.

A USEFUL TOOL

To proceed further, recall the natural transformation

$$\mu_A : A \otimes \_ \longrightarrow (A^*, \_)$$

which, on the module M, evaluates to

$$A \otimes M \longrightarrow (A^*, M) : a \otimes m \mapsto \big(F_{a,m} : \lambda \mapsto \lambda(a)m\big)$$

EXAMPLE

Evaluating μ_A on Λ, we recover the canonical evaluation map

$$A \longrightarrow A^{**}$$

FURTHER PROPERTIES OF $\mathfrak{s}$

Now evaluate μ_A on the injective envelope

$$\iota : \Lambda \rightarrowtail I$$

This yields a commutative square

$$\begin{tikzcd} A \otimes \Lambda \arrow[r, "\mu_A(\Lambda)"] \arrow[d, "1\otimes\iota"'] & (A^*, \Lambda) \arrow[d, tail, "(1,\iota)"] \\ A \otimes I \arrow[r, "\mu_A(I)"'] & (A^*, I) \end{tikzcd}$$

COROLLARY

$\mathfrak{s}$ is a subfunctor of 1-torsion: $\mathfrak{s} \subseteq \mathfrak{t}$.

FURTHER PROPERTIES OF $\mathfrak{s}$

Observation The natural transformation $A \otimes \_ \xrightarrow{\mu_A} (A^*, \_)$ is an isomorphism when A is a finitely generated projective.

This, together with the fact that $\mathrm{Hom}(\_, I)$ is an exact functor, imply that, when A is finitely presented, the map

$$\mu_A(I) : A \otimes I \longrightarrow (A^*, I)$$

is an isomorphism.

COROLLARY

$\mathfrak{s} = \mathfrak{t}$ *on finitely presented modules.*

FURTHER PROPERTIES OF $\mathfrak{s}$

- $\mathfrak{s}$ preserves filtered colimits and coproducts.

  This is true for the tensor product. The claim now follows from the defining copresentation

  $$0 \longrightarrow \mathfrak{s} \longrightarrow \_ \otimes \Lambda \longrightarrow \_ \otimes I$$

  Since any module is a filtered colimit of finitely presented modules,

  $$\mathfrak{s} = \varinjlim \mathfrak{s}|_{FP}$$

  i.e., $\mathfrak{s}$ is the colimit of its own restriction to finitely presented modules.

WHAT WAS WRONG WITH 1-TORSION?

EXAMPLE

As opposed to $\mathfrak{s}$, 1-torsion does not preserve filtered colimits.

To see that, represent $\mathbb{Q}$ as a filtered colimit of its finitely generated submodules: $\mathbb{Q} = \varinjlim A_i$. If $\mathfrak{t}$ preserved colimits, we would have had

$$\mathfrak{t}(\mathbb{Q}) = \mathfrak{t}(\varinjlim A_i) = \varinjlim(\mathfrak{t}(A_i)) = \varinjlim\{0\} = \{0\},$$

a contradiction.

FURTHER PROPERTIES OF $\mathfrak{s}$

- The torsion functor $\mathfrak{s}$ is the largest subfunctor of the 1-torsion functor $\mathfrak{t}$ that preserves filtered colimits.

PROOF.

Since $\mathfrak{s} = \mathfrak{t}$ on finitely presented modules, for any subfunctor $\mathfrak{u} \subseteq \mathfrak{t}$ we have $\mathfrak{u}|_{FP} \subseteq \mathfrak{s}|_{FP}$. Assuming that $\mathfrak{u}$ preserves filtered colimits, apply the exact functor $\varinjlim$ to get $\mathfrak{u} \subseteq \mathfrak{s}$. $\square$

Further properties of $\mathfrak{s}$

Similar to classical torsion and 1-torsion, we have

Theorem

$\mathfrak{s}$ is a radical, i.e., $\mathfrak{s}(A/\mathfrak{s}(A)) = \{0\}$ for any module A.

Proof.

We know that $\mathfrak{t}$ is a radical: $\mathfrak{t}(A/\mathfrak{t}(A)) = \{0\}$. Since $\mathfrak{s} \subseteq \mathfrak{t}$, we also have

$$\mathfrak{s}(A/\mathfrak{t}(A)) = \{0\}$$

If A is finitely presented, we are done because $\mathfrak{t}(A) = \mathfrak{s}(A)$. For an arbitrary A, write, as before, $A \simeq \varinjlim A_i$, with the A_i finitely presented and apply the exact functor $\varinjlim$. $\square$

Familiar definitions

Notation: Set $\mathfrak{s}^{-1} := \mathbf{1}/\mathfrak{s}$.

Definition

$\mathfrak{s}^{-1}(A)$ is called the torsion-free quotient module of A.

FURTHER PROPERTIES OF $\mathfrak{s}$

PROPOSITION

The following conditions are equivalent:

A) $\mathfrak{s}$ *preserves epimorphisms;*

B) $\mathfrak{s}$ *is the zero functor;*

C) ${}_\Lambda\Lambda$ *is absolutely pure;*

D) Λ *is* `left` *FP-injective, i.e.,* $\mathrm{Ext}^1_\Lambda(M, \Lambda) = \{0\}$ *for all finitely presented left* Λ*-modules* M*.*

In particular, if Λ *is selfinjective on the left, then* $\mathfrak{s}$ *is the zero functor.*

FURTHER PROPERTIES OF $\mathfrak{s}$

PROPOSITION

If

$$0 \to A' \to A \to A'' \to 0$$

is a pure exact sequence, then the sequence

$$0 \to \mathfrak{s}(A') \to \mathfrak{s}(A) \to \mathfrak{s}(A'') \to 0$$

is exact.

Further properties of $\mathfrak{s}$

Proposition

Suppose the injective envelope of ${}_{\Lambda}\Lambda$ is flat. Then

- $\mathfrak{s} \simeq \mathrm{Tor}_1(\_, \Sigma\Lambda)$.
- *If* $0 \to A' \to A \to A'' \to 0$ *is a short exact sequence of right* Λ*-modules, then the induced sequence*

$$0 \to \mathfrak{s}(A') \to \mathfrak{s}(A) \to \mathfrak{s}(A'') \to A' \otimes \Sigma\Lambda \to A \otimes \Sigma\Lambda \to A'' \otimes \Sigma\Lambda \to 0$$

  is exact. In particular, $\mathfrak{s}$ *is left-exact.*
- $\mathfrak{s}^2 = \mathfrak{s}$, *i.e.,* $\mathfrak{s}$ *is the torsion class of a torsion theory.*
- *The torsion-free class is closed under extensions.*

FURTHER PROPERTIES OF $\mathfrak{s}$

1-Torsion of a finitely presented right module over a left semihereditary ring splits off. In general, since $\mathfrak{s}$ preserves filtered colimits, we have

PROPOSITION

Suppose Λ is left simihereditary and A is a right Λ-module. Then the inclusion $\mathfrak{s}(A) \subseteq A$ is pure.

ANOTHER DESCRIPTION OF $\mathfrak{s}$

Let $\mathcal{F}$ denote the class of flat (right) Λ-modules, $Rej(A, \mathcal{F})$ – the reject of $\mathcal{F}$ in the right module A, and $rej(A, \mathcal{F})$ the restriction of the reject to the finitely presented modules. Then

$$\mathfrak{s} \subseteq Rej(\_, \mathcal{F}) \subseteq \mathfrak{t}$$

Restricting to finitely presented modules, we get equalities. Whence

PROPOSITION

$\mathfrak{s} \simeq \overrightarrow{rej}(\_, \mathcal{F})$, i.e., the torsion functor is isomorphic to the colimit extension of the reject of flats restricted to finitely presented modules.

Exercise

Exercise

Most of the basic results about classical torsion carry over, in one form or another, to the new setting. State such results and prove them.

Part 3. A functorial interlude

INJECTIVE STABILIZATION OF AN ADDITIVE FUNCTOR

Let $F : \Lambda\text{-Mod} \to \mathtt{Ab}$ be an additive covariant functor.

DEFINITION

The injective stabilization $\overline{F}$ of F is defined by the exact sequence

$$0 \longrightarrow \overline{F} \longrightarrow F \longrightarrow R^0 F$$

where $F \longrightarrow R^0 F$ is the canonical morphism of functors.

REMARK

Since $\overline{F}$ is a subfunctor of the additive functor F, it is itself additive.

HOW TO COMPUTE THE INJECTIVE STABILIZATION

Let B be a left Λ-module. Three easy steps to compute $\overline{F}(B)$:

- embed B in an injective module: $\iota : B \to I$,
- apply F to ι, and
- take the kernel of $F(\iota)$.

Thus $\overline{F}(B)$ is defined by the exact sequence

$$0 \longrightarrow \overline{F}(B) \longrightarrow F(B) \xrightarrow{F(\iota)} F(I)$$

EXAMPLE

EXAMPLE

Let A be a right Λ-module. Associated with A, is the functor

$$F := A \otimes \_$$

on the category of left Λ-modules. Its injective stabilization is denoted by $A \vec{\otimes} \_$. Evaluating it on $B := {}_\Lambda\Lambda$, we have

$$A \vec{\otimes}_\Lambda \Lambda = \mathfrak{s}(A)$$

THE CASE A IS FINITELY PRESENTED

When A is finitely presented, the isomorphism type of each component of $A \vec{\otimes} \_$ can be computed as follows.

PROPOSITION (A-B, 1969)

If A is finitely presented, then

$$A \vec{\otimes} \_ \simeq \mathrm{Ext}^1(\mathrm{Tr} A, \_)$$

DIGRESSION: INJECTIVELY STABLE FUNCTORS

DEFINITION

F is said to be injectively stable if the inclusion $\overline{F} \longrightarrow F$ is an isomorphism.

PROPOSITION (A-B, 1969)

The following conditions are equivalent:

- F *is injectively stable,*
- F *vanishes on injectives,*
- $R^0 F = 0$.

REMARK

Injectively stable functors are precisely the *effaceable* functors of Grothendieck.

INJECTIVE STABILIZATION: MISCELLANEA

- $\overline{F}$ is the largest subfunctor of F vanishing on injectives.
- $\overline{F} = 0$ if and only if F preserves monomorphisms.
- Injectively stable functors form a Serre class.
- Let
  - $\mathcal{T}$ be the class of injectively stable functors, and
  - $\mathcal{F}$ be the class of mono-preserving functors.

  Then $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory in the (quasi)category of additive functors.

Part 4. Definition of cotorsion

HOW DO WE DEFINE COTORSION?

In the absence of a classical prototype, we try and dualize the definition of torsion.

Start with a simple question:

- Why is $\mathfrak{s}(A)$ a subset of A?

WHY $\mathfrak{s}(A)$ IS A SUBSET OF A

The answer is obvious: because, by definition, $\mathfrak{s}(A)$ is a subset of $A \otimes \Lambda$ and there is a canonical isomorphism

$$A \otimes \Lambda \xrightarrow{\cong} A$$

Question Is there a "dual" canonical isomorphism?

Answer Yes, there is:

$$\mathrm{Hom}(\Lambda, C) \xrightarrow{\cong} C$$

Dualizing the definition of torsion

The torsion of A was defined as the value of the **injective** stabilization of the tensor product functor $A \otimes \_$ on Λ.

Dually, we should define the cotorsion of C as the value of the **projective** stabilization of the Hom functor $\mathrm{Hom}(\_, C)$ on Λ.

Remark

The notion of projective stabilization is formally dual to that of injective stabilization. Injective containers should be replaced by projective ancestors, R^0 by L_0, arrows should be reversed, Ker replaced by Coker, etc.

An obstacle: $\mathrm{Hom}(\_, C)$ is contravariant

In this brave new world of stable functors we are facing a problem: the original definition of the injective stabilization assumed that the functor was covariant. The same applies to projective stabilization. But $\mathrm{Hom}(\_, C)$ is contravariant . . .

To deal with this, we view a contravariant functor as a covariant functor on the opposite category. It is never a module category, but it is an abelian category, and all the requisite operations are still possible.

PROJECTIVE STABILIZATION OF A CONTRAVARIANT FUNCTOR

Let F be a contravariant additive functor, $B \in \Lambda\text{-Mod}$, and

$$0 \longrightarrow B \xrightarrow{\iota} I$$

an injective container.

In $(\Lambda\text{-Mod})^{op}$, this is a projective ancestor of B:

$$I \xrightarrow{\iota^{op}} B \longrightarrow 0$$

The value of the projective stabilization of the covariant functor F^{op} on B is defined as $\mathrm{Coker}\, F^{op}(\iota^{op}) = \mathrm{Coker}(F(\iota))$.

Thus, the value of the projective stabilization of F on B should be defined by the exact sequence

$$F(I) \xrightarrow{F(\iota)} F(B) \longrightarrow \underline{F}(B) \longrightarrow 0$$

EXAMPLE: $F = \mathrm{Hom}(\_, C)$ AND $B := \Lambda$

EXAMPLE

Let

$$0 \longrightarrow \Lambda \xrightarrow{\iota} I$$

be an injective container of Λ.

Apply $\mathrm{Hom}(\_, C)$ and pass to the cokernel:

$$(I, C) \xrightarrow{(\iota, C)} (\Lambda, C) \longrightarrow \underline{\mathrm{Hom}(\_, C)}(\Lambda) \longrightarrow 0$$

The image $I(\Lambda, C)$ of (ι, C) consists of all maps $\Lambda \to C$ factoring through injectives. Hence

$$\underline{\mathrm{Hom}(\_, C)}(\Lambda) = \overline{\mathrm{Hom}}(\Lambda, C)$$

DEFINITION OF COTORSION

DEFINITION

Let C be a (left) Λ-module. The cotorsion quotient module of C is

$$\mathfrak{q}(C) := \underline{\mathrm{Hom}(\_, C)}(\Lambda) = \overline{\mathrm{Hom}}(\Lambda, C)$$

Thus $\mathfrak{q} = \overline{\mathrm{Hom}}(\Lambda, \_)$ is a quotient of the identity functor.

FIRST OBSERVATIONS

- The short exact sequences

$$0 \longrightarrow I(\Lambda, C) \longrightarrow (\Lambda, C) \longrightarrow (\overline{\Lambda, C}) \longrightarrow 0$$

give rise to a short exact sequence of endofunctors on Λ-Mod

$$0 \longrightarrow \mathfrak{q}^{-1} \longrightarrow \mathbf{1} \longrightarrow \mathfrak{q} \longrightarrow 0$$

- $\mathfrak{q}$ preserves epimorphisms.
- $\mathfrak{q}$ is finitely presented (see the defining sequence on page 134):

$$(I, \_) \xrightarrow{(\iota, \_)} (\Lambda, \_) \longrightarrow \mathfrak{q} \longrightarrow 0$$

TRACE OF INJECTIVES COMES INTO PLAY

LEMMA

Under the canonical isomorphism

$$(\Lambda, C) \cong C : f \mapsto f(1),$$

$I(\Lambda, C)$ *identifies with* $Tr(\mathcal{I}, C)$, *the trace in* C *of the class* $\mathcal{I}$ *of injective* Λ*-modules.*

PROPOSITION

$\mathfrak{q}$ *is a coradical, i.e.,* $\mathfrak{q}(\mathfrak{q}^{-1}(C)) = \{0\}$ *for any* C.

COTORSION MODULES

DEFINITION

The module C is cotorsion if $C \to \mathfrak{q}(C)$ is an isomorphism. In other words, C is cotorsion if no map $\Lambda \to C$ factors through an injective. Equivalently, $Tr(\mathcal{I}, C) = \{0\}$.

EXAMPLE

Any PID which is not a field, viewed as a module over itself, is cotorsion (as it has no nonzero divisible elements).

Cotorsion-free modules

Definition

The module C is cotorsion-free if $C \to \mathfrak{q}(C)$ is the zero map, i.e., any map $\Lambda \to C$ factors through an injective. Equivalently, $Tr(\mathcal{I}, C) = C$.

Example

Any injective module is cotorsion-free.

Obviously, $\{0\}$ is the only module which is cotorsion and cotorsion-free.

PRIOR ATTEMPTS

Unlike for the concepts of the **cotorsion module of a module** and **cotorsion-free module of a module**, there have been attempts to define notions of **cotorsion module** and **cotorsion-free module.**

PRIOR ATTEMPTS: MATLIS

Matlis calls a module C over a commutative domain a cotorsion module if

$$\mathrm{Hom}(I, C) = \{0\} = \mathrm{Ext}^1(I, C)$$

Here I is the *field of fractions* of the domain.

Comparing these conditions with the defining sequence

$$(I, C) \xrightarrow{(\iota, C)} (\Lambda, C) \longrightarrow (\overline{\Lambda, C}) \longrightarrow 0$$

we see that the first condition alone guarantees that

- cotorsion modules in the sense of Matlis are cotorsion in our sense.

Prior attempts: Harrison, Warfield, Fuchs, Enochs-Jenda

Enochs and Jenda call a module C over an arbitrary ring cotorsion if

$$\mathrm{Ext}^1(F, C) = \{0\}$$

for any *flat* module F.

They remark that their definition generalizes the definitions of Harrison and Warfield and agrees with that of Fuchs but differs from the definition of Matlis.

According to this definition, injectives are cotorsion, but such modules are cotorsion-free in our sense.

Prior attempts: Harrison, Warfield, Fuchs, Enochs-Jenda

On the other hand, Enochs and Jenda show that pure injectives are cotorsion in their sense.

Now let $\mathbb{k}$ be a field. It is known that $\mathbb{k}[[X]]$ is pure injective as a module over itself. But this is a PID and, as we remarked before, $\mathbb{k}[[X]]$ is cotorsion in our sense.

Thus the class of cotorsion modules in the sense of Enoch and Jenda contains both cotorsion modules and cotorsion-free modules in our sense.

Expected properties hold

Exercise

Formulate and prove basic properties of cotorsion (Hint: dualize the properties of torsion).

Part 5. Duality and the exchange formula

THE AUSLANDER-GRUSON-JENSEN FUNCTOR

The Auslander-Gruson-Jensen duality, discovered by Auslander and independently by Gruson and Jensen, is a pair of exact contravariant functors

$$\begin{tikzcd} \mathrm{fp}(\mathrm{mod}(\Lambda^{op}), \mathtt{Ab}) \arrow[rr, bend left, "D"] & & \mathrm{fp}(\mathrm{mod}(\Lambda), \mathtt{Ab}) \arrow[ll, bend left, "D"] \end{tikzcd}$$

each of which interchanges the tensor product and the Hom functor when the fixed argument is a finitely presented module.

GENERAL PICTURE

There is an exact contravariant functor

$$D_A : \mathrm{fp}(\mathrm{Mod}\,(\Lambda^{op}), \mathtt{Ab}) \to (\mathrm{mod}(\Lambda), \mathtt{Ab})$$

defined by

$$D_A := R_0(\epsilon \circ w),$$

where ϵ is the tensor embedding

$$\epsilon : \mathrm{Mod}\,(\Lambda^{op}) \to (\mathrm{mod}(\Lambda), \mathtt{Ab}) : M \mapsto \_ \otimes M$$

and w is the defect functor. For any representable functor $(M, \_)$

$$D_A(M, \_) = \_ \otimes M$$

As is shown by Dean-Russell (2016), the functor D_A is completely determined by this property and by being exact.

AN EXTENSION OF THE AGJ FUNCTOR

THEOREM (S. DEAN - J. RUSSELL, 2016)

The functor

$$D_A : \mathrm{fp}(\mathrm{Mod}\,(\Lambda^{op}), \mathcal{A}b) \to (\mathrm{mod}(\Lambda), \mathcal{A}b)$$

admits a left adjoint D_L and a right adjoint D_R, both of which are fully faithful. The functors D_R and D_A restrict to the AGJ duality D on the full subcategories of pp-functors.

GENERAL PICTURE

The foregoing statement is part of the following diagram of functors

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{fp}(\mathrm{Mod}(\Lambda^{op}), \mathtt{Ab}) \arrow[rr, "D_A"] \arrow[dr, "w"] & & (\mathrm{mod}(\Lambda), \mathtt{Ab}) \arrow[ll, bend right, "D_L"'] \arrow[ll, bend left, "D_R"] \arrow[dl, "\mathrm{ev}_\Lambda"] \\
& \mathrm{Mod}(\Lambda^{op}) \arrow[ul, bend left, "\mathsf{Y}"] \arrow[ul, bend right, "L^0\mathsf{Y}"'] \arrow[ur, bend left, "R_0\epsilon"] \arrow[ur, bend right, "\epsilon"'] &
\end{tikzcd}
$$

SENDING COTORSION TO TORSION

THEOREM

For any module B

$$D_A\overline{\mathrm{Hom}}(B, \_) \simeq \_ \overrightarrow{\otimes} B$$

COROLLARY

The Auslander-Gruson-Jensen functor sends the cotorsion functor on left (right) modules to the torsion functor on right (left) modules. In short,

$$D_A(\mathfrak{q}) \simeq \mathfrak{s}$$

Equivalently,

$$D_A(Tr(\mathcal{I}, \_)^{-1}) \simeq \overrightarrow{rej}(\_, \mathcal{F})$$

GOING BACK: SENDING TORSION TO COTORSION

PROPOSITION

For any pure injective left Λ-module M,

$$\overline{\mathrm{Hom}}(M, \_) \simeq D_L(\_ \overrightarrow{\otimes} M)$$

COROLLARY

If ${}_\Lambda\Lambda$ is pure injective, then $\mathfrak{q} \simeq D_L(\mathfrak{s})$.

GOING BACK: ANOTHER OPTION

THEOREM

Suppose the injective envelope of Λ is finitely presented. Then the notions of torsion and cotorsion are dual. More precisely, the right adjoint

$$D_R : (\mathrm{mod}(\Lambda), \mathcal{A}b) \to \mathrm{fp}(\mathrm{Mod}\,(\Lambda^{op}), \mathcal{A}b)$$

of D_A carries the torsion functor to the cotorsion functor, i.e.,

$$D_R(\mathfrak{s}) \simeq \mathfrak{q}$$

COROLLARY

Let Λ be an artin algebra. Then $D_R(\mathfrak{s}) \simeq \mathfrak{q}$.

AN AUSLANDER-REITEN FORMULA FOR ARBITRARY MODULES

The foregoing isomorphisms are between functors, with no apparent connections between their arguments. We can do better.

Let Λ be an algebra over a commutative ring R. Choose an injective R-module $\mathbf{J}$ and let $D_{\mathbf{J}} := \mathrm{Hom}_R(\_, \mathbf{J})$.

PROPOSITION (AN AR FORMULA FOR ARBITRARY MODULES)

Let A be a right Λ-module and B a left Λ-module. There is an isomorphism

$$D_{\mathbf{J}}(A \overset{\rightarrow}{\otimes} B) \simeq \overline{\mathrm{Hom}}(B, D_{\mathbf{J}}(A)),$$

functorial in A and B.

EXCHANGE FORMULA

Specializing to the case $B =_\Lambda \Lambda$, we have

PROPOSITION

In the above notation,

$$D_{\mathbf{J}} \circ \mathfrak{s} \simeq \mathfrak{q} \circ D_{\mathbf{J}}$$

i.e., for each injective R-module $\mathbf{J}$ and each right Λ-module A, we have an isomorphism

$$D_{\mathbf{J}}(\mathfrak{s}(A)) \simeq \mathfrak{q}(D_{\mathbf{J}}(A))$$

which is functorial in A.

Exchange formula

Corollary

Let Λ be an arbitrary ring, $R := \mathbb{Z}$, and, for any right Λ-module A, let $A^+ := \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ be the character module of A. Then

$$\mathfrak{s}(A)^+ \simeq \mathfrak{q}(A^+)$$

Thank you