Model-theoretic imaginaries and localisation for additive categories

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$$\mathbb{ABEX} \overset{\simeq^{\mathrm{op}}}{\text{———}} \mathbb{DEF}, \quad \mathbb{ABEX} \overset{\simeq^{\mathrm{op}}}{\text{———}} \mathbb{COH}, \quad \mathbb{DEF} \overset{\simeq}{\text{———}} \mathbb{COH}$$

$$\mathcal{A} = \mathbb{L}^{\mathrm{eq}+}(\mathcal{D}) = \mathcal{G}^{\mathrm{fp}} \rightleftarrows \mathcal{D} = \mathrm{Abs}(\mathcal{G}) = \mathrm{Ex}(\mathcal{A},$$

$$\mathcal{G} = \text{Flat-}\mathcal{A} = \mathrm{Ind}\big(\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})\big)$$

$\mathbb{ABEX}$ - the 2-category of skeletally small abelian categories and exact functors (and natural transformations); $\mathbb{COH}$ - the 2-category of locally coherent Grothendieck categories and coherent morphisms; $\mathbb{DEF}$ - the 2-category of definable additive categories and interpretation functors (see [1]).

This picture has mostly been developed within representation theory/model theory of modules but it is an analogue of the picture for toposes (the **Set**-, as opposed to **Ab**-based world) and is in part a special case of the regular logic picture. But with special features. In particular, everything has a dual.

For example (the original example): take $\mathcal{D} =$ the category of modules over a ring and consider the model theory of these structures.

Fix a ring R with 1 (or any skeletally small preadditive category, such as $\mathrm{mod}\text{-}R$).
$\mathrm{Mod}\text{-}R = (R^{\mathrm{op}}, \mathbf{Ab})$ is the category of right R-modules
$\mathrm{mod}\text{-}R$ is the category of finitely presented right R-modules

The model theory of modules (over a fixed ring) started with Smielew's proof ([5], 1954) of decidability for $\mathbf{Ab} = \mathrm{Mod}\text{-}\mathbb{Z}$; it was developed through the early/mid 70s (Fisher, Eklof and Sabbagh, Garavaglia), was transformed by Baur's proof (1976) of pp-elimination of quantifiers and then by Ziegler. The model theory *per se*, applications (especially to the representation theory of finite-dimensional algebras) and links with the functor-category approach (of Auslander and Reiten, Gruson, Jensen and Lenzing) to modules developed through the 80s. The subject was transformed again in the early 90s by Herzog's elementary duality and the subsequent amalgamation of the model-theoretic ("imaginaries") and functor-category-theoretic viewpoints/techniques. The above picture, which essentially comes from a 2010 paper [2] of Prest and Rajani, has *very* slowly emerged.

As usual in model theory, we are interested in the **definable subsets** of modules: the solution sets of equations, the boolean combinations (finite intersections, unions, complements) of these and their projections.

To retain an induced additive structure, we should restrict to those which are **pp-definable**, that is, solution sets of **pp** (positive primitive, = regular) **formulas** - those of the form $\exists \overline{y}\ \overline{xy}H = \overline{0}$ where H is a matrix over R (thus: projected systems of linear equations). The solution set of a pp formula in any module is a subgroup.

Theorem (pp-elimination of quantifiers for modules; Baur [4] and others)

(1) Every definable subset of a module M is a finite boolean combination of cosets of pp-definable subgroups.
(We don't need (2).)

We order pp formulas (in the same free variables) by $\psi(\overline{x}) \leq \varphi(\overline{x})$ if $\psi(M) \leq \varphi(M)$ for every R-module M; we then say that φ, ψ form a **pp-pair**.

Corollary

The model theory of modules fits well with the algebra.

Specifically, because pp formulas are preserved by homomorphisms (that is, for φ pp, the assignment $M \mapsto \varphi(M)$ gives a functor from $\mathrm{Mod}\text{-}R$ to $\mathbf{Ab}$), the ordinary algebraic category of modules (with all homomorphisms for the maps, rather than that just the, model-theoretically nice, elementary embeddings) is a good context for the model theory of modules.
Furthermore, the elementary embeddings can be replaced by the pure embeddings (and these are algebraically natural), and the pure-injective (=algebraically compact = saturated for sets of pp formulas) modules play the role usually taken by the saturated models.

An embedding of modules $A \to B$ is **pure** if for every pp formula φ we have $\varphi(A) = A \cap \varphi(B)$.
A module N is **pure-injective** if it is injective over pure embeddings. This is equivalent to the notion of **algebraic compactness** (the aforementioned pp-saturation).

Imaginaries

We enrich the usual 1-sorted language for modules by adding new ("imaginary") sorts and define $\mathbb{L}_R^{\mathrm{eq}+}$ to be the category with objects the (sorts given by) pp-pairs ϕ/ψ and with arrows from ϕ/ψ to ϕ'/ψ' being the pp-definable maps from ϕ/ψ to ϕ'/ψ'. Then $\mathbb{L}_R^{\mathrm{eq}+}$, the **category of pp-imaginaries** for R-modules, is a small abelian category (Herzog [6]) - which turns out to be equivalent to the category of finitely presented functors on finitely presented modules.

Theorem (Burke, [5])

$\mathbb{L}_R^{\mathrm{eq}+} \simeq (\mathrm{mod}\text{-}R, \mathbf{Ab})^{\mathrm{fp}}$.

Herzog noticed that this is also equivalent to the opposite of the free abelian category on R (Freyd [4], Adelman [1]) though neither of those descriptions of $\mathbb{L}_R^{\mathrm{eq}+}$ applies to general definable categories (which need not be locally finitely presented).

Here's the picture for this case (at $\mathcal{D} = \mathrm{Mod}\text{-}R$).

$$\mathrm{Ab}(R^{\mathrm{op}}) = \mathbb{L}_R^{\mathrm{eq}+} = (\mathrm{mod}\text{-}R, \mathbf{Ab})^{\mathrm{fp}} \;\text{———}\; \mathrm{Mod}\text{-}R$$

$$(\mathrm{mod}\text{-}R, \mathbf{Ab})$$

The Ziegler spectrum

Let pinj_R be the set of isomorphism classes of (direct-sum-)indecomposable pure-injective modules. This is the underlying set of a topological space, a basis of (compact) open sets being the collection of sets of the form

$$(\varphi/\psi) = \{N \in \mathrm{pinj}_R : |\varphi(N)/\psi(N)| > 1\}$$

with $\psi \leq \varphi$ a pp-pair. This - the (right) **Ziegler spectrum** of R - is (if R is a ring) a (quasi)compact space which sees a lot of the model theory of R-modules.

The right and left Ziegler spectra of R are homeomorphic at the level of topology (i.e. as locales) and often (always?) at the level of points.

Equipped with the Hochster dual (complements of compact open sets for a basis of opens) of the Ziegler topology, pinj_R is the Gabriel spectrum of the (Grothendieck abelian) category $(R\text{-}\mathrm{mod}, \mathbf{Ab})$.

The lattice of pp formulas

Another useful structure is the lattice, pp_R of pp formulas (in one or, more generally, pp_R^n, in n, free variables). This is equivalent to the lattice of finitely generated subfunctors of the (nth power of the) forgetful functor from $\mathrm{mod}\text{-}R$ to $\mathbf{Ab}$ and is also equivalent to the lattice of (n-)pointed finitely presented modules.

Duality also appears here: if $\varphi(\overline{x})$ is a pp formula, say it is $\exists \overline{y}\ (\overline{x}, \overline{y}) \begin{pmatrix} A \\ B \end{pmatrix} = 0$ where A, B are matrices with entried in R, then the **elementary dual** of φ is the pp formula for left modules which is $\exists \overline{z} \begin{pmatrix} I_n & A \\ 0 & B \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{z} \end{pmatrix} = 0$ where I_n is the $n \times n$ identity matrix and 0 denotes a zero matrix of the correct size.

Theorem (Prest [5])

Elementary duality of pp formulas induces an anti-isomorphism of lattices of pp formulas $\mathrm{pp}_R^n \simeq \left(\mathrm{pp}_{R^{\mathrm{op}}}^n\right)^{\mathrm{op}}$. *In particular* $D^2\varphi(M) = \varphi(M)$ *for every* M, *and* $\psi \leq \varphi$ *iff* $D\varphi \leq D\psi$.

A **definable subcategory** of $\mathrm{Mod}\text{-}R$ is the full subcategory consisting of those objects which satisfy a given set of conditions of the form $\varphi(-) = \psi(-)$ where $\psi \leq \varphi$ is a pp-pair. These subcategories have alternative descriptions.

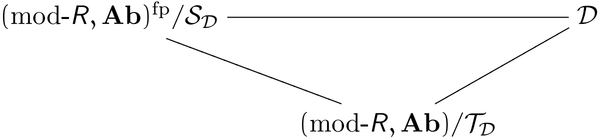
Theorem

A subcategory of $\mathrm{Mod}\text{-}R$ is definable iff it is closed under direct products, directed colimits and pure submodules.

Theorem (Ziegler [6])

If $\mathcal{D}$ is a definable subcategory of $\mathrm{Mod}\text{-}R$ then the indecomposable pure-injectives in it form a closed subset of the Ziegler spectrum. This gives a bijection between closed subsets of the spectrum and definable subcategories of $\mathrm{Mod}\text{-}R$.

The picture for $\mathrm{Mod}\text{-}R$ localises to that for a definable subcategory $\mathcal{D}$.



Here $\mathcal{S}_{\mathcal{D}}$ is the Serre subcategory consisting of those functors $F \in (\mathrm{mod}\text{-}R, \mathbf{Ab})^{\mathrm{fp}}$ (rather, the $\varinjlim$-commuting extensions of these to functors on $\mathrm{Mod}\text{-}R$) which are 0 on $\mathcal{D}$, and then $\mathcal{T}_{\mathcal{D}}$ is the hereditary (finite type) torsion subcategory of $(\mathrm{mod}\text{-}R, \mathbf{Ab})$ generated by this.

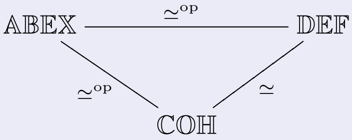
The "imaginaries" interpretation of the associated small abelian category also localises. Denote by $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$ the category of pp-imaginaries for $\mathcal{D}$: this has the same objects as $\mathbb{L}_R^{\mathrm{eq}+} = \mathbb{L}^{\mathrm{eq}+}(\mathrm{Mod}\text{-}R)$ but now the arrows are the pp-definable relations which are functional when restricted to $\mathcal{D}$. This is the quotient of $\mathbb{L}_R^{\mathrm{eq}+}$ by the Serre subcategory consisting of those pp-sorts which are 0 on (every object of) $\mathcal{D}$.

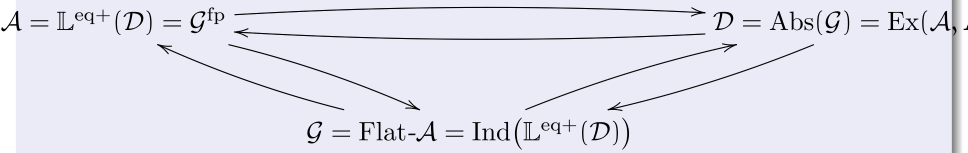
Every small abelian category arises in this way.

So let's return to the general picture. Recall that the notation R stands for any skeletally small preadditive category.

Theorem

There is a diagram of equivalences and anti-equivalences between $\mathbb{ABEX}$, $\mathbb{COH}$ *and* $\mathbb{DEF}$ *as follows.*





Theorem (variously Herzog, Hu, Krause, Makkai, Prest ([8], [1], [4], [8]))

(1) A definable category $\mathcal{D}$ *can be recovered from its category of imaginaries as* $\mathrm{Ex}(\mathbb{L}^{\mathrm{eq}+}(\mathcal{D}), \mathbf{Ab})$ *- the category of exact additive functors from* $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$ *to* $\mathbf{Ab}$*.*
(2) In the other direction, we have an intrinsic definition of the "finitely presented functor" category associated with a definable category $\mathcal{D}$*, namely* $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D}) \simeq (\mathcal{D}, \mathbf{Ab})^{\prod, \underrightarrow{\lim}}$ *- the category of functors on* $\mathcal{D}$ *which commute with direct products and directed colimits - we also write this as* $\mathrm{fun}(\mathcal{D})$*.*
More generally, if $\mathcal{C}$ *and* $\mathcal{D}$ *are definable categories, then the interpretation functors* $I : \mathcal{D} \to \mathcal{C}$ *are in natural (2-categorical) bijection with the exact functors* $\mathrm{fun}(\mathcal{C}) \to \mathrm{fun}(\mathcal{D})$ *(alternatively written* $\mathbb{L}^{\mathrm{eq}+}(\mathcal{C}) \to \mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$*).*

Here an **interpretation functor** between definable categories is one which commutes with direct products and direct limits (equivalently, whence the name, it is an (additive) interpretation in the model-theoretic sense).

Duality

Duality runs through the whole picture. It is obvious for $\mathbb{ABEX}$, on which it is the 2-category equivalence which takes each abelian category to its opposite. It follows that there is a corresponding self-equivalence on each of the other two categories. Thus, for every definable category, there is a "dual" definable category, in particular the categories of right and left modules are dual in this sense. In the context of the model theory of modules, this duality was found first for pp formulas (as seen above); later this was extended to the category of pp-pairs and the Ziegler spectrum by Herzog [6]. In an algebraic form it is the (literal) duality of "finitely presented functor" categories, $\big((R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}\big)^{\mathrm{op}} \simeq (\mathrm{mod}\text{-}R, \mathbf{Ab})^{\mathrm{fp}}$, due to Auslander [3], and Gruson and Jensen [3].

Duality plays a key role in the additive theory.

Using this

Let's continue in the context of representations of (= modules over) finite-dimensional algebras, where a main concern is description/classification of the modules and morphisms between them. That usually means the finite-dimensional modules but the methods here lead naturally to some of the infinite-dimensional modules.

The finite-dimensional modules are pure-injective but, unless the algebra is of finite representation type, there are more points in the Ziegler spectrum than the finite-dimensional indecomposables. So description of the Ziegler spectrum is an obvious goal.

Another standard problem is determination of the difficulty of classification/parametrisation of the finite-dimensional indecomposables (tame and wild representation type and refinements of these). So the computation of invariants which are some measure of complexity of the category of modules is another natural problem.

One measure of complexity is the **m-dimension** of the lattice, pp_R, of pp formulas in one free variable = the lattice of finitely generated subfunctors of the forgetful functor. Suppose that L is a modular lattice. At the first stage of the m-dimension analysis of L we factor L by the congruence relation generated by the intervals of finite length: points a and b will be identified iff the interval $[a+b, a \wedge b]$ is of finite length. The result is again a modular lattice, and we repeat the process, transfinitely, where at limit stages we factor L by the union of the inverse images in L of the congruence relations generated so far. If the procedure stabilises with a nontrivial lattice (necessarily densely ordered) then we say that the m-dimension of L is ∞ or undefined. Otherwise, provided L has a top and a bottom, the first ordinal μ such that the μ-collapse of L is trivial is not a limit, and then we say that the m-dimension of L is $\mu - 1$.

Another measure of complexity of $\mathrm{Mod}\text{-}R$ is the **Krull-Gabriel dimension** (KG-dimension) of the functor category $(\mathrm{mod}\text{-}R, \mathbf{Ab})^{\mathrm{fp}}$. A finitely presented functor of finite length is assigned KG-dimension 0. We then localise, forming the quotient of the category $(\mathrm{mod}\text{-}R, \mathbf{Ab})^{\mathrm{fp}}$ by the Serre subcategory consisting of the functors of finite length. If the image of a functor in the resulting quotient category is of finite length (but non-zero) then we assign that functor KG-dimension 1. The process continues, transfinitely. The KG-dimension of R is the maximum (if R is a ring) of ranks of functors, or ∞ if one of the localised categories is nonzero but has no functors of finite length.

The **Cantor-Bendixson rank** of Zg_R is a measure of complexity of that space, and hence of $\mathrm{Mod}\text{-}R$: isolated points of a topological space X are assigned rank 0; these points are removed and isolated points in the remaining space are assigned rank 1; etc. (transfinitely); if, at some stage, there are remaining points but no isolated ones then all those points are assigned rank ∞. The Cantor-Bendixson rank of X, $\mathrm{CB}(X)$ is the maximum rank of points of X.

Theorem

For any ring R, we have $\mathrm{KG}(R) = \mathrm{mdim}(\mathrm{pp}_R)$. If this value is $< \infty$ then it also equals $\mathrm{CB}(\mathrm{Zg}_R)$ (this equality of KG-dimension with CB-rank also is true under various other conditions - maybe always?).

If R is a finite-dimensional algebra the finite-dimensional indecomposables are exactly the isolated points of Zg_R and, together, they are dense in the space. In fact, a module N is pure-injective iff N is a direct summand of a direct product of finite-dimensional modules.

For such algebras, the value of $\mathrm{KG}(R)$ does, as far as has been verified, reflect the algebraic complexity of mod-R (more precisely, the representation type of R). For a finite-dimensional algebra (more generally for an artin algebra) R:

• $\mathrm{KG}(R) = 0$ iff R is of finite representation type;
• $\mathrm{KG}(R) = 1$ is impossible for such rings (Herzog [7], Krause [2])
• $\mathrm{KG}(R) = 2$ for tame hereditary (Prest [6], Ringel [4] at the end of a long chain of people), and some other, algebras;
• $\mathrm{KG}(R)$ can be any finite value ≥ 2 and, for domestic string algebras, must be finite (Laking, Prest, Puninski [3])
• Conjecturally, $\mathrm{KG}(R)$ is finite iff it is defined iff R is of domestic representation type;
• $\mathrm{KG}(R) = \infty$ for some tame algebras (nondomestic string algebras, nondomestic tubular algebras, Harland, Prest, Puninski [5], [3])
• $\mathrm{KG}(R) = \infty$ for wild algebras (Gregory and Prest [2], at the end of a long chain of people).

Suppose that R and S are finite-dimensional algebras.

A **representation embedding** from $\mathrm{mod}\text{-}S$ to $\mathrm{mod}\text{-}R$ is a functor $F : \mathrm{mod}\text{-}S \to \mathrm{mod}\text{-}R$ which is exact, preserves indecomposability and reflects isomorphism. Such a functor has the form $M \mapsto M \otimes_S B_R$ for an (S, R)-bimodule B which is finitely generated and projective as a left S-module. The idea is that if there is such a representation embedding then it is at least as hard to understand R-modules as it is to understand S-modules.

We can also ask whether such a functor "preserves the complexity of the category of S-modules" in other senses. For example, it should be non-decreasing on dimensions which are some measure of this complexity (Krull-Gabriel dimension = m-dimension; uniserial dimension = breadth/width). Indeed, this is so by the following result.

Theorem (Gregory and Prest, [2])

If R, S are finite-dimensional algebras then any representation embedding from $\mathrm{mod}\text{-}S$ to $\mathrm{mod}\text{-}R$ induces, for some n, an embedding of lattices $\mathrm{pp}_S^1 \to \mathrm{pp}_S^n$ and hence is non-decreasing on the above dimensions.

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