Definability in Module Theory

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Contents

We will concentrate on explaining some ideas and methods from model theory and the particular form they take in the context of modules, as well as how they can contribute to our understanding of representations, especially certain nice infinite-dimensional ones. We will, in particular, discuss definability - perhaps the central concept in model theory, and use of the Compactness Theorem/ultraproducts in order to realise types - use of the Compactness Theorem is the central method in model theory.

The aim of the lectures is to enable one to engage with the details of papers which use these concepts and techniques, specifically to:

• gain some familiarity with pp formulas;
• see something of pp-types and how they can produce indecomposable pure-injectives;
• then, since the first three lectures will be rather focussed, gain some broader perspective at the end.

1 Lecture 1: Definable sets

1.1 Definable sets

At least, definable sets in algebraic structures - we'll ignore relations, such as $\leq$ on structures.

Definable sets are what we get when we close solution sets of equations under the boolean set-theoretic operations of (finite) intersection $\cap$, union $\cup$, complement $(-)^c$ and projection. **Formulas** are what we use to describe these sets.

Formally: Given an algebraic structure M, the **definable subsets** of M or, more precisely, the subsets of M^n (various n) definable in M are the sets obtained inductively as follows:

• the solution set X in M of any equation is a definable subset (if the equation has n variables, then $X \subseteq M^n$) ;

• the intersection $X \cap Y$ of any two definable subsets X, Y ($X, Y \subseteq M^n$ for some n) is a definable subset;

• the complement $M^n \setminus X$ of any definable subset $X \subseteq M^n$ is a definable subset;

• if $X \subseteq M^n$ is a definable subset then the projection $\pi_{\widehat{i}} X \subseteq M^{n-1} = \{(a_1, \dots, a_{i-1}, a_{i+1}, \dots a_n) : (a_1, \dots a_n) \in X\}$ of X along the ith coordinate is a definable subset.

Example 1.1. In commutative k-algebras, where k is a commutative ring (a field, the ring $\mathbb{Z}$ of integers,...). So an equation has the form $f(\overline{x}) = 0$ where $f(\overline{x}) \in k[x_1, \dots, x_n]$ and the solution set of this formula in a k-algebra R is the subset $\{\overline{a} \in R^n : f(\overline{a}) = 0\}$ of R^n.

In particular, if k is a field and L/k an extension of fields, then an affine subvariety of L^n is the solution set in L of a finite conjunction

$$\bigwedge_{j=1}^{m} f_j(\overline{x}) = 0$$

of such equations. Note that this is a formula, where $\wedge$ (and $\bigwedge$, cf. $+$ and $\sum$) is introduced to describe the intersection of solution sets of equations.

For instance, the set of units in a commutative ring R is the solution set of the formula $\phi(x) \equiv \exists y\,(xy = 1)$. We get this from the equation $xy = 1$ by existentially quantifying out y. In terms of solutions sets: let $\theta(x, y)$ be the equation $xy = 1$, so $\theta(R) = \{(a, b) : ab = 1\}$, so then $\phi(R)$ is the projection of $\theta(R)$ to the first coordinate. "*existential quantification = projection*"

The set of non-units is then defined by the *negation* $\neg\phi$, that is $\neg\exists y\,(xy = 1)$, of ϕ, where we use $\neg$ "not" to describe complement: $\neg\phi(R) = R^n \setminus \phi(R)$ if ϕ has n free variables.

We use $\vee$ to express "or" - note that $\phi \vee \psi$ is equivalent to $\neg(\neg\phi \wedge \neg\psi)$ and we use $\to$ to express "implies" - with $\phi \to \psi$ being equivalent to $(\neg\phi) \vee \psi$. We can also introduce the universal quantifier $\forall x$ and note $\forall x \phi$ is equivalent to $\neg\exists x(\neg\phi)$.

So, for instance, to define the set of irreducible elements we can use the formula

$$\forall y, z\,((x = yz) \to (x|y \vee x|z)),$$

with $x|y$ just being a shorthand for $\exists u(xu = y)$; an equivalent formula is

$$\forall y, z\,(x \neq yz \vee x|y \vee x|z).$$

Now that we have formulas to describe definable sets, we can also use them to write down conditions on structures. For instance $\forall x\,(x = 0 \vee \exists y(xy = 1))$ is either true in a commutative ring (exactly if it's a field) or false.

So we have seen some examples of definable sets, the formulas we can use to define/specify these, and the additional use of sentences (formula without free variables) to express conditions on structures (as opposed to conditions on elements of structures).

Example 1.2. The *Chevalley-Tarski Theorem* says that, if k is an algebraically closed field and $V \subseteq k^n$ is an affine variety (that is a subset defined by a conjunction of equations) then the image of V under any projection (indeed under any polynomial map) is *constructible*, meaning that it is a finite boolean combination of solution sets of equations.

In particular, that projection can be defined by a formula that does not use quantifiers, even though projection usually means introducing an existential quantifier.

This theorem says that algebraically closed fields have **quantifier elimination**: every definable set for an algebraically closed can be defined by a formula that does not use quantifiers.

In general, definable sets can be arbitrarily complex in the sense of there being no bound on the number of alterations of quantifiers needed to define them. So it is good when we can say that every definable set has a (relatively) simple definition.

(Much) more information on definable sets and on what is discussed below can be found in [12] (and [11] for a more model-theoretic approach). A set of notes overlapping with these notes but with a little more detail on some things is [14].

1.2 Definable sets in modules

Fix the ring R and look at, say, right R-modules. In this case, every equation is (equivalent to one of) the form $\sum_{i=1}^n x_i r_i = 0$ with the $r_i \in R$. So a conjunction

$$\bigwedge_{j=1}^{m} \sum_{i=1}^{n} x_i r_{ij} = 0$$

of these is a system of R-linear equations, which we may write more compactly as $\overline{x}H = 0$ where $H = (r_{ij})_{ij}$ is a matrix over R, and the projection of the solution set to such a system of R-linear equations is defined by a formula of the form

$$\exists x_{k+1}, \ldots, x_n \,\big(\bigwedge_{j=1}^{m} \sum_{i=1}^{n} x_i r_{ij} = 0\big).$$

A formula (equivalent to one) of this form is a **positive primitive**, or just **pp**, formula (the term **regular** formula also is used in some parts of the literature). We can write a pp formula more compactly as $\exists \overline{y}\ (\overline{x}\ \overline{y})G = 0$, or $\exists \overline{y}\ (\overline{x}\ \overline{y}) \begin{pmatrix} G' \\ G'' \end{pmatrix} = 0$ if we want to partition the matrix G.

That is, pp formulas are those we use to define projected solution sets of systems of R-linear equations.

Note that if $\phi(\overline{x})$ is a pp formula then, in any module M, its solution set $\phi(M)$ is a subgroup (of M^n if n is the length of $\overline{x}$, that is, the number of free variables). We refer to this as a **pp-definable subgroup** of M (more accurately, as subgroup of M^n pp-definable in M). (Other terminologies that have been used are "subgroup of finite definition" and "finitely matrizable subgroup".) Each such subgroup $\phi(M)$ is a submodule of M^n under the (diagonal) action of $\mathrm{End}(M)$, in particular is a module over the centre of R, though not necessarily an R-submodule.

Theorem 1.3. *(***Pp-elimination of quantifiers for modules***)*
(1) Every definable subset of a module is a (finite) boolean combination of pp-definable subgroups.

(If we allow parameters from M into our definitions so, for instance, allowing inhomogeneous systems of linear equations $\bigwedge_{j=1}^m \sum_{i=1}^n x_i r_{ij} = a_j$ with the $a_j \in M$, then every definable subset will be a finite boolean combination of cosets of pp-definable subgroups, or $\emptyset$.)
(2) Every sentence is equivalent to a finite boolean combination of conditions of the form

$$\big|\phi(-)/\psi(-)\big| > t,$$

where $t \geq 1$ and where ψ, ϕ are pp with $\psi \leq \phi$.

If R is an algebra over an infinite field, then these simply are boolean combinations of formulas of the form $\phi(-) = \psi(-)$ and $\phi(-) \neq \psi(-)$.

In the above we write $\psi \leq \phi$ to mean that ψ implies ϕ, that is $\psi(M) \leq \phi(M)$ for every module M - we refer to this pair of formulas as a **pp-pair**.

Example 1.4. Let's try to express injectivity of a module (M). We can use Baer's Criterion: given any right ideal I of R and any morphism $f : I \to M$, there is a morphism $g : R \to M$ with $gj = f$, where j is the inclusion of I into R.

Fix I and choose a generating set, say $(r_\lambda)_\lambda$, for I. So f is determined by the images $(fr_\lambda)_\lambda$. What are the conditions on a tuple $(a_\lambda)_\lambda$ from M to be of the form $(fr_\lambda)_\lambda$ for some morphism f?

Choose a generating set $(\overline{r}\cdot\overline{s_\alpha} = 0)_\alpha$ of the relations on the chosen generators ($\overline{r} \cdot \overline{s_\alpha}$ expanding to $\sum_\lambda r_\lambda s_{\alpha\lambda} = 0$ where $s_{\alpha\lambda} = 0$ for almost all λ).

Then the required condition on a tuple $\overline{a} = (a_\lambda)_\lambda$ is that, for each α, we have $\overline{a} \cdot \overline{s_\alpha} = 0$.

So, if I is finitely generated and finitely related, then there is a pp formula $\phi(\overline{x})$ such that $\phi(M) = \{\overline{a} \in M^{(-)} : \exists f : I \to M \text{ such that } \overline{a} = f\overline{r}\}$, namely $\bigwedge_\alpha \overline{x} \cdot \overline{s_\alpha} = 0$ where we are now assuming that we have chosen the tuple $\overline{r}$ to be finite and have chosen finitely many generating relations.

How do we say that there is $g : R \to M$ such that $gj = f$? Note that $g(1)r_\lambda = f(r_\lambda)$, for all λ, so a formula expressing existence of such g is $\exists y \left(\bigwedge_\lambda yr_\lambda = x_\lambda \right)$.

Therefore, if I is finitely presented, then the sentence

$$\forall \overline{x} \, \big(\bigwedge_\alpha \overline{x} \cdot \overline{s_\alpha} = 0 \to \exists y \bigwedge_\lambda yr_\lambda = x_\lambda \big)$$

expresses injectivity of M over the inclusion of I into R. Note that this is exactly expressing closure of a pp-pair, namely, it is expressing that

$$\big|(\bigwedge_\alpha \overline{x} \cdot \overline{s_\alpha})/(\exists y \bigwedge_\lambda yr_\lambda = x_\lambda)\big| = 1.$$

If R is right noetherian, then every right ideal is finitely presented, so the collection of all these sentences (that is, closure of a certain set of pp-pairs) cuts out the injective modules. In fact, if R is any ring, then the set of all these sentences as I ranges over the finitely presented ideals, cuts out the absolutely pure modules iff R is right coherent.

2 Lecture 2: Pp formulas and purity

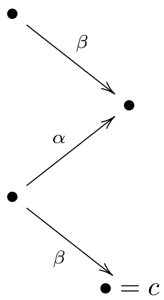
2.1 Free realisations and finitely presented modules

Suppose that A is a finitely presented (right R-)module and that $\overline{c} = (c_1, \dots, c_n)$ is a tuple from A. Then there is a pp formula ϕ such that $\overline{c} \in \phi(A)$ and such that, for every module M and n-tuple $\overline{b} \in \phi(M)$, there is $f : A \to M$ such that $f\overline{c} = \overline{b}$. That is, in every module M, the solution set $\phi(M)$ is the set of images of $\overline{c}$ under morphisms $A \to M$. We say that the pair $(A, \overline{c})$ is a **free realisation** of ϕ.

The formula ϕ describes how $\overline{c}$ sits in A: choose a finite generating set $\overline{a}$ for A and suppose that the system of equations $\overline{a}H = 0$ generates the relations on $\overline{a}$. Also take a matrix G such that $\overline{c} = \overline{a}G$. Then take $\phi(\overline{x})$ to be the formula $\exists \overline{y}\, (\overline{x} = \overline{y}G \wedge \overline{y}H = 0)$.

One may check that every pp formula is freely realised in this sense in some finitely presented module (use elementary duality, discussed below, to write any pp-formula in the form $\exists \overline{y}\, (\overline{x} = \overline{y}G \wedge \overline{y}H = 0)$ and reverse the above outlined argument).

Example 2.1. Take R to be the path algebra of the Kronecker quiver $\widetilde{A_1}$ $1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2$ and take A to be the (regular) module and the element c to be as indicated.



Then (A, c) is a free realisation of the formula

$$\exists y\, (x = y\beta \wedge \exists z\, (z\beta = y\alpha \wedge z\alpha = 0)).$$

2.2 The lattice of pp formulas

We (pre-)order pp formulas in the same (number of) free variables by $\psi(\overline{x}) \leq \phi(\overline{x})$ iff $\psi(M) \leq \phi(M)$ for every module M (since taking solution sets of pp formulas commutes with direct limits, it is enough that this be so in every finitely presented module M).

Then the equivalence classes of such pp formulas, with this order, form a lattice which we denote by pp_R^n, or just pp^n, if n is the number of free variables. The operations are meet, given by $\phi(\overline{x}) \wedge \psi(\overline{x})$, and join, given by $\exists \overline{y}\, (\phi(\overline{y}) \wedge \psi(\overline{x} - \overline{y}))$, corresponding respectively to intersection and sum in the lattice of (pp-definable) subgroups of any module.

If M is any module, then taking solution sets in M, $\phi(\overline{x}) \mapsto \phi(M)$, is a surjective lattice homomorphism from pp^n to the lattice of subgroups of M^n pp-definable in M. The kernel equivalence relation is the set of pairs $(\phi(\overline{x}), \psi(\overline{x}))$ of pp formulas such that $\phi(M) = \psi(M)$ and is generated by the **pp-pairs** ϕ/ψ

that is, $\psi \leq \phi$) which are **closed on** M (that is, $|\phi(M)/\psi(M)| = 1$ as in the statement of the pp-elimination of quantifiers theorem). A pp-pair is **open** on M if it is not closed on M.

Note that if $(A, \overline{a})$ is a free realisation of a pp formula ϕ, then, for every pp formula ψ, $\overline{a} \in \psi(A)$ iff $\phi \leq \psi$ (using terminology that we will introduce later, this says that ϕ **generates the pp-type** of $\overline{a}$ in A). For the direction which is not the definition, if $\overline{a} \in \psi(A)$ take any module M and tuple $\overline{b} \in \phi(M)$. Then there is a morphism $A \to M$ taking $\overline{a}$ to $\overline{b}$ and so, since *morphisms preserve pp formulas* ($f\phi(M) \subseteq \phi(N)$ whenever $f : M \to N$ is a morphism and ϕ is pp - exercise), $\overline{b} \in \psi(M)$, as required to show that $\phi \leq \psi$.

2.3 Duality of pp formulas

If $\phi(\overline{x})$ is the pp formula, for right R-modules,

$$\exists \overline{y}\, (\overline{x}\ \overline{y}) \begin{pmatrix} G \\ H \end{pmatrix} = 0,$$

define the pp formula $D\phi(\overline{x})$, for *left* R-modules - the (**elementary**) **dual** of ϕ to be

$$\exists \overline{z} \begin{pmatrix} I_n & G \\ 0 & H \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{z} \end{pmatrix} = 0.$$

Similarly (taking transposes to interchange the roles of rows and columns, left and right) we define the dual of a pp formula for left modules.

Then, for every pp formula ϕ and module M, we have $DD\phi(M) = \phi(M)$, that is, ϕ is equivalent to $DD\phi$ (exercise).

Moreover, if $\psi \leq \phi$ then $D\phi \leq D\psi$, $D(\phi \wedge \psi) = D\phi + D\psi$ and $D(\phi + \psi) = D\phi \wedge D\psi$.

Thus D gives an anti-isomorphism, "elementary duality", of lattices $\mathrm{pp}_R^n \simeq (\mathrm{pp}^n + R^{\mathrm{op}})^{\mathrm{op}}$ of pp formulas, in n free variables, for right and left modules, for each n.

It connects nicely to the algebraic duality $M_R \mapsto ({}_RM^*) = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$:

$$D\phi({}_RM^*) = \mathrm{ann}_{M^*}(\phi(M)).$$

Exercise 2.2. Compute the dual of the formula we saw in the example over the Kronecker algebra [after simplification it can be written as $\exists v, w\, (\beta x = \alpha v \wedge \beta v = \alpha w)$] and also compute a free realisation of this dual pp formula [a minimal free realisation is a 7-dimensional left module].

Theorem 2.3. *(***Herzog's criterion** *[6]) Let $\overline{a} \in M_R^n$, $\overline{l} \in_R L^n$. Then $\overline{a} \otimes_R \overline{l} = 0$ (that is $\sum_{i=1}^n a_i \otimes l_i = 0$) in $M \otimes_R L$ iff there is a pp formula $\phi(\overline{x})$ such that $\overline{a} \in \phi(M)$ and $\overline{l} \in D\phi(L)$.*

Exercise 2.4. Check this for (M, c) being the Kronecker algebra free realisation seen in the earlier example and (L, l) being the free realisation of the dual formula. That is, check directly why $c \otimes l = 0$ in $M \otimes_R L$.

If $\phi(\overline{x})$ is $\exists \overline{y}\, (\overline{x}\ \overline{y}) \begin{pmatrix} G \\ H \end{pmatrix} = 0$ then its dual has the form

$$\exists \overline{z}\, (\overline{x} = G\overline{(z)} \wedge H\overline{z} = 0).$$

Since every pp formula is a dual formula, this allows us to write a typical pp formula in the form $\exists \overline{y}(\overline{x} = \overline{y}G \wedge \overline{y}H = 0)$, that is 'generalised divisibility formula'.

2.4 Purity

Every morphism between modules preserves pp-definable subgroups: if $f : L \to M$ then $f\phi(L) \subseteq \phi(M)$. A morphism, necessarily injective, is **pure** if it also reflects pp-definable subgroups. That is a monomorphism $j : L \to M$ is pure if, for every pp formula $\phi(x_1, \ldots, x_n)$ we have

$$j\phi(L) = L^n \cap \phi(M)$$

(in fact it is enough to require this for $n = 1$).

For instance, the inclusion of $2\mathbb{Z}$ into $\mathbb{Z}$ is not pure - take $\phi(x)$ to be $\exists y\, x = y + y$ and $a = 2 \in 2\mathbb{Z}$.

We then say that $0 \to L \to M \to M/L \to 0$ is a **pure-exact sequence**.

A module N is said to be **pure-injective** if it is injective over pure embeddings, that is, if whenever we have a pure embedding $L \to M$ and a morphism $g : L \to N$, then there is a lifting of g to a morphism $M \to N$ as shown.

$$\begin{tikzcd} L \arrow[r] \arrow[d, "g"'] & M \arrow[dl, dotted] \\ N & \end{tikzcd}$$

There are many equivalent ways to define pure monomorphisms, for instance: $j : M \to N$ is pure iff $j \otimes 1_L : M \otimes_R L \to N \otimes_R L$ is monic for every (finitely presented) ${}_RL$.

We can see this using Herzog's Criterion:

(pp-pure $\Rightarrow$ $\otimes$-pure): if $\overline{a} \otimes \overline{l} = 0$ in $N \otimes L$, then, by Herzog's Criterion, there is some pp ϕ such that $\overline{a} \in \phi(N)$ and $\overline{l} \in D\phi(L)$, but pp-purity implies $\overline{a} \in \phi(M)$, so $\overline{a} \otimes \overline{l} = 0$ in $M \otimes L$.

($\otimes$-pure $\Rightarrow$ pp-pure): Suppose $\overline{a} \in \phi(N)$. Take $({}_RL, \overline{l})$ to be a free realisation of $D\phi$. Then, by Herzog's Criterion, $\overline{a} \otimes \overline{l} = 0$ in $N \otimes L$ so, by assumption, $\overline{a} \otimes \overline{l} = 0$ in $M \otimes L$. So by Herzog's criterion, there is ψ such that $\overline{a} \in \psi(M)$ and $\overline{l} \in D\psi(L)$. Since $(L, \overline{l})$ freely realises $D\phi$ it follows that (as observed earlier) that $D\phi \leq D\psi$ and hence, since D is a duality, that $\psi \leq \phi$, in particular that $\overline{a} \in \phi(M)$, as required.

3 Lecture 3: Pp-types, compactness and pure-injectivity

3.1 The Compactness Theorem

If $\overline{a} \in \phi(M)$ then this fact is a piece of information about how $\overline{a}$ sits in M (namely that it satisfies a certain projected system of linear equations, equivalently that is satisfies a certain generalised divisibility condition). The **pp-type** of $\overline{a}$ in M, $\mathrm{pp}^M(\overline{a})$ is the set of all these pieces of information:

$$\mathrm{pp}^M(\overline{a}) = \{\phi(\overline{x}) \text{ pp} : \overline{a} \in \phi(M)\}.$$

This is, note, a filter in the lattice pp_R^n of pp formulas (where $\overline{a} = (a_1, \ldots, a_n)$).

Recall that, a **filter** on a (modular) lattice (with top and bottom elements) is an upwards-closed, non-empty proper subset which is closed under finite meet.

In fact, every filter in pp_R^n arises in this way. To prove that we use the Compactness Theorem and ultraproducts.

Take an indexed set $(M_i)_{i \in I}$ of structures, all of the same kind (groups, rings, R-modules, ...). Choose a filter $\mathcal{F}$ on the set I of indices and form the **reduced product**

$$M^* = \prod_{i \in I} M_i/\mathcal{F} = \varinjlim_{J \in \mathcal{F}} \prod_{j \in J} M_j$$

(in the case of modules the directed colimit and products are taken in the category of modules, in general it has to be taken in the category of structures for a certain language). Elements of M^* have the form $(a_i)_i/\sim$ where $(a_i)_i \sim (b_i)_i$ iff $\{i \in I : a_i = b_i\} \in \mathcal{F}$.

Theorem 3.1. *(Los' Theorem)*
(1a) Let $\phi(x_1, \ldots, x_n)$ be a pp formula and $\overline{a} = (a^1, \ldots, a^n)$, with $a^j = (a_i^j)_i/\sim$, be elements of M^*. Then:

$$\overline{a} \in \phi(M^*) \text{ iff } \{i \in I : (a_i^1, \ldots, a_i^n) \in \phi(M_i)\} \in \mathcal{F}.$$

(1b) If $\mathcal{F}$ is an ultrafilter then (1a) holds for arbitrary formulas ϕ.
(2a) If $\phi(\overline{x})$ and $\psi(\overline{x})$ are pp formulas then:

$$\phi(M^*) \subseteq \psi(M^*) \text{ iff } \{i \in I : \phi(M_i) \subseteq \psi(M_i)\} \in \mathcal{F}.$$

(2b) If $\mathcal{F}$ is an ultrafilter and σ is any sentence (formula without quantifiers), then:

$$\sigma \text{ is true in } M^* \text{ iff } \{i \in I : \sigma \text{ is true in } M_i\} \in \mathcal{F}.$$

A filter $\mathcal{F}$ on I is an **ultrafilter** if it is a maximal (with respect to inclusion) proper filter (equivalently if, for every $J \subseteq I$, either $J \in \mathcal{F}$ or $J^c \in \mathcal{F}$). Then we refer to the reduced product as an **ultraproduct** (or **ultrapower** if all the structures are the same).

(In all this there is a formal language in the background - describing that involves giving the precise rules used to produce the formulas that we are using to describe definable sets. This does have to be done carefully but the details are easily available.)

A corollary is the Compactness Theorem - the basic theorem of Model Theory. First we extend the definition of pp-type.

Suppose that M is a structure and $\overline{a} \in M^n$. Then the **type** of $\overline{a}$ in M is

$$\mathrm{tp}^M(\overline{a}) = \{\phi(\overline{x}) \text{ a formula} : \overline{a} \in \phi(M)\}.$$

More generally, a **partial type** for M is a collection $\Phi(\overline{x})$ of formulas in the free variables $\overline{x}$ such that, for every finite subset Φ' of Φ, we have $\bigcap_{\phi \in \Phi'} \phi(M) \neq \emptyset$. A **type** for M is a partial type which is maximal, with respect to inclusion, among partial types for M (equivalently an ultrafilter in the lattice of definable subsets of M). If we allow the elements of M to appear as parameters (constants) in our formulas, then we will refer to a (partial) type with parameters from M.

We say that a (partial) type Φ is **realised** in M if $\bigcap_{\phi \in \Phi} \phi(M) \neq \emptyset$. The type of any tuple from M is a type but not every type for M may be realised in M. Consider for example, the abelian group $M = \mathbb{Z}$ and $\Phi(x) = \{\exists y\, x = ty : t \in \mathbb{Z}, t \neq 0\} \cup \{x \neq 0\}$.

Theorem 3.2. *(Compactness Theorem) If M is a structure and $\Phi(\overline{x})$ is a (partial) type for M (possibly with parameters), then there is an ultraproduct M^* of M which contains a realisation of Φ.*

Note that there is the diagonal map embedding M naturally into M^*. This will be a pure embedding (in fact an elementary one, meaning as in the definition of pure embedding but allowing arbitrary formulas in place of pp ones).

In particular, if $p(\overline{x})$ is a filter in the lattice pp_R^n of pp formulas for R-modules, then there is a module M^* and $\overline{a} \in M^*$ with $\mathrm{pp}^{M^*}(\overline{a}) = p$.

Example 3.3. If we take M to be the ring of integers, then we can produce a ring M^* with comparatively bizzare behaviour: new primes; non-zero elements divisible by infinitely many distinct primes; non-zero elements divisible by arbitrarily high powers of a particular prime, yet M^* will be a Bezout ring in that every finitely generated ideal is principal.

3.2 Algebraic compactness and pure-injectivity

A module M is **algebraically compact** if every filter of cosets of pp-definable subgroups has non-empty intersection, that is, if every pp-type, consisting of formulas *with parameters from* M is realised in M.

Note that if R is a k-algebra where k is a field, then any finite-dimensional R-module will be algebraically compact (since every pp-definable subgroup is a k-subspace).

Example 3.4. Consider the localisation $M = \mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at some prime p (as a module over itself or over $\mathbb{Z}$). The pp-definable subgroups of M are the $p^n M$ ($n \geq 0$) and 0. So the pp-definable cosets form a p-branching tree of infinite depth and every branch along this tree is a filter/pp-type with parameters from M. Regarded as descriptions of potential elements, these pp-types are mutually inconsistent so, since there are uncountable many branches but M is countable, they are not all realised in M. Hence M is not algebraically compact.

Theorem 3.5. *A module is algebraically compact iff it is pure-injective, that is, injective over pure embeddings.*

Every module M has a **pure-injective hull** (or **pure-injective envelope**), that is, a pure embedding $i : M \to H(M)$ into a pure-injective module with the property that if $j : M \to N$ is a pure embedding and N is pure-injective, then there is a split (in particular, pure) embedding $f : M \to N$ with $fi = j$.

In the example above, the pure-injective hull of M is the embedding of the localisation $\mathbb{Z}_{(p)}$ into its p-adic completion - the ring $\overline{\mathbb{Z}_{(p)}}$ of p-adic integers.

Here are some of the (many) equivalents to pure-injectivity.

Theorem 3.6. *The following are equivalent for any module N:*
(i) N is pure-injective;
(ii) N is algebraically compact
(iii) N is a direct summand of N^{**};
(iv) for any (index) set I, the summation map $N^{(I)} \to N$ factors through the canonical embedding $N^{(I)} \to N^I$.

3.3 Structure of pure-injective modules and hulls

Pure-injective modules form a class which extends the finite-dimensional modules into the infinite-dimensional ones but which still has a nice structure theory and which, over some rings are even classifiable. (Recall that over almost all rings there is no structure theorem for infinitely generated modules.)

Say that a (nonzero) module is **superdecomposable** if it has no indecomposable direct summands. For instance the ring of endomorphisms of a countably-infinite-dimensional vector space modulo the ideal of finite rank endomorphisms, is superdecomposable as a module over itself.

Theorem 3.7. *(***Structure theorem for pure-injectives***) If N is a pure-injective module then*

$$N = H(\bigoplus_{\lambda} N_\lambda) \oplus N'$$

where each N_λ is an indecomposable pure-injective direct summand of N and where N' is superdecomposable. In all such decompositions, the factors N_λ are the same up to isomorphism and multiplicity and the superdecomposable factor is the same up to isomorphism.

Just as finitely presented modules are the 'correct contexts' in which to realise **finitely generated** pp-types (those which, as a filter in pp_R^n are finitely=singly generated), pure-injectives give the correct contexts for general pp-types, in the sense that if $\overline{a}$ is a tuple from a module M and if $p = \mathrm{pp}^M(\overline{a})$, then, for every (matching) tuple $\overline{b}$ from a pure-injective module N, there is a morphism $f : M \to N$ with $f\overline{a} = \overline{b}$ iff $p \subseteq \mathrm{pp}^N(\overline{b})$.

Let p be a pp-type. We have seen that p is realised in some module M and, since the embedding of M into $H(M)$ is pure, we may assume that M is pure-injective.

Theorem 3.8. *(Fisher, mid70s) Let $\overline{a}$ be a tuple from a pure-injective module N. Then there is a pure-injective direct summand, $H(\overline{a})$, of N which is minimal with respect to containing all the entries of $\overline{a}$. The module $H(\overline{a})$ is unique to isomorphism over $\overline{a}$.*

Indeed, if N' is any pure-injective module and $\overline{a'}$ is a tuple from N' such that $\mathrm{pp}^N(\overline{a}) = \mathrm{pp}^{N'}(\overline{a'})$, then there is a split embedding from $H(\overline{a})$ to N' taking $\overline{a}$ to $\overline{a'}$.

In view of this, if p is a pp-type and $\overline{a}$ (from a pure-injective module) is a realisation of p, then we set $H(p) = H(\overline{a})$, referring to this as the **hull** of p.

There is the following criterion on p for being realised in an indecomposable pure-injective (reflecting that indecomposable pure-injectives have local endomorphism rings); we say that p is **irreducible** if the hull $H(p)$ of p is indecomposable.

Theorem 3.9. *(***Ziegler's Criterion** *[18]) Let $p(\overline{x})$ be a pp-type. Then p is irreducible iff, for every $\psi_1(\overline{x}), \psi_2(\overline{x})$ not in p, there is $\phi \in p$ such that*

$$(\phi \wedge \psi_1) + (\phi \wedge \psi_2) \notin p.$$

If we embed Mod-R in the functor category $(R\text{-mod}, \mathbf{Ab})$ using the tensor embedding $M_R \mapsto M \otimes_R -$, discussed briefly below, then this criterion becomes an expression of the fact that an indecomposable injective object in a Grothendieck abelian category is uniform (i.e., the intersection of two non-zero subobjects is non-zero).

4 Lecture 4: The Ziegler spectrum, definable categories, functor categories, multisorted modules

4.1 The Ziegler spectrum

The (right) **Ziegler spectrum** of R, Zg_R, is a topological space whose points are the isomorphism classes of indecomposable pure-injective right R-modules. A basis of open sets consists of the sets

$$(\phi/\psi) = \{N \in \mathrm{Zg}_R : \phi(N) > \psi(N)\}$$

- the set of indecomposable pure-injectives which open the pp-pair ϕ/ψ. Note that, because every indecomposable pure-injective is the hull of some pp-type, we do have just a set of these.

Ziegler proved that this basis consists precisely of the compact open sets. In particular, the whole space is compact, being $(x = x/x = 0)$, but seldom Hausdorff (if R is a finite-dimensional algebra then Zg_R is Hausdorff, indeed discrete, iff R is of finite representation type).

If $N \in \mathrm{Zg}_R$ then a basis of open neighbourhoods of N can be obtained by choosing any pp-type p such that $N = H(p)$ - the corresponding basis then consists of the sets (ϕ/ψ), with $\psi < \phi$, such that $\phi \in p$, $\psi \notin p$.

Example 4.1. If R is a finite-dimensional algebra then, direct from the existence of almost split sequences, the finite-dimensional indecomposable modules are precisely the isolated points of Zg_R (note that these are pure-injective, hence points of Zg_R). So, if R is a finite-dimensional algebra of infinite representation type then, by compactness, there is a non-isolated point, that is, an infinite-dimensional indecomposable (pure-injective) module. It is also the case that, over a finite-dimensional algebra, the finite-dimensional=isolated points are dense.

The closed sets are particularly significant in that they parametrise the *definable subcategories* - these are discussed in the next section.

Example 4.2. Ziegler spectra of tame hereditary algebras are described, see [**?**], [**?**]. It turns out that the infinite-dimensional indecomposable pure-injectives all are associated to tubes in the regular part of the Auslander-Reiten quiver of such an algebra.

More precisely, and taking the Kronecker algebra for the simplest example, we have, associated to each 'quasisimple' regular module S, an adic and a Prüfer module, obtained respectively by taking the inverse limit of the coray ending at S and the direct limit of the ray starting at S; plus a generic module which has the structure $K(T) \underset{\beta=T}{\overset{\alpha=1}{\rightrightarrows}} K(T)$, where $K(T)$ denotes the ring of rational functions in T over the base field K.

Example 4.3. Ziegler spectra of domestic string algebras have been described, see [**?**], [**?**], [9]. Roughly, there are infinite-dimensional band modules analogous to those seen in the tame hereditary case, plus infinite-dimensional string modules.

4.2 Definable subcategories

A closed set of the Ziegler spectrum has the form $\bigcap_\lambda [\phi_\lambda/\psi_\lambda]$, so consists of the indecomposable pure-injectives N such that $\phi_\lambda(N) = \psi_\lambda(N)$ for each λ. We may look at the class of *all* modules M satisfying these closure conditions: this class is referred to as a **definable subcategory** of Mod-R. (The name comes about because such classes of modules are axiomatised, or defined in the precise model-theoretic sense, by the corresponding set of sentences $\forall \overline{x}\, (\phi_\lambda(\overline{x}) \to \psi_\lambda(\overline{x}))$.) This is the sense in which the closed subsets of Zg_R parametrise the definable subcategories of Mod-R.

There are various equivalent characterisations of definable subcategories.

Theorem 4.4. *For a class $\mathcal{D}$ of R-modules, closed in* Mod-R*, under isomorphism, the following are equivalent:*
(i) $\mathcal{D}$ forms a definable subcategory of Mod-R*;*
(ii) $\mathcal{D}$ is closed in Mod-R *under direct products, direct limits and pure submodules;*
(iii) $\mathcal{D}$ is closed in Mod-R *under ultraproducts, finite direct sums and pure submodules;*
(iv) there is a set $f_\lambda : A_\lambda \to B_\lambda$ of morphisms in mod-R *such that*
$\mathcal{D} = \{M : \mathrm{Hom}(f_\lambda, M) : \mathrm{Hom}(B_\lambda, M) \to \mathrm{Hom}(A_\lambda, M)$ *is surjective* $\forall \lambda\}$.

These categories also can be characterised as the categories of exact functors on small abelian categories, see [7], [13].

It is also the case that any definable subcategory is closed under pure quotients; indeed, if $0 \to L \to M \to N \to 0$ is a pure-exact sequence, then $M \in \mathcal{D}$ iff $L, N \in \mathcal{D}$.

Furthermore, if $M \in \mathcal{D}$ then the pure-injective hull $H(M)$ of M is also in $\mathcal{D}$.

Every definable subcategory is determined by the pure-injective modules in it, indeed by the set of indecomposable pure-injective modules in it. We say that a pure-injective module N is an **elementary cogenerator** for the definable subcategory $\mathcal{D}$ if every module in $\mathcal{D}$ purely embeds in some power N^I of $\mathcal{D}$. Every definable category has an elementary cogenerator, which may be taken, minimally, to be the pure-injective hull of the direct sum of the *neg-isolated* pure-injectives in (the closed set corresponding to) $\mathcal{D}$ - for these see below.

Example 4.5. Let R be a tubular canonical algebra. Every finite-dimensional indecomposable module has a slope (indeed, every indecomposable module has a slope [17]) which (with some exceptions that we can ignore) is a non-negative rational or ∞, such that if there is a non-zero morphism $A \to B$ with A, B finite-dimensional indecomposables, then the slope of A is $\leq$ the slope of B.

Then we can use the Compactness Theorem to show that there are modules of irrational slope (see [5]), indeed the modules of irrational slope form a non-zero definable subcategory and so there are, for each positive irrational r, indecomposable pure-injective modules, necessarily infinite-dimensional, of slope r.

4.3 Functor categories

There are many useful functor categories around; here we very briefly discuss two.

$(\text{mod-}R, \mathbf{Ab})^{\text{fp}}$ - the category of finitely presented functors (we always mean additive functors) on finitely presented right modules.

Note that every pp formula ϕ, and hence every pp-pair ϕ/ψ, defines a functor, let us denote it F_ϕ, on Mod-R which, as already remarked, is determined by its restriction to mod-R (since it commutes with direct limits and since every module is a direct limit of finitely presented modules). In fact, each of these (restricted) functors is finitely presented in the large functor category $(\text{mod-}R, \mathbf{Ab})$ *and conversely*, every finitely presented functor on mod-R has this form. (Indeed, more is true, the natural transformations also are definable by pp formulas.)

We can interpret a free realisation $(A, \overline{c})$ of a pp formula ϕ in this category as being a projective precover $(A, -) \to F_\phi \to 0$ of the functor F_ϕ, where c appears through the morphism $c : R_R \to A$ Yoneda-transformed to $(c, -) : (A, -) \to (R, -)$, with F_ϕ being exactly the image of $(c, -)$.

$(R\text{-mod}, \mathbf{Ab})$ - the category of all functors on finitely presented left R-modules. There is a full and faithful embedding Mod-$R \to (R\text{-mod}, \mathbf{Ab})$ given on objects by $M \to (M \otimes_R -)$. This functor takes pure-exact sequences in the module category to exact sequences in the functor category and takes pure-injective modules to injective functors.

Indeed, the indecomposable pure-injectives are taken to exactly the indecomposable injective functors, and so the Ziegler topology becomes a topology on the indecomposable injectives of the functor category (the Zariski topology on the set of those injectives is, in fact, the dual of the Ziegler topology).

A number of results about pure-injectives that we have mentioned become corollaries of corresponding results on injectives (giving easy proofs) and, for instance, Ziegler's criterion becomes an expression of the fact that an indecomposable injective is uniform.

We say that an indecomposable pure-injective N is **neg-isolated** if $N \otimes -$ is the injective hull of a simple functor. This is weaker than being isolated in Zg_R, which is equivalent to $N \otimes -$ being the injective hull of a finitely presented simple functor.

4.4 Multisorted modules

So far, everything has been said for modules over a ring but, in fact, essentially everything holds true, and by essentially the same proofs, for "multisorted modules" by which we mean additive functors from a skeletally small preadditive category to **Ab** (see [10]). For example, the objects of the functor category (mod-R, $\mathbf{Ab}$) are multisorted modules, so all the above applies also to this category in place of Mod-R (that is the case where the preadditive category has a single object). The only point to note is that the whole space need no longer be compact, because in general no single compact open set is equal to the whole space.

This viewpoint, and fact that multisorted modules, additive functors on small preadditive categories, and representations of quivers are all equivalent, is explained in [15].

In particular, any finitely accessible category (see [1]) with products falls under the description of being a category of multisorted modules, so all the above applies to comodules, sheaves of modules, quasicoherent sheaves, and a variety of other types of structure (in general with some sort of condition to ensure the finite accessibility, though definable categories of multisorted modules are more general: though accessible, they need not be *finitely* accessible).

Finally, we point out that a lot of this can be developed for triangulated categories, see [2], [3], [4], [8].

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