ZIEGLER SPECTRA OVER SERIAL RINGS AND COALGEBRAS

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Contents

Abstract

The *Ziegler spectrum* of a locally finitely presented category $\mathcal{C}$ is a topological space, denoted by $Zg(\mathcal{C})$, the points of which are the isomorphism classes of the *indecomposable pure-injective* objects. This notion is a generalisation of what is known as the Ziegler spectrum over a ring; the right (left) Ziegler spectrum over a ring R is a topological space, denoted by Zg_R $({}_RZg)$ the points of which are the indecomposable pure-injective right (left) R-modules. Ziegler ([37]) showed that there is a one-to-one correspondence between the complete theories for right (left) R-modules closed under products and the closed subsets of Zg_R $({}_RZg)$.

In Part I we compare the complexity of the Ziegler spectrum over a *serial* ring R (*CB-rank*) with the complexity of the ring R (*Krull dimension*). Using results obtained by Eklof, Herzog and Puninski ([5] and [22]), we show that if R is a serial ring with Krull dimension, then Zg_R has CB-rank and for every point in Zg_R, its CB-rank (which measures its "degree of isolation" in Zg_R) is equal to its m-dimension (which measures its "model-theoretic complexity"). We also characterise the serial *Krull-Schmidt* rings with Krull dimension in terms of CB-rank.

In Part II we take the first steps into model theory of *comodules*. The category

of right comodules over a coalgebra C, denoted by $\mathcal{M}^C$, is equivalent to the category of rational left C^*-modules (where C^* is the dual algebra of C), which we denote by C^*-Rat. We will see that the category C^*-Rat (or, equivalently, $\mathcal{M}^C$) is locally finitely presented (and Abelian), and therefore *axiomatisable* in some finitary language. However, we can find a condition on the algebra C^* under which the class of rational left C^*-modules is not elementary, i.e. the rational C^*-modules are not generally axiomatisable in a finitary language for left C^*-modules. It is interesting that under the same condition on the algebra C^*, $Zg(C^*\text{-}Rat)$ is embedded (as a topological space) in ${}_{C^*}Zg$.

Declaration

No part of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or institute of learning.

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The Author

I graduated in 1995 at the *Katholieke Universiteit Leuven* (Belgium) in pure mathematics. The title of my dissertation, written under the supervision of Alfred Warrinnier, was *Finite Numbers According to Gottlob Frege's Grundlagen der Arithmetik*. Prior to my graduation, I spent three months at *The University of Manchester* as an *Erasmus* student.

I started my research at *The University of Manchester* in September 1995. A slightly more concise version of Part I of this thesis will be published in *Communications in Algebra*, under the title *Ziegler Spectra of Serial Rings with Krull Dimension*.

Basics

The material presented in this chapter is relevant for both Part I and Part II. With a little background from model theory for modules it is however possible to read Part I without first reading this chapter.

0.1 Category Theory

For further details about the basic category-theoretic notions presented in this section, we refer the reader to [30], [17] or any textbook on category theory.

By a *functor* (between categories) we mean a covariant functor. Let $\alpha : A \longrightarrow B$ be a morphism in a category. Its kernel (if it exists) is denoted by $ker(\alpha) : Ker(\alpha) \longrightarrow A$ and its cokernel (if it exists) is denoted by $coker(\alpha) : B \longrightarrow Coker(\alpha)$. $Ker(coker(\alpha))$ (if it exists), called the *image of* α, is denoted by $Im(\alpha)$.

Let $\{A_i\}$ be a family of objects in a category. The *direct sum* (= coproduct) of the objects A_i is denoted by $\oplus_i A_i$ (if it exists) and their product (= direct product) is denoted by $\prod_i A_i$ (if it exists). By an *indecomposable* object we mean a direct sum-indecomposable object, i.e. an object which is not a direct sum of proper subobjects.

Recall that a category is called *Abelian* if

(1) $\mathcal{C}$ is preadditive (i.e. for every pair A, B of objects, the set of morphisms from A to B, $Hom(A, B)$, has the structure of an Abelian group such that for all objects A, B, C and for all $\alpha, \alpha' \in Hom(A, B)$ and all $\beta, \beta' \in Hom(B, C)$, $\beta \circ (\alpha + \alpha') = \beta \circ \alpha + \beta \circ \alpha'$ and $(\beta + \beta') \circ \alpha = \beta \circ \alpha + \beta' \circ \alpha$);

(2) every finite family of objects has a product (or, equivalently, every finite family of objects has a direct sum);

(3) every morphism has a kernel and a cokernel;

(4) every morphism α has a factorisation $\alpha = \gamma \circ \beta$ where β is a cokernel and γ is a kernel.

A category $\mathcal{C}$ is said to be *complete* if every functor $\mathbf{F} : \mathcal{D} \longrightarrow \mathcal{C}$ with $\mathcal{D}$ a small category (that is, the objects of $\mathcal{D}$ form a set) has a limit (= projective limit) which we denote by $lim\mathbf{F}$, and $\mathcal{C}$ is said to be *cocomplete* if every such $\mathbf{F}$ has a colimit (= inductive limit) which we denote by $colim\mathbf{F}$.

It is well-known that *an Abelian category is complete iff every family of objects has a product* and that *an Abelian category is cocomplete iff every family of objects has a direct sum.*

A poset $(P, \leq)$ is called *directed* if for each $x, y \in P$, there is $z \in P$ with $x \leq z$ and $y \leq z$. Every poset $(P, \leq)$ can be regarded as a category by associating to each pair $x, y \in P$ with $x \leq y$ a unique morphism $x \longrightarrow y$. A functor from a directed poset (regarded as a category) to a category $\mathcal{C}$ is called a *direct system* in $\mathcal{C}$. By a *direct limit* we mean a colimit of a direct system. A direct system $\mathbf{F} : (P, \leq) \longrightarrow \mathcal{C}$ is called a *direct system of monomorphisms* if for each morphism $x \longrightarrow y$ in the category $(P, \leq)$ (i.e. for each pair $x, y \in P$

with $x \leq y$), $\mathbf{F}(x \longrightarrow y)$ is a monomorphism in $\mathcal{C}$. A colimit of a direct system of monomorphisms is called a *direct limit of monomorphisms.*

An object in a category $\mathcal{C}$ is called *finitely presented* (*finitely generated*) if the functor $Hom(A,-) : \mathcal{C} \longrightarrow Set$ (where Set denotes the category of sets), preserves direct limits (direct limits of monomorphisms).

Below we will give more tangible descriptions for finitely presented and finitely generated objects in "nice" categories.

A monomorphism $\alpha : A \longrightarrow B$ (in any category) is said to be *split* if there is a morphism $\beta : B \longrightarrow A$ with $\beta \circ \alpha = id_A$.

Let $\{A_i\}$ be a family of subobjects of an object A in some category. The *sum* of the A_i, which we denote by $\sum_i A_i$, is defined as the image of the morphism $\oplus_i A_i \longrightarrow A$ (if the direct sum exists) which is induced by the monomorphisms $A_i \longrightarrow A$. The *intersection* of the A_i is defined as the kernel of the morphism $A \longrightarrow \prod_i A/A_i$ (if the product exists) which is induced by the epimorphisms $A \longrightarrow A/A_i$ (where $A/A_i := Coker(A_i \longrightarrow A)$).

A family $\{G_i\}$ of objects in a cocomplete Abelian category $\mathcal{C}$ is a *generating* family if for every object A of $\mathcal{C}$, there is an epimorphism $\oplus_{j \in J}(\oplus_i G_i) \longrightarrow A$, for some index set J. A cocomplete Abelian category is called *locally finitely generated* (*locally finitely presented*) if it has a generating family of finitely generated (finitely presented) objects.

A cocomplete Abelian category is called a *Grothendieck* category if it has a generating family and if it satisfies the following condition:

(Ab5) If A is a subobject of an object B and $\{B_i\}$ is a directed set of

subobjects of B, then

$$(\sum_i B_i) \cap A = \sum_i (B_i \cap A).$$

An object A of a Grothendieck category is finitely generated iff for every family $\{A_i\}$ of subobjects of A with $A = \sum_i A_i$ there is an index j with $A_j = A$.

An object A of a locally finitely generated Grothendieck category is finitely presented iff A is finitely generated and if every epimorphism $B \longrightarrow A$, with B finitely generated, has finitely generated kernel.

A monomorphism $\alpha : A \longrightarrow B$ (in any category) is said to be *split* if there is a morphism $\beta : B \longrightarrow A$ with $\beta \circ \alpha = id_A$.

A monomorphism $\alpha : A \longrightarrow B$ in a locally finitely presented Grothendieck category is said to be *pure* if for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\gamma} \\ A' & \xrightarrow[\alpha']{} & B' \end{array}$$

with A' and B' finitely presented objects, β factorises through α' (i.e. there exists a morphism $\kappa : B' \longrightarrow A$ with $\kappa \circ \alpha' = \beta$). It is clear what is meant by a *pure subobject.*

Let $\mathfrak{M}$ be a class of monomorphisms (in some category).

A monomorphism $\alpha : A \longrightarrow B$ in $\mathfrak{M}$ is called $\mathfrak{M}$-*essential* if for every morphism $\beta : B \longrightarrow C$, $\beta \in \mathfrak{M}$, whenever $\beta \circ \alpha \in \mathfrak{M}$. For $\mathfrak{M}$ the class of all monomorphisms (in a category), an $\mathfrak{M}$-essential monomorphism is called an *essential* monomorphism.

An object E is called $\mathfrak{M}$-*injective* if for every monomorphism $\alpha : A \longrightarrow B$ in $\mathfrak{M}$ and for every morphism $\beta : A \longrightarrow E$, there is a morphism $\gamma : B \longrightarrow E$ with $\gamma \circ \alpha = \beta$ (i.e. γ extends β via α). If $\mathfrak{M}$ is the class of all monomorphisms, then an $\mathfrak{M}$-injective object is called an *injective* object; a *pure-injective* object is an $\mathfrak{M}$-injective object where $\mathfrak{M}$ is the class of all pure monomorphisms.

An $\mathfrak{M}$-*injective hull* of an object A is defined as an $\mathfrak{M}$-injective object E for which there is a monomorphism $\alpha : A \longrightarrow E$ in $\mathfrak{M}$ such that whenever E' is another $\mathfrak{M}$-injective object for which there is a monomorphism $\beta : A \longrightarrow E'$ in $\mathfrak{M}$, then there is a monomorphism $\gamma : E \longrightarrow E'$ in $\mathfrak{M}$ with $\gamma \circ \alpha = \beta$ (or, equivalently, an $\mathfrak{M}$-injective hull of an object A is an $\mathfrak{M}$-injective object E for which there is an $\mathfrak{M}$-essential monomorphism $A \longrightarrow E$). For $\mathfrak{M}$ the class of monomorphisms (pure monomorphisms) in some category, an $\mathfrak{M}$-injective hull is called an *injective hull* (a *pure-injective hull*). $\mathfrak{M}$-injective hulls (if they exist) are unique up to isomorphism and in a Grothendieck category every object has an injective hull.

Throughout, all rings are associative and have a unit, and all modules are unital. In Part I we will work in *Mod-R*, the category of right modules over a fixed ring R. In Part II we will work with left modules; the category of left modules over a fixed ring R is denoted by *R-Mod*. For a ring R, *Mod-R* (*R-Mod*) is locally finitely presented Abelian (which is equivalent to being locally finitely presented Grothendieck) with R as finitely presented generator.

The above category-theoretic notions are generalisations of the module theoretic notions known under the same name. For the module-theoretic basics we refer the reader to [1] and [30].

0.2 Model Theory

In this section we introduce some basics from multi-sorted model theory. They are a generalisation of the perhaps more familiar one-sorted model-theoretic basics which can be found in [10] or in any textbook on model theory.

For a nonempty set S, which will be called the *set of sorts*, a *finitary S-sorted signature* $\mathcal{L}$ is specified by giving:

• for each $s \in S$ the set of constant symbols of sort s;

• for each finite subset $\{s_1, \ldots, s_n, s\}$ of S, the set of operation symbols of arity $s_1 \times \cdots \times s_n \longrightarrow s$;

• for each finite subset $\{s_1, \ldots, s_n\}$ of S, the set of relation symbols of arity $s_1 \times \cdots \times s_n$.

By a *finitary signature* we mean a finitary S-sorted signature for some nonempty set S; a *finitary multi-sorted signature* is a finitary S-sorted signature, where S has at least two elements; a *finitary λ-sorted signature* is a finitary S-sorted signature, where the cardinality of S is λ.

For the remainder of this section, let $\mathcal{L}$ be a finitary S-sorted signature.

An $\mathcal{L}$-*structure* A is an S-sorted set $|A| = (A_s)_{s\in S}$ (i.e. a collection of sets indexed by S) together with

• for each sort s and for each constant symbol c in $\mathcal{L}$ of sort s, a constant $c^A \in A_s$;

• for each operation symbol α in $\mathcal{L}$ of arity $s_1 \times \cdots \times s_n \longrightarrow s$, an operation $\alpha^A : A_{s_1} \times \cdots \times A_{s_n} \longrightarrow A_s$;

• for each relation symbol R in $\mathcal{L}$ of arity $s_1 \times \cdots \times s_n$, a relation $R^A \subseteq A_{s_1} \times \cdots \times A_{s_n}$.

For an element a of an $\mathcal{L}$-structure, $s(a)$ denotes the sort of a.

A *homomorphism* from an $\mathcal{L}$-structure A to an $\mathcal{L}$-structure B is an S-sorted function $f : |A| \longrightarrow |B|$ (i.e. $f = (f_s)_{s\in S}$ and $f_s : A_s \longrightarrow B_s$ for every $s \in S$) which preserves the operations and the relations, i.e.:

• for every operation symbol α in $\mathcal{L}$ of arity $s_1 \times \cdots \times s_n \longrightarrow s$,
$f_s^A(\alpha^A(a_1, \ldots, a_n)) = \alpha^B(f_{s_1}(a_1), \ldots, f_{s_n}(a_n))$;

• for every relation symbol R in $\mathcal{L}$ of arity $s_1 \times \cdots \times s_n$,
$(f_{s_1}^A \times \cdots \times f_{s_n}^A)(R^A) \subseteq R^B$.

The category of $\mathcal{L}$-structures, with objects the $\mathcal{L}$-structures and morphisms the homomorphisms of $\mathcal{L}$-structures, will be denoted by $Str(\mathcal{L})$. The category $Str(\mathcal{L})$ is known to be locally finitely presented (see [2] page 201).

If A is an $\mathcal{L}$-structure and B a subset of $|A|$, then the closure of B under the operations in $\mathcal{L}$ is denoted by $\langle B\rangle$; if $\langle B\rangle$ is equal to $|A|$, then A is said to be *generated by* B.

Using $\wedge$, $\vee$, $\neg$ and $\rightarrow$, together with the quantifiers $\exists$ and $\forall$, one builds up formulae from $\mathfrak{L}$ in the same way as one does from a 1-sorted signature, and one generalises the terminology (such as "variable", "term", "atomic formula", "sentence", "(consistent, complete, ...) theory", "model", "elementary equivalence", ...) well-known from the 1-sorted case to the multi-sorted case. We omit the details. One important aspect of multi-sorted languages one has to bear in mind is that to each variable x occurring in a formula is associated a certain sort $s(x)$, so that for example "$\exists x \varphi(x)$" is to mean "there exists an element of the sort $s(x)$ which satisfies φ".

The language built up from $\mathfrak{L}$ by allowing only finitely many conjunctions (and disjunctions), finitely many quantifiers and finitely many (free) variables in a formula, will be denoted by $\mathfrak{L}_\omega$; $\mathfrak{L}_\omega$ is called a *finitary language.*

Throughout, by a tuple (of elements of an $\mathfrak{L}$-structure or of variables in $\mathfrak{L}_\omega$) we mean a finite tuple.

Let T be a theory (i.e. a collection of sentences) in $\mathfrak{L}_\omega$. The full subcategory of $Str(\mathfrak{L})$, the objects of which are the models of T, will be denoted by $Mod(T)$. If $A \in Str(\mathfrak{L})$, then $Th(A)$ denotes the theory $\{\sigma : \sigma$ is a sentence in $\mathfrak{L}_\omega$ for which $A \models \sigma\}$.

Until further notice, we assume that the finitary S-sorted signature $\mathfrak{L}$ has no relation symbols.

Let A be an $\mathfrak{L}$-structure. A collection $\sim = (\sim_s)_{s \in S}$ of equivalence relations on A_s ($s \in S$) is called a *congruence relation* on A, if for every operation symbol α in $\mathfrak{L}$ of arity $s_1 \times \cdots \times s_n \longrightarrow s$, $\alpha^A(a_1, \ldots, a_n) \sim_s \alpha^A(a'_1, \ldots, a'_n)$, whenever $a_i \sim_{s_i} a'_i$ ($1 \leq i \leq n$), i.e. the operations respect the equivalence relations $\sim_s$. If $\sim$ is a congruence relation on A, then $A/\sim$ denotes the S-

sorted set $A_s/\sim_s)_{s\in S}$, where $A_s/\sim_s$ denotes the set of $\sim_s$-equivalence classes of A_s, with $\mathfrak{L}$-structure given as follows: if α is an operation symbol in $\mathfrak{L}$ of arity $s_1 \times \cdots \times s_n \longrightarrow s$, then $\alpha^{A/\sim} : A_{s_1}/\sim_{s_1} \times \cdots \times A_{s_n}/\sim_{s_n} \longrightarrow A_s/\sim_s$ takes $([a_1], \ldots, [a_n])$ to $[\alpha^A(a_1, \ldots, a_n)]$, where $[\quad]$ is used to denote the appropriate equivalence classes.

For X a set of variables in the language $\mathfrak{L}_\omega$, $\langle X\rangle$ denotes the formal closure of X under the operations in $\mathfrak{L}$, $\langle X\rangle$ is the set of terms in $\mathfrak{L}_\omega$ built up from the elements of X (since we assume that $\mathfrak{L}$ has no relation symbols, $\langle X\rangle$ is in fact an $\mathfrak{L}$-structure).

A consistent theory in $\mathfrak{L}_\omega$ whose sentences are of the form $\forall\overline{x}(t(\overline{x}) = t'(\overline{x}))$, where $t(\overline{x})$ and $t'(\overline{x})$ are terms in $\mathfrak{L}_\omega$, is called a (finitary) *equational theory.* Let T be an equational theory in $\mathfrak{L}_\omega$ and let X be a set of variables in $\mathfrak{L}_\omega$. By $\langle X\rangle_T$ we denote the quotient of the $\mathfrak{L}$-structure $\langle X\rangle$ modulo the smallest congruence relation $\sim$ with $t(\overline{x}) \sim t'(\overline{x})$ (where $\overline{x}$ is a tuple in X) for every sentence "$\forall\overline{y}t(\overline{y}) = t'(\overline{y})$" in T with $s(\overline{x}) = s(\overline{y})$. For $x \in X$, we will not introduce a new notation to denote its equivalence class modulo the congruence relation $\sim$ based on the equational theory T. $\langle X\rangle_T$ is a model of T and it is *free* in the sense that for every $M \in Mod(T)$, if $\overline{f} : X \longrightarrow |M|$ is an S-sorted function, then there is a unique morphism $f : \langle X\rangle_T \longrightarrow M$ in $Mod(T)$ extending $\overline{f}$. If T is a finitary equational theory, then the class of models of T in $Str(\mathfrak{L})$ is called a *variety* in $Str(\mathfrak{L})$. A variety in $Str(\mathfrak{L})$ is, as a category, locally finitely presented ([2](3.7)), and ([2](3.13)), the finitely presented objects in a variety $Mod(T)$ are the objects (which are isomorphic to objects) of the form $\langle X\rangle_T/\sim$, where $X = \{x_1, \ldots, x_n\}$ is a finite set of variables in $\mathfrak{L}_\omega$ and $\sim$ is the smallest congruence relation on $\langle X\rangle_T$ with

$t_i(\overline{x}) \sim t'_i(\overline{x})$ for $1 \leq i \leq m$, with the t_i and t'_i terms in $\mathfrak{L}_\omega$ and m some positive integer (i.e. the finitely presented objects in a variety $Mod(T)$ are the objects in $Mod(T)$ which are presented by finitely many generators and finitely many equations in $Mod(T)$). A class in $Str(\mathfrak{L})$ is a variety in $Str(\mathfrak{L})$ iff it is closed under subobjects, quotient objects and products in $Str(\mathfrak{L})$ ([2](3.9)).

We allow again relation symbols in $\mathfrak{L}$.
A formula in $\mathfrak{L}_\omega$ is said to be a *positive primitive formula* (in n free variables), or a *pp-formula* for short, if it is of the form

$$\exists y_1, \dots, y_k \psi(x_1, \dots, x_n, y_1, \dots, y_k)$$

where $\psi(x_1, \dots, x_n, y_1, \dots, y_k)$ is a conjunction of atomic formulae.
It is easy to see that if $\alpha : A \longrightarrow B$ is a morphism in $Str(\mathfrak{L})$, then for every *pp*-formula $\varphi(\overline{x})$, $B \models \varphi(\alpha(\overline{a}))$ whenever $A \models \varphi(\overline{a})$.
pp-formulae will be of great interest throughout these notes. We state the following result which describes purity in $Str(\mathfrak{L})$ in terms of *pp*-formulae:

Proposition 0.2.1 ([2](5.15))
Let $\mathfrak{L}$ be a finitary signature and let A be a substructure of an $\mathfrak{L}$-structure B. Then A is a pure subobject of B in $Str(\mathfrak{L})$ iff for every pp-formula $\varphi(\overline{x})$ in $\mathfrak{L}_\omega$ and for every tuple $\overline{a}$ in A, $A \models \varphi(\overline{a}) \Leftrightarrow B \models \varphi(\overline{a})$.

In case $\mathfrak{L}$ has no relation symbols, we can show the following more general result, the proof of which is essentially the same as the proof of [2](5.15).

Lemma 0.2.2 *Let $\mathfrak{L}$ be a finitary S-signature with no relation symbols and let T be an equational theory in $\mathfrak{L}_\omega$.*
Then, if A and B are models of T with A a substructure of B, A is a pure subobject of B in $Mod(T)$ iff A is a pure subobject of B in $Str(\mathfrak{L})$.

Proof. ($\Rightarrow$) Let A be a pure subobject of B in $Mod(T)$ and let

$$\varphi(\overline{x}) \equiv \exists y_1, \dots, y_m \bigwedge_{i=1}^{k} t_i(x_1, \dots, x_n, y_1, \dots, y_m) = t'_i(x_1, \dots, x_n, y_1, \dots, y_m)$$

be a *pp*-formula in $\mathfrak{L}_\omega$ (so the t_i and t'_i are terms in $\mathfrak{L}$) with

$$B \models \bigwedge_{i=1}^{k} t_i(a_1, \dots, a_n, b_1, \dots, b_m) = t'_i(a_1, \dots, a_n, b_1, \dots, b_m)$$

for some tuple $\overline{a} = (a_1, \dots, a_n)$ in A and some tuple $\overline{b} = (b_1, \dots, b_m)$ in B.
Let $\sim$ be the smallest congruence relation on $\langle \overline{x}, \overline{y} \rangle_T$ with $t_i(\overline{x}, \overline{y}) \sim t'_i(\overline{x}, \overline{y})$ for all $1 \leq i \leq k$. Put $Q = \langle \overline{x}, \overline{y} \rangle_T / \sim$ and put $P = \langle \overline{x} \rangle_T$.
Define the function $\overline{g} : \{\overline{x}, \overline{y}\} \longrightarrow |B|$ by $\overline{g}(x_i) = a_i$ ($1 \leq i \leq n$) and $\overline{g}(y_i) = b_i$ ($1 \leq i \leq m$). Then there is a unique morphism $g : Q \longrightarrow B$ in $Mod(T)$ extending $\overline{g}$ (because $B \in Mod(T)$ and $B \models \bigwedge_{i=1}^{k} t_i(a_1, \dots, a_n, b_1, \dots, b_m) = t'_i(a_1, \dots, a_n, b_1, \dots, b_m)$).
Define the function $\overline{g'} : \{\overline{x}\} \longrightarrow |A|$ by $\overline{g'}(x_i) = a_i$. Then there is a unique morphism $g' : P \longrightarrow A$ in $Mod(T)$ extending $\overline{g'}$ (since $A \in Mod(T)$).
Let $f : P \longrightarrow Q$ denote the morphism in $Mod(T)$ given by $f(x_i) = x_i$.
Then the diagram

$$\begin{tikzcd} A \arrow[r] & B \\ P \arrow[u, "g'"] \arrow[r, "f"'] & Q \arrow[u, "g"'] \end{tikzcd}$$

(where the morphism on top is the embedding of A in B) is commutative, so that, since both P and Q are finitely presented objects in $Mod(T)$ and A is a pure subobject of B in $Mod(T)$, there is a morphism $h : Q \longrightarrow A$ with $h \circ f = g'$. Since

$$Q \models \bigwedge_{i=1}^{k} t_i(x_1, \ldots, x_n, y_1, \ldots, y_m) = t'_i(x_1, \ldots, x_n, y_1, \ldots, y_m),$$

this implies that $A \models \varphi(\overline{a})$.

We conclude that, by Proposition 0.2.1, A is a pure subobject of B in $Str(\mathfrak{L})$.

($\Leftarrow$) Let A be a pure subobject of B in $Str(\mathfrak{L})$ and let

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g'} \\ P & \underset{f}{\longrightarrow} & Q \end{array}$$

be a commutative diagram where P and Q are finitely presented objects in $Mod(T)$ (and the morphism on top is the embedding of A in B).

We can assume that $Q = \langle\{y_1, \ldots, y_m\}\rangle_T / \sim_Q$, where $\sim_Q$ is the smallest congruence relation on $\langle\{y_1, \ldots, y_m\}\rangle_T$ with $t_i(\overline{y}) \sim_Q t'_i(\overline{y})$ for $1 \leq i \leq k$, where the t_i and t'_i are terms in $\mathfrak{L}_\omega$ and k is some positive integer. Similarly, we can assume that $P = \langle\{x_1, \ldots, x_n\}\rangle_T / \sim_P$ with $\sim_P$ the smallest congruence relation on $\langle\{x_1, \ldots, x_n\}\rangle_T$ satisfying finitely many conditions on certain terms in $\mathfrak{L}_\omega$ built up from $\overline{x}$. Let $[x_i]_P$ denote the equivalence class of x_i in $\langle\{x_1, \ldots, x_n\}\rangle_T$ modulo $\sim_{P_{s_i}}$, where $s(x_i) = s_i$ and let $[y_j]_Q$ denote the equivalence class of y_j in $\langle\{y_1, \ldots, y_m\}\rangle_T$ modulo $\sim_{Q_{s_j}}$, where $s(y_j) = s_j$. For each $1 \leq i \leq n$, there is a term τ_i in $\mathfrak{L}_\omega$ with $f([x_i]_P) = \tau_i([y_1]_Q, \ldots, [y_m]_Q)$. Put $\psi(z_1, \ldots, z_n, z'1, \ldots, z'_m) = \bigwedge_{i=1}^{n}(z_i = \tau_i(\overline{z'})) \wedge \bigwedge_{i=1}^{k}(t_i(\overline{z'}) = t'_i(\overline{z'}))$.

Then

$$Q \models \psi(f([x_1]_P), \dots, f([x_n]_P), [y_1]_Q, \dots, [y_m]_Q)$$

so that

$$B \models \psi(g([x_1]_P), \dots, g([x_n]_P), g'([y_1]_Q), \dots, g'([y_m]_Q))$$

(since $g' \circ f = g$) and therefore, with $a_i = g([x_i]_P)$,

$$A \models \exists z'_1, \dots, z'_m \psi(a_1, \dots, a_n, z'_1, \dots, z'_m)$$

(by Proposition 0.2.1, since A is a pure subobject of B in $Str(\mathcal{L})$). Say $A \models \psi(a_1, \dots, a_n, a'_1, \dots, a'_m)$ Define the function $\overline{h} : \{y_1, \dots, y_m\} \longrightarrow |A|$ by $\overline{h}(y_j) = a'_j$ $(1 \leq j \leq m)$. Then there is a unique morphism $h : Q \longrightarrow A$ in $Mod(T)$ extending $\overline{h}$. So for all $1 \leq i \leq n$ we have that

$$\begin{aligned} h \circ f([x_i]_P) &= h(\tau_i([y_1]_q, \dots, [y_m]_Q)) \\ &= \tau_i(a'_i, \dots, a'_m) \\ &= a_i \\ &= g([x_i]_P) \end{aligned}$$

i.e. $h \circ f = g$. □

If A is an $\mathcal{L}$-structure and $\varphi(\overline{x}) = \exists y_1, \dots, y_k \psi(x_1, \dots, x_n, y_1, \dots, y_k)$ a *pp*-formula in $\mathcal{L}_\omega$, then $\varphi(A)$ is defined as the subset

$$\{\overline{a} \in A^n : A \models \exists y_1, \dots, y_k \psi(a_1, \dots, a_n, y_1, \dots, y_k)\}$$

of A^n.

We say that a category $\mathcal{C}$ is *finitarily axiomatisable* if $\mathcal{C}$ is equivalent to a category $Mod(T)$ where T is a theory in some finitary language. By a *finitary signature for a category* $\mathcal{C}$ we mean a finitary signature $\mathcal{L}$ for which $\mathcal{C}$ is embedded in $Str(\mathcal{L})$.
It is known (see [2] or [20], [21]), that *every locally finitely presented category is finitarily axiomatisable.*

For the remainder of this section, we assume that $\mathcal{C}$ is a locally finitely presented additive category.

To any generating set $\mathcal{G}$ of finitely presented objects of $\mathcal{C}$, one associates the finitary signature $\mathcal{L}(\mathcal{G})$ which is defined as follows:
• for each $G \in \mathcal{G}$, there is a sort s_G;
• for each sort s_G, there is an operation symbol $+_G$ of arity $s_G \times s_G \longrightarrow s_G$, an operation symbol $-_G$ of arity $s_G \longrightarrow s_G$ and a constant symbol 0_G;
• for each morphism $\alpha : G \longrightarrow H$ in the full subcategory $\mathcal{G}$ of $\mathcal{C}$, there is an operation symbol of arity $s_H \longrightarrow s_G$ (!) and, for convenience, the symbol for the morphism in $\mathcal{G}$ is taken as the corresponding operation symbol in $\mathcal{L}(\mathcal{G})$.
$\mathcal{L}(\mathcal{G})$ is such that the category $\mathcal{C}$ is axiomatisable in $\mathcal{L}(\mathcal{G})_\omega$. (See [21] for details.)
If the locally finitely presented category $\mathcal{C}$ is Abelian, then one associates to $\mathcal{C}$ a topological space $Zg(\mathcal{C})$, called the *Ziegler spectrum of* $\mathcal{C}$, the points of which are the isomorphism classes of the indecomposable pure-injective objects of $Mod(T)$, where T is a theory in $\mathcal{L}(\mathcal{G})_\omega$ for which $Mod(T) \cong \mathcal{C}$,

and with basic open sets the sets of the form

$$(\varphi/\psi) := \{M \in Zg(\mathcal{C}) : |\varphi(M)/(\varphi \wedge \psi)(M)| > 1\}$$

where φ and ψ are *pp*-formulae in $\mathcal{L}(\mathcal{G})_\omega$ in one free variable (see [21]).
(It is convenient not to distinguish between an indecomposable pure-injective object in $Mod(T)$ and its isomorphism class in $Mod(T)$.)
The notation $Zg(\mathcal{C})$ (and the terminology "the Ziegler spectrum of $\mathcal{C}$") is justified by the fact that if $\mathcal{G}'$ is another generating set of finitely presented objects, then every formula in $\mathcal{L}(\mathcal{G})_\omega$ is equivalent to a formula in $\mathcal{L}(\mathcal{G}')_\omega$ (so that, in particular, $Zg(\mathcal{C})$ is, up to homeomorphism, independent on the choice of generating family of finitely presented objects). (See [21] for details.)

Let R be a ring. The locally finitely presented Abelian category Mod-R has R as a finitely presented generating object and the corresponding signature is $\mathcal{L}(R) = (0, +, -, r)_{r \in R}$, where 0 is a constant symbol, + is a binary operation symbol, $-$ is a unary operation symbol and each $r \in R$ plays the role of a unary operation symbol. In building up formulae from the signature $\mathcal{L}(R)$ for right R-modules, it is appropriate to write the 1-ary operation symbols on the right-hand side of the variables; the resulting finitary language will be denoted by $\mathcal{L}_R$ (in case of R-Mod, one takes the same signature, but in the finitary language for left R-modules built up from $\mathcal{L}(R)$, one writes the operation symbols on the left-hand side of the variables; this finitary language for left R-modules is denoted by ${}_R\mathcal{L}$).
One can immediately write down a theory in $\mathcal{L}_R$ for which its category of models equals Mod-R: take the standard axioms for right R-modules; this theory will be denoted by T_R (in the case of left R-modules, this theory will

be denoted by ${}_RT$).

For background (and a lot more) on model theory of modules, we refer the reader to [18]. In model theory of (right) modules over a fixed ring R, purity is defined as follows:

> a submodule N of a module M is said to be a *pure submodule* of M if for every *pp*-formula $\varphi(\overline{x})$ in $\mathcal{L}_R$ and for every tuple $\overline{a}$ in N, $N \models \varphi(\overline{a}) \Leftrightarrow M \models \varphi(\overline{a})$.

That this definition agrees with the category theoretic definition of purity in *Mod-R* follows from Lemma 0.2.2, since T_R is an equational theory in the finitary language $\mathcal{L}_R$ whose underlying one-sorted signature has no relation symbols.

The finitely presented right (left) R-modules, i.e. the finitely presented objects in *Mod-R* (*R-Mod*), are the right (left) R-modules M for which there exist positive integers n, m and an exact sequence $R^n \longrightarrow R^m \longrightarrow M \longrightarrow 0$ in *Mod-R* (*R-Mod*) (this is a translation – to the category of modules over a fixed ring – of the fact that the finitely presented objects in a finitary variety are the objects which are presented by finitely many generators and finitely many equations in the variety).

In [33] (see also [18](4.20)) it is shown that every right (left) module M over a ring R has a pure-injective hull in *Mod-R* (*R-Mod*), which we denote by $PE(M)$. The injective hull of a right (left) module M over a ring R in *Mod-R* (*R-Mod*) will be denoted by $E(M)$.

The Ziegler spectrum of *Mod-R* (*R-Mod*) will be denoted by Zg_R (${}_RZg$), rather than by $Zg(Mod\text{-}R)$ ($Zg(R\text{-}Mod)$); Zg_R (${}_RZg$) is called the *right (left) Ziegler spectrum over R*.

Part I

Ziegler Spectra over Serial Rings With Krull Dimension

Chapter 1

Preliminaries

1.1 Notations and Conventions for Part I

Throughout, all rings are associative and have a unit (denoted by 1), and all modules are unital modules. The letter R is reserved to denote a ring. Throughout Part I, modules are right modules (unless stated otherwise). If M is a submodule (proper submodule) of a module N, we write $M \leq N$ ($M < N$).

Unless stated otherwise, the letters i, j, k, l, m and n denote positive integers and the letters $\alpha, \beta, \gamma, \delta$ and λ will be reserved to denote ordinals.

The letters U, V, W will be reserved to denote indecomposable pure-injective modules. Morphisms will be denoted in bold face, and, since we consider right modules, we will write the morphisms on the left-hand side of the arguments so that by $\mathbf{fh}$ we mean $\mathbf{f} \circ \mathbf{h}$.

Throughout Part I, we will be working in $\mathcal{L}_R$. In $\mathcal{L}_R$, *pp*-formulae are the formulae of the form

$$\exists y_1, \dots, y_k \bigwedge_{i=1}^{m} (\sum_{j=1}^{n} x_j a_{ji} + \sum_{l=1}^{k} y_l b_{li} = 0)$$

where the a_{ji} and the b_{li} are elements of the ring R.

Also formulae which are, modulo T_R, equivalent to a formula of the above form are called *pp*-formulae. By pp_R^n we denote the set of *pp*-formulae in $\mathcal{L}_R$ in n free variables. This set forms a modular lattice as follows: for $\varphi, \psi \in pp_R^n$, $\varphi \leq \psi$ iff $T_R \models \varphi \to \psi$, and $(\varphi + \psi)(\overline{x})$ is defined as $\exists \overline{y}, \overline{z}(\overline{x} = \overline{y} + \overline{z} \wedge \varphi(\overline{y}) \wedge \psi(\overline{z}))$. One can easily verify that $\leq$ defined in this way is a partial order on pp_R^n and that, given $\varphi, \psi \in pp_R^n$, $inf(\varphi, \psi) = \varphi \wedge \psi$ and $sup(\varphi, \psi) = \varphi + \psi$. If $\varphi \leq \psi$ (in pp_R^n), then $[\varphi, \psi]$ denotes the set $\{\theta \in pp_R^n : \varphi \leq \theta \leq \psi\}$; φ, ψ expresses that $\varphi \leq \psi$ and not $\psi \leq \varphi$.

The set of *pp*-formulae in any (finite) number of free variables is denoted by pp_R. L_R^n denotes the set $pp_R^n \cup \{\neg\varphi : \varphi \in pp_R^n\}$. For $\varphi(\overline{x})$ a *pp-n*-formula and M a module, $\varphi(M) = \{\overline{a} \in M^n : M \models \varphi(\overline{a})\}$ is a subgroup of M^n; a subgroup of this form is called a *pp-n-definable subgroup* of M and $pp_R^n(M)$ denotes the lattice of *pp-n*-definable subgroups of M.

An arbitrary subset of L_R^n is called a *pp-n-type* (or *pp-type* for short). A *pp*-type $p(\overline{x})$ is said to be *consistent* if there exists an R-module M and a tuple $\overline{a}$ in M with $M \models \varphi(\overline{a})$ for every $\varphi(\overline{x}) \in p(\overline{x})$; such a pair $(M, \overline{a})$ is called a *realisation* of $p(\overline{x})$. We will make no distinction between a *pp-n*-type $p(\overline{x})$ and the set of formulae in L_R^n which are, modulo T_R, implied by a finite conjunction of formulae from $p(\overline{x})$. A consistent *pp-n*-type $p(\overline{x})$ is said to be *pp-complete* if for every $\varphi(\overline{x}) \in pp_R^n$, either $\varphi(\overline{x}) \in p(\overline{x})$ or $\neg\varphi(\overline{x}) \in p(\overline{x})$. For a module M and a tuple $\overline{a}$ in M, the *pp-type of $\overline{a}$ in M* is defined as

$tp^{\pm}_M(\overline{a}) = \{\varphi(\overline{x}) \in L^n_R : M \models \varphi(\overline{a})\}$. The *positive part* of a *pp-n*-type $p(\overline{x})$ is defined as $p^+(\overline{x}) = p(\overline{x}) \cap pp^n_R$; by an *exact realisation* of $p^+(\overline{x})$ we mean a pair $(M, \overline{a})$ (where M is a module and $\overline{a}$ is an n-tuple in M) for which the positive part of $tp^{\pm}_M(\overline{a})$ is equal to $p^+(\overline{x})$.

For a subset A of a pure-injective module M, the *hull of A in M*, which we denote by $H(A)$, is defined as the minimal direct summand of M containing A; $H(A)$ is unique up to A-isomorphism, moreover, if $\overline{a}$ is a tuple in a pure-injective module M and $\overline{b}$ is a tuple in a pure-injective module N such that $tp^+_M(\overline{a}) = tp^+_N(\overline{b})$, then $(H(\overline{a}), \overline{a}) \cong (H(\overline{b}), \overline{b})$, i.e. there is an isomorphism $H(\overline{a}) \longrightarrow H(\overline{b})$ taking $\overline{a}$ to $\overline{b}$ (in fact, the same holds for any subset A of M and any subset B of N). This allows us to define the hull of a *pp*-complete *pp*-type as follows: for a *pp*-complete *pp*-type $p(\overline{x})$, the *hull of* $p(\overline{x})$, which we denote by $H(p(\overline{x}))$, is defined as the hull of any exact realisation of $p^+(\overline{x})$ in some pure-injective module. A *pp*-complete *pp*-type is called *indecomposable* in case its hull is indecomposable. Note that every indecomposable pure-injective module is the hull of any of its nonzero elements. For further details about the above definitions we refer the reader to [18].
If $p(\overline{x})$ is a *pp*-type and φ, ψ are *pp*-formulae with $\psi < \varphi$ and $\varphi, \neg\psi \in p(\overline{x})$, then we write $\varphi/\psi \in p(\overline{x})$.
We say that a consistent theory T in $\mathfrak{L}_R$ is a *theory for R-modules* if every model of T is a model of T_R.

1.2 Ziegler Spectra over Rings

Details about the material presented in this section can be found in [37] or [18].

Lemma 1.2.1 ([37] (4.9))
Let R be a ring. If $p(x)$ is an indecomposable pp-type in $\mathcal{L}_R$, then for every $\overline{\varphi}/\overline{\psi} \in p(x)$, the set

$$\{(\varphi/\psi) : \varphi/\psi \in p(x) \text{ and } \overline{\psi} \leq \psi < \varphi \leq \overline{\varphi}\}$$

forms a neighbourhood-basis in Zg_R for the point $H(p(x))$.

Let T be a complete theory of R-modules. By Zg_R^T we denote the set of all isomorphism types of indecomposable pure-injective direct summands of models of T. So $Zg_R^{\mathbf{T}} = Zg_R$ (the right Ziegler spectrum over the ring R) where $\mathbf{T}$ is defined as

$Th(\oplus\{M_T : T$ a complete theory for R-modules and M_T some model of $T\})$,

which is the largest complete theory of R-modules in the sense that if T is a complete theory for R-modules, then every model of T is a pure substructure of some model of $\mathbf{T}$ (in [18] $\mathbf{T}$ is denoted by T^* and on the set of complete theories for R-modules, the following partial order is defined: $T_1 \leq T_2$ iff every model of T_1 is purely embedded in some model of T_2). For every theory T for R-modules, Zg_R^T is a quasi-compact topological space (i.e. every open cover of Zg_R^T has a finite subcover).

It is known that every module is elementarily equivalent to a direct sum of indecomposable pure-injective modules. There is a one-to-one correspondence between closed sets in the Ziegler spectrum over a ring R and complete theories of R-modules which are closed under products (i.e. the set of models of the theory is closed under arbitrary direct products): the map $T \mapsto Zg_R^T$ is a bijection from the set of complete theories for R-modules which are closed under products to the set of closed subsets of Zg_R (this map is in fact order preserving, where the partial order on the set of complete theories closed under products is the one given above, and the partial order on the closed subsets of Zg_R is the obvious inclusion).

If φ/ψ is a pair of *pp*-formulae with $\psi <_T \varphi$ (i.e. $T \models \psi \to \varphi$ and not $T \models \varphi \to \psi$) and such that, modulo T, there is no *pp*-formula strictly between φ and ψ, then φ/ψ is called a *T-minimal pair.* If $p(x)$ is a *pp*-complete *pp*-type containing a T-minimal pair, then $H(p(\overline{x}))$ is isolated in the subspace Zg_R^T of Zg_R.
By a *minimal pair* we mean a $\mathbf{T}$-minimal pair.

Lemma 1.2.2 ([37] (8.10))
Let T be a complete theory of R-modules. If $p(x)$ and $q(x)$ are pp-complete pp-types with $H(p(x)) \cong H(q(x)) \in Zg_R^T$, then $p(x)$ contains a T-minimal pair iff $q(x)$ contains a T-minimal pair.

The *Cantor-Bendixson derivatives* of a topological space are defined as follows:
If $\mathcal{T}$ is a topological space, then $\mathcal{T}' = \mathcal{T} \setminus \{x \in \mathcal{T} : x \text{ is an isolated point in } \mathcal{T}\}$

with the subset topology, and $\mathcal{T}^{(0)} = \mathcal{T}$. For $\mathcal{T}$ a topological space and α an ordinal, $\mathcal{T}^{(\alpha+1)} = (\mathcal{T}^{(\alpha)})'$; and if λ is a limit ordinal, then $\mathcal{T}^{(\lambda)} = \bigcap\{\mathcal{T}^{(\beta)} : \beta < \lambda\}$.

Note that the CB-derivatives of a topological space $\mathcal{T}$ are closed subspaces of $\mathcal{T}$ (so that the CB-derivatives of Zg_R correspond to complete theories of R-modules closed under products). Therefore, from the fact that Zg_R is a quasi-compact topological space, it follows that the CB-derivatives of Zg_R are quasi-compact.

The *derivatives* of the theory $\mathbf{T}$ are defined as follows:

for each ordinal α, $\mathbf{T}^{(\alpha)}$ is defined as the complete theory which corresponds to the closed subspace $Zg_R^{(\alpha)}$ of Zg_R.

If $\mathcal{T}$ is a quasi-compact topological space for which there is an ordinal β with $\mathcal{T}^{(\beta)} = \emptyset$, then the *Cantor-Bendixson rank* of $\mathcal{T}$, denoted by $CB(\mathcal{T})$, is defined as $\alpha - 1$, where α is the least ordinal for which $\mathcal{T}^{(\alpha)} = \emptyset$ (note that $\alpha - 1$ is defined since, by quasi-compactness of $\mathcal{T}$, α cannot be a limit ordinal). If x is a point in $\mathcal{T}$ with $x \in \mathcal{T}^{(\beta)} \setminus \mathcal{T}^{(\beta+1)}$, then the *Cantor-Bendixson rank of* x, denoted by $CB(x)$, is defined as β.

If the topological space $\mathcal{T}$ is such that $\mathcal{T}^{(\alpha)} \neq \emptyset$ for every ordinal α, then we write $CB(\mathcal{T}) = \infty$ to express that the CB-rank of $\mathcal{T}$ is not defined.

Given an R-module M, the *m-dimension (modulo M)* of a pair φ/ψ of *pp*-formulae is denoted by $m\text{-}dim_M(\varphi/\psi)$ and is defined by transfinite recursion as follows:

$m\text{-}dim_M(\varphi/\psi) = -1$ if $\varphi(M) = \psi(M)$;

If α is an ordinal with $m\text{-}dim_M(\varphi/\psi) \not< \alpha$, then $m\text{-}dim_M(\varphi/\psi) = \alpha$ provided for every infinite ascending chain $\varphi_0 < \varphi_1 < \cdots$ in $[\psi, \varphi]$ there exists $i \geq 1$ with $m\text{-}dim_M(\varphi_i/\varphi_{i-1}) < \alpha$ and for every infinite descending chain $\varphi_0 > \varphi_1 > \cdots$ in $[\psi, \varphi]$ there exists $i \geq 1$ with $m\text{-}dim_M(\varphi_{i-1}/\varphi_i) < \alpha$.

If φ/ψ is a pair of *pp*-formulae for which there is no ordinal α with $m\text{-}dim_M(\varphi/\psi) = \alpha$, then we say that φ/ψ has no m-dimension (modulo M) and we write $m\text{-}dim_M(\varphi/\psi) = \infty$.

The *m-dimension* of a module M is denoted by $m\text{-}dim(M)$ and defined as $m\text{-}dim_M(x = x/x = 0)$.

The *m-dimension* of a complete theory T of R-modules is denoted by $m\text{-}dim(T)$ and defined as $m\text{-}dim(M)$, where M is any model of T.

Chapter 2

Serial Rings

2.1 Definitions, Notations and Basic Facts

The Jacobson radical of a ring R is denoted by $Jac(R)$. The *transfinite powers* of $Jac(R)$ are defined by transfinite recursion as follows:
$Jac(0) = Jac(R)$, and, for an ordinal $\alpha > 0$, $Jac(\alpha) = \bigcap_{i\geq 1, \beta<\alpha} Jac(\beta)^i$.
So if α is a nonzero nonlimit ordinal, then $Jac(\alpha) = \bigcap_{i\geq 1} Jac(\alpha - 1)^i$, and if α is a nonzero limit ordinal, then $Jac(\alpha) = \bigcap_{\beta<\alpha} Jac(\beta)$.
We adopt the following notational convention: we put, for an ordinal α, $Jac(\alpha)^0 = R$.
The transfinite powers of the Jacobson radical will pay an important role in our discussion on the Ziegler spectra over serial rings (with Krull dimension).

If e is an idempotent in R (i.e. $e^2 = e$), then eR and Re are projective modules (since eR, resp. Re, is a direct summand of the free, and therefore projective, module R_R, resp. ${}_RR$).

A set $\{e_1, \ldots, e_n\}$ of idempotents in R is called *complete* if $1 = e_1 + \cdots + e_n$, and it is called *orthogonal* if the e_i are pairwise orthogonal (i.e. $e_i e_j = 0$ for all $i \neq j$). An idempotent e of R is called *primitive* in case $e \neq 0$ and eR is an indecomposable direct summand of R_R (or, equivalently, Re is an indecomposable direct summand of ${}_RR$. An orthogonal set $\{e_1, \cdots, e_n\}$ of primitive idempotents in R is called *basic* if for every primitive idempotent $e \in R$, $eR \cong e_i R$ for some e_i.

A nonzero cyclic module with a unique maximal submodule is called *local*. A (nonzero) element of a module is called *local* if the cyclic submodule generated by that element is local.

R is called *semiperfect* if R is semisimple modulo its Jacobson radical and if idempotents lift modulo its Jacobson radical (i.e. $R/Jac(R)$ is a ring direct sum of a finite number of simple Artinian rings, and, if $g + Jac(R)$ is an idempotent in $R/Jac(R)$, then there is an idempotent e of R with $g + Jac(R) = e + Jac(R)$). The following gives some characterisations of semiperfect rings.

Theorem 2.1.1 ([1] (27.6),(27.11) and [35] (42.6))

For a ring R the following are equivalent:

(1) R is semiperfect;

(2) R has a complete orthogonal set $\{e_1, \ldots, e_n\}$ of local idempotents (so that $R_R = e_1R \oplus \cdots \oplus e_nR$ and ${}_RR = Re_1 \oplus \ldots \oplus Re_n$);

(3) Every simple right R-module has a projective cover;

(4) Every finitely generated right R-module has a projective cover.

The following are standard results on semiperfect rings.

Lemma 2.1.2 ([1] (15.18))

If R is a ring which is semisimple modulo its Jacobson radical, and if M is a right R-module, then $MJac(R) = \cap\{N \leq M : N$ is maximal in $M\}$ (i.e. $MJac(R)$ is equal to the radical of M), and $M/MJac(R)$ is semisimple.

Proposition 2.1.3 ([1](27.10),(27.11))

Let R be a semiperfect ring. Then every complete orthogonal set of primitive idempotents in R contains a basic set. And, if $\{e_1, \cdots, e_m\}$ is an orthogonal set of primitive idempotents in R, then the following are equivalent:

(1) $\{e_1, \cdots e_m\}$ is a basic set in R;

(2) Every indecomposable projective right R-module is isomorphic to some e_iR;

(3) Every simple right R-module is isomorphic to some $e_iR/e_iJac(R)$.

Lemma 2.1.4 ([1] (27.13))

Let R be a semiperfect ring with basic set $\{e_1, \cdots, e_m\}$. If P is the projective cover of a finitely generated module M, then $P \cong (e_1R)^{(k_1)} \oplus \cdots \oplus (e_mR)^{(k_m)}$, where the k_i are the unique non-negative (but possibly zero) integers for which $M/MJac(R) \cong (e_1R/e_1Jac(R))^{(k_1)} \oplus \cdots \oplus (e_mR/e_mJac(R))^{(k_m)}$.

A (right or left) module is called *uniserial* if its submodules are linearly ordered by inclusion (i.e. they form a chain), and a *serial* module is a module which is a direct sum of uniserial modules. A ring is called *right (resp. left) serial* if it is serial as a right (resp. left) module over itself. By a *serial* ring we mean a ring which is both right and left serial.

Right (or left) serial rings are examples of semiperfect rings (for if $R_R = I_1 \oplus \cdots \oplus I_n$ with each I_i uniserial, then (see [1](7.3)) for each i there is some idempotent e_i of R such that $I_i = e_iR$, so, since every finitely generated module has a maximal submodule, there is a complete orthogonal set of, necessarily local and primitive, idempotents in R).

Given a serial ring R we will fix an orthogonal set $\mathcal{E}(R) = \{e_1, \cdots, e_n\}$ of primitive idempotents with each e_iR and each Re_i uniserial (and therefore local) and such that $R_R = e_1R \oplus \ldots \oplus e_nR$ and ${}_RR = Re_1 \oplus \ldots \oplus Re_n$. If there is no danger for confusion, $\mathcal{E}(R)$ will be abbreviated to $\mathcal{E}$. The letters e, f, g and h will be reserved to denote elements of $\mathcal{E}$. We also adopt the convention that, unless specified otherwise, "for every e-pair" ("for some e-pair") means "for every $e \in \mathcal{E}$ and for every e-pair" ("for some $e \in \mathcal{E}$ and some e-pair").

Lemma 2.1.5 ([5](1.3))

Let R be a serial ring. Every finitely presented right R-module is a direct sum of modules of the form eR/rR, where $e \in \mathcal{E}$ and $r \in eR$.

Proposition 2.1.6 ([29](2.1), [33](5.11))

Let R be a serial ring. Then

(1) $Jac(0)$ is nilpotent iff R is (two-sided) Artinian;

(2) $Jac(1) = 0$ iff R is two-sided Noetherian;

(3) If R is two-sided Noetherian, then R is the product of an Artinian serial ring and finitely many prime Noetherian serial rings.

A *domain* is defined as a ring with no nonzero divisors. It is clear that a serial domain is uniserial; a uniserial domain is called a *valuation domain.*

2.2 Serial Rings and Krull Dimension

Let $(P, \leq)$ (or P for short) be a poset. For $a, b \in P$ we write $a < b$ in case $a \leq b$ and $a \neq b$. The lattice of submodules of a module M is denoted by $Latt(M)$.

The *Krull dimension* of $(P, \leq)$, denoted by $Kdim(P)$, is defined by transfinite recursion as follows:

If P is a singleton, then $Kdim(P) = -1$;

If α is an ordinal and $Kdim(P) \not< \alpha$, then $Kdim(P) = \alpha$, provided for every infinite descending chain $a_0 > a_1 > a_2 > \ldots$ in P there exists a nonzero $i \geq 1$ with $Kdim([a_{i-1}, a_i]) < \alpha$;

If there is no ordinal α with $Kdim(P) = \alpha$, then we say that P has no Krull dimension and we write $Kdim(P) = \infty$.

Clearly, if Q is a subset of the poset P, then $Kdim(Q) \leq Kdim(P)$.

By the Krull dimension of a module M we mean the Krull dimension of $Latt(M)$, and by the (right) Krull dimension of a ring R we mean the Krull dimension of $Latt(R_R)$. These notions are in accordance with the definition of Krull dimension formulated in [7]. By the *left Krull dimension* of a ring R we mean $Kdim({}_RR)$.

A ring with (right) Krull dimension needn't have left Krull dimension, and conversely, a ring with left Krull dimension needn't have (right) Krull dimension. Even if R has both right and left Krull dimension, the two dimensions needn't be the same. As we shall see, serial rings are examples for which having (right) Krull dimension is equivalent to having left Krull dimension; moreover, (for serial rings) $Kdim(R_R) = Kdim({}_RR)$. This was first shown by Wright in [36]. We will however follow [16], since the proof of Müller and

Singh is much more elementary; moreover, they proved a slightly stronger result. One part of the proof is due to Krause and Lenagan and is stated below.

Proposition 2.2.1 ([14]Thm.8) *If R is a ring with right (or left) Krull dimension α, then $Jac(\alpha)$ is nilpotent.*

In case of serial rings there is a stronger connection between the transfinite powers of the Jacobson radical and the Krull dimension. This connection will imply that $Kdim(R_R) = Kdim(_RR)$.
We need a few lemmas.

Lemma 2.2.2 ([16] (1.5))
Let R be a semiperfect ring. If rR is local, then there is a primitive idempotent $e \in R$ for which $r = re$.

Proof. Since R is semiperfect, the cyclic module rR has a projective cover $\mathbf{f} : e'R \longrightarrow rR$, where e' is a primitive idempotent (by Lemma 2.1.4). Define an epimorphism $\mathbf{g} : R \longrightarrow rR$ via $\mathbf{g}(1) = r$. Then there exists $\mathbf{h} : R \longrightarrow e'R$ with $\mathbf{fh} = \mathbf{g}$. Moreover, $\mathbf{h}$ is an epimorphism (since $\mathbf{g}$ is an epimorphism and $\mathbf{f}$ is a superfluous epimorphism; [1] (5.15)). Since $e'R$ is projective, $\mathbf{h}$ splits, i.e. there is a monomorphism $\mathbf{k} : e'R \longrightarrow R$ with $\mathbf{hk} = 1_R$, so that $R = Im(\mathbf{k}) \oplus Ker(\mathbf{h})$ and $Im(\mathbf{k}) = eR$ for some idempotent $e \in R$. Since $e'R \cong Im(\mathbf{k})$, e is primitive. Finally, $r = \mathbf{g}(1) = \mathbf{fh}(1) = \mathbf{fh}(e) = \mathbf{fh}(1)e = re$. □

Lemma 2.2.3 ([16](1.6))

Let $\{I_\alpha\}$ be a family of two-sided ideals of a right serial ring, and let $I = \bigcap_\alpha I_\alpha$. Then, for every local element r of R, $rI = \bigcap_\alpha rI_\alpha$. If M is a uniserial module over R and if $I \leq I_\alpha I_\beta$ holds for all α, β, then $aI = MI$ holds for every $a \in M \backslash MI$.

Proof. First note that $eI = \bigcap_\alpha eI_\alpha$ holds for any idempotent e and for any family $\{I_\alpha\}$ of two-sided ideals in any ring.
If rR is local, then, by Lemma 2.2.2, $r = re$ for some primitive idempotent $e \in R$. Clearly, $rI \leq \bigcap_\alpha rI_\alpha$. Define an epimorphism $\mathbf{f} : eR \longrightarrow rR$ via $\mathbf{f}(e) = r$. If $Ker(\mathbf{f}) \leq eI$, then $eR/eI \cong rR/rI$ (because, if $\mathbf{p}$ denotes the natural epimorphism $eR \longrightarrow eR/eI$, then $\mathbf{p} = \mathbf{hf}$ for a unique $\mathbf{h}$ and $Ker(\mathbf{h}) = \mathbf{f}(Ker(\mathbf{p})) = \mathbf{f}(eI) = rI$; [1](3.6)), and the result follows from the first observation. If $Ker(\mathbf{f}) > eI$, then $Ker(\mathbf{f}) > eI_\beta$ for some β. But then $rI_\beta = 0$ (by definition of $\mathbf{f}$), and therefore $\bigcap_\alpha rI_\alpha = 0$ so that $rI = 0$ and $rI = \bigcap_\alpha rI_\alpha$.
For the last part, let $a \in M \backslash MI$. If $aI < MI$, then there is some $a' \in M$ with $a'I > aI = \bigcap_\alpha aI_\alpha$. So there is some β with $a'I > aI_\beta$. And $aR > MI \geq a'I = \bigcap_\alpha a'I_\alpha$ yields $aR > a'I_\alpha$ for some α so that (since $I \leq I_\alpha I_\beta$) $a'I > aI_\beta \geq a'I_\alpha I_\beta \geq a'I$, a contradiction. □

Lemma 2.2.4 ([16](5.4))

Let R be a serial ring and α an ordinal. If M is a uniserial module and $MJac(\alpha)^k = 0$ for some $k \geq 1$, then $Kdim(M) \leq \alpha$.

Proof. We give a proof by induction on α.

Let M be a (nonzero) uniserial module with $MJac(R)^k = 0$ for some $k \geq 1$. Lemma 2.1.2 implies that $M/MJac(R)$ and $MJac(R)^i/MJac(R)^{i+1}$ (for $i \geq 1$) are semisimple (but possibly zero), and therefore simple, since M is uniserial. So M has a finite composition series (of length at most k), and therefore M is Artinian, i.e. $Kdim(M) = 0$. This establishes the basis of the induction argument.

Suppose that, whenever N is a uniserial module with $NJac(\beta)^l = 0$ for some $\beta < \alpha$ and some $l \geq 1$, then $Kdim(N) \leq \beta$.

We show that for a uniserial module M with $MJac(\alpha) = 0$, $Kdim(M) \leq \alpha$. Since for every $i \geq 1$ there is a short exact sequence

$$0 \longrightarrow MJac(\alpha)^i \longrightarrow MJac(\alpha)^{i-1} \longrightarrow MJac(\alpha)^{i-1}/MJac(\alpha)^i \longrightarrow 0$$

and because Krull dimension behaves well under short exact sequences ([7] (1.1)) the proof will then be complete.

Let $M = M_0 > M_1 > M_2 > \cdots$ be a descending chain of submodules of M. Take $a \in M \setminus M_1$. Then, by Lemma 2.2.3, $0 = aJac(\alpha) = \bigcap_{\beta<\alpha, i\geq 1} aJac(\beta)^i$. Therefore, for every $i \geq 1$, there exist $j \geq 1$ and $\beta < \alpha$ with $aR > M_i > M_{i+1} \geq aJac(\beta)^j$ and so $(M_i/M_{i+1})Jac(\beta)^j = 0$. The induction hypothesis implies that $Kdim(M/M_{i+1}) \leq \beta < \alpha$ for every $i \geq 1$. This shows that $Kdim(M) \leq \alpha$. $\square$

Theorem 2.2.5 ([16](5.5),(5.6))

Let R be a serial ring. Then $Kdim(R_R) \leq \alpha$ iff $Jac(\alpha)$ is nilpotent. Therefore, the following are equivalent:

$Kdim(R_R) = \alpha$;

α is the least ordinal for which $Jac(\alpha)$ is nilpotent;

$Kdim({}_R R) = \alpha$.

Proof. Since R is serial, there is a complete orthogonal set of primitive (and local) idempotents $\mathcal{E}$ such that $R_R = \bigoplus_{e\in\mathcal{E}} eR$. If $Jac(\alpha)$ is nilpotent, then there is some $k \geq 1$ such that, for each $e \in \mathcal{E}$, $eRJac(\alpha)^k = 0$. Lemma 2.2.4 implies that, for each $e \in \mathcal{E}$, $Kdim(eR) \leq \alpha$, and therefore (by [7]) $Kdim(R_R) \leq \alpha$. Together with Proposition 2.2.1, this proves the first part.

The second part is now clear. $\square$

We conclude this chapter with an easy lemma.

Lemma 2.2.6 *If R is a valuation domain, then for every ordinal β, the two-sided ideal $Jac(\beta)$ is completely prime.*

Proof. Let β be an ordinal and $r, s \in R$ with $rs \in Jac(\beta)$.

If $Jac(\beta) = Jac(0)$ and $r \notin Jac(\beta)$, then $Jac(0) < Rr$ so that $Rr = R$ and therefore $tr = 1$ for some $t \in R$ which implies that $s \in Jac(0)$.

Suppose $Jac(\beta) < Jac(0)$ and $r \notin Jac(\beta)$. Then $r \in R\backslash Jac(\beta)$ so that, by Lemma 2.2.3, $rJac(\beta) = Jac(\beta)$ and therefore $rs = rt$ for some $t \in Jac(\beta)$ which implies that $s \in Jac(\beta)$ (since R is a domain). $\square$

Chapter 3

Ziegler Spectra over Serial Rings

3.1 Model Theory of Modules over Serial Rings

In this section we list some results by Eklof, Herzog and Puninski on model theory of modules over a serial ring.

Throughout the remainder of Part I, we let R be a serial ring.

In [5] a simple description is given for *pp*-formulae in ${}_R\mathcal{L}$. This result and all the other results from [5] can immediately be translated into results in $\mathcal{L}_R$. We will only make use of *pp*-formulae in one free variable in the language $\mathcal{L}_R$.

Proposition 3.1.1 ([5] (1.6), (1.12))
Let R be a serial ring. If $\varphi(x) \in pp^1_R$, then

$$T_R \models \varphi(x) \leftrightarrow \sum_{i=1}^{n}(xr_i = 0 \wedge s_i|x)$$

for some $r_i, s_i \in R$*; and if* $e \in \mathcal{E}$ *with* $\varphi(x) \leq e|x$*, then we can take* $r_i \in eR$ *and* $s_i \in Re$*. Also*

$$T_R \models \varphi(x) \leftrightarrow \bigwedge_{i=1}^{m}(xr_i = 0 + s_i|x)$$

for some $r_i, s_i \in R$*.*

Also in [5], the points in Zg_R are classified via a classification of indecomposable "e-types". For convenience we slightly modify the terminology from [5].

If $e \in \mathcal{E}$ and if $p(x)$ is a *pp*-type with $e|x \in p(x)$, then $p(x)$ is called an *e-type*. If $e \in \mathcal{E}$ and $I < eR$, $J < Re$, then to the pair $\langle I, J\rangle$ is associated the following e-type:

$$\begin{aligned} J^*/I(x) &= \{xr = 0 \wedge s^*|x : r \in I, s^* \in J^*\} \\ &\quad \cup \{\neg(xr^* = 0 + s|x) : r^* \in I^*, s \in J\} \end{aligned}$$

where I^* is defined as $eR \setminus I$ and J^* is defined as $Re \setminus J$. We will call a pair $\langle I, J\rangle$ an *e-pair* in case the e-type $J^*/I(x)$ is consistent.

For S a subset of R and a an element of an R-module, $ann_S(a)$ denotes $\{s \in S : as = 0\}$, the annihilator of a in S.

The following theorem describes the indecomposable e-types.

Theorem 3.1.2 ([5] (2.7))

If $e \in \mathcal{E}$ and $I < eR, J < Re$, then $J^*/I(x)$ is pp-complete and indecomposable whenever it is consistent.

Conversely, every indecomposable e-type is of the form $J^*/I(x)$ (where $I < eR, J < Re$).

We will also make use of the following fact which is mentioned in [25]:

Corollary 3.1.3 ([25])

Let $p(x)$ be a pp-complete e-type ($e \in \mathcal{E}$). Then the following are equivalent:

(1) $p(x)$ is indecomposable;

(2) For all $r \in eR, s \in Re$, if $xr = 0 + s|x \in p(x)$, then $xr = 0 \in p(x)$ or $s|x \in p(x)$.

Proof.

(1) $\Rightarrow$ (2) If $p(x)$ is indecomposable, then (by Theorem 3.1.2) $p(x)$ is equivalent to $J^*/I(x)$ for some e-pair $\langle I, J\rangle$. Suppose $r \in eR$, $s \in Re$ and $xr = 0 + s|x \in p(x)$. Then, by the definition of $J^*/I(x)$ and since $p(x)$ is consistent, $r \in I$ or $s \in J^*$, i.e. $xr = 0 \in J^*/I(x)$ or $s|x \in J^*/I(x)$.

(2) $\Rightarrow$ (1) By Theorem 3.1.2 it suffices to show that there are $I < eR, J < Re$ with $J^*/I(x) \subseteq p(x)$. Let $U = H(p(x))$ and let $0 \neq a \in U$ with $U \models p(a)$. Put $I = ann_{eR}(a)$ and $J = \{s \in Re : a \notin Us\}$. Then $I < eR$, $J < Re$ and, for every $r \in I$ and for every $s^* \in J^*$, $xr = 0 \wedge s|x \in p(x)$. And if $xr^* = 0 + s|x \in p(x)$ for some $r^* \in I^*$ and some $s \in J$, then, by the hypothesis, $xr^* = 0 \in p(x)$ or $s|x \in p(x)$, i.e. $r^* \in I$ or $s \in J^*$, a contradiction. We conclude that $J^*/I(x) \subseteq p(x)$. $\square$

From the fact that for every indecomposable pure-injective R-module U, $U = \bigoplus_{e \in \mathcal{E}} Ue$ as a left $End_R(U)$-module ([5] (2.5)), it is clear that every indecomposable pure-injective R-module is the hull of an indecomposable e-type (for some $e \in \mathcal{E}$).

Therefore, the next theorem (which is due to Puninski) together with the previous theorem gives a description of the indecomposable pure-injective R-modules.

Theorem 3.1.4 ([5] (2.8))

For $e \in \mathcal{E}$ and $I < eR, J < Re$, $\langle I, J \rangle$ is an e-pair (i.e. $J^*/I(x)$ is consistent) iff $s^*r^* \neq s^*r + sr^*$ for every $r \in I, r^* \in I^*, s \in J, s^* \in J^*$.

The following theorem completes the classification of the points in Zg_R.

Theorem 3.1.5 ([5] (2.11))

Let $\langle I, J \rangle$ be an e-pair and let $\langle K, L \rangle$ be an f-pair ($e, f \in \mathcal{E}$). Then $H(J^*/I(x)) \cong H(L^*/K(x))$ iff one of the following conditions hold:

(a) There exists $u \in K^* \cap J^*$ (so $u \in fRe \setminus \{0\}$) with $uI = K$ and $Lu = J$;

(b) There exists $v \in I^* \cap L^*$ (so $v \in eRf \setminus \{0\}$) with $vK = I$ and $Jv = L$.

If $\langle I, J \rangle$ is an e-pair and $\langle K, L \rangle$ an f-pair with $H(J^*/I(x)) \cong H(L^*/K(x))$ then we write $\langle I, J \rangle \sim \langle K, L \rangle$ and say that the two pairs are *equivalent*.

If U is an indecomposable pure-injective R-module and $\langle I, J \rangle$ is an e-pair with $H(J^*/I(x)) \cong U$, then we say that $\langle I, J \rangle$ *is an e-pair of U*.

Puninski considered the following interesting class of serial rings:

Definition 3.1.6 By $\mathcal{P}$ we denote the class of serial rings R which satisfy the following condition:
for every $e \in \mathcal{E}$, if $I < eR$ and $J < Re$, then $J^*/I(x)$ is consistent iff for every $r^* \in I^*$ and for every $s^* \in J^*$, $s^*r^* \neq 0$.

Puninski ([25]) showed that if R is a serial ring, then $R \in \mathcal{P}$ iff for every $e \in \mathcal{E}$ and for every $r \in eR$, $End(eR/rR)$ is local (i.e. $\mathcal{P}$ is the class of serial *Krull-Schmidt* rings) iff for every $e, f \in \mathcal{E}$ and for every $0 \neq r \in eRf$, $(e + Jac(eRe))r \subseteq rU(fRf)$ where $U(fRf)$ denotes the set of the invertible elements of the uniserial ring fRf. The class $\mathcal{P}$ includes the commutative serial rings (easy to see with the latter criterion) and the one-sided Noetherian serial rings ([25]). We will include the proof.

Lemma 3.1.7 *Let R be a serial ring and let $e \in \mathcal{E}$.*
(1) If $r \in eR$, then there exists $g \in \mathcal{E}$ with $rR = rgR$. And, similarly, if $r \in Re$, then there exists $g \in \mathcal{E}$ with $Rgr = Rr$.
(2) If $I < eR$, $r \in I$ and $r^* \in I^*$, then there exist $f \in \mathcal{E}$ and $t \in Jac(fRf)$ with $rf = r^*t \in I^*$. And, similarly, if $J < Re$, $r \in J$ and $r^* \in J^*$, then there exist $f \in \mathcal{E}$ and $t \in Jac(fRf)$ with $fr = tr^* \in J^*$.

Proof. *(1)* Let $r \in eR$. Then $r = \sum_{f\in\mathcal{E}} rf$ and for every $f \in \mathcal{E}$, $rfR \leq rR$ so that, since rR is uniserial, $\sum_{f\in\mathcal{E}} rfR = rgR$ for some $g \in \mathcal{E}$, and therefore $rR = rgR$.
(2) By *(1)* we can pick $f \in \mathcal{E}$ with $r^*R = r^*fR$. Then $r^*f \in I^*$ (for $r^*f \in I$ would imply $r^* \in I$) so that, since eR is uniserial, $rfR < r^*fR$, which implies that $rf = r^*t$ for some $t \in fRf$. Suppose $t \notin Jac(fRf)$. Then, since

fRf is uniserial, $ts = f$ for some $s \in fRf$ so that $rfs = r^*f$, contradicting $rfR < r^*fR$. $\square$

Lemma 3.1.8 ([25])

Every one-sided Noetherian serial ring belongs to $\mathcal{P}$.

Proof. Suppose that R is right Noetherian and that R does not belong to $\mathcal{P}$. Then there exist $e \in \mathcal{E}$ and $I < eR, J < Re$ such that $w^*v^* \neq 0$ for all $v^* \in I^*, w^* \in J^*$, and $s^*r^* = s^*r + sr^*$ for some $r \in I, r^* \in I^*, s \in J, s^* \in J^*$. By Lemma 3.1.7 we can pick $f, g \in \mathcal{E}$ with $rf = r^*v$ and $gs = ws^*$, where $v \in Jac(fRf)$ and $w \in Jac(gRg)$. Then $gs^*r^*f = gs^*rf + gsr^*f$, with $gs^* \in J^*, r^*f \in I^*, gs \in J$ and $rf \in I$, so that, by the assumption that R does not belong to $\mathcal{P}$, $gs^*r^*f \neq 0$.

Let, for every $n \geq 1$, $\mathbf{h}_n$ be the endomorphism in $End(gR/gs^*r^*fR)$ induced by $g \mapsto (g-w)^n$. Since $w \in Jac(gRg)$ we have that $(g-w)^n$ has a left inverse in gRg (for every $n \geq 1$) which we denote by $(g-w)^{-n}$; since $\mathbf{h}_n(g + gs^*r^*fR) = (g-w)^n g = (g-w)^n$, this implies that, for every $n \geq 1$, $\mathbf{h}_n$ is an epimorphism.

Clearly, for every $n \geq 1$, $(g-w)^{-n}s^*r^*f + gs^*r^*fR \in Ker(\mathbf{h}_n)$. We will show that for all $n \geq 2$, $(g-w)^{-n}s^*r^*f + gs^*r^*fR \notin Ker(\mathbf{h}_{n-1})$. It then follows that $Ker(\mathbf{h}_n) < Ker(\mathbf{h}_{n+1})$ for all $n \geq 1$, contradicting the assumption that R is right Noetherian.

Suppose $(g-w)^{-n}s^*r^*f \in Ker(\mathbf{h}_{n-1})$. Then $(g-w)^{-1}s^*r^*f = gs^*r^*t$ for some $t \in fRf$. Together with $gs^*r^*f = gs^*r^*v + ws^*r^*f$ we get $(g-w)^{-1}s^*r^*f = gs^*r^*t = (g-w)^{-1}s^*r^*vt$ from which it follows that $gs^*r^*f = 0$ (since

$vt \in Jac(fRf)$), a contradiction.

A similar proof can be given in case R is left Noetherian. $\square$

Also the following characterisation of the class $\mathcal{P}$ is due to Puninski.

Theorem 3.1.9 ([25])

For a serial ring R, the following are equivalent:

(1) $R \in \mathcal{P}$;

(2) Every indecomposable finitely presented R-module has a local endomorphism ring.

Proof.

(1) $\Rightarrow$ (2) Suppose that R does not satisfy *(2)*. Then, by Lemma 2.1.5, there are $e \in \mathcal{E}$ and $r \in eR$ for which, with $M = eR/rR$, $End(M)$ is not local and therefore, by [9] (where it is shown that the pure-injective hull of an indecomposable finitely presented module M is indecomposable iff the endomorphism ring of M is local) $PE(M)$ is decomposable, i.e. the e-type $p(x) := tp^{\pm}_{M}(e)$ is decomposable. Corollary 3.1.3 implies that we can pick some $r^* \in eR$ and $s \in Re$ with $xr^* = 0 \notin p(x)$, $s|x \notin p(x)$ and $xr^* = 0 + s|x \in p(x)$. Clearly $r^* \notin rR$ so that ($rR < r^*R$ and) $r \in r^*Jac(R)$.

Since $T_R \models xr^* = 0 + s|x \leftrightarrow sr^*|xr^*$ and $xr^* = 0 + s|x \in p(x)$ we have that $tsr^* = r^* + rv$ for some $t, v \in R$. Put $J = Jac(Re)$, $I = rR$ and $s^* = e$. Then $r \in I$, $r^* \in I^*$, $s \in J$ (since $s|x \notin p(x)$ so that $e \notin Rs$, i.e. $Rs < Re$) and $s^* \in J^*$ (since $e \in Jac(Re)$ would imply $Re = Jac(Re)$). Moreover, $s^*r^* = r^* = tsr^* - rv = tsr^* - s^*rv$, $ts \in J$ and $rv \in I$, and therefore, by Theorem 3.1.4, the e-type $J^*/I(x)$ is inconsistent. Suppose, for a contradiction, that $R \in \mathcal{P}$. Then, the inconsistency of $J^*/I(x)$ implies that there are

$u^* \in I^*$,$w^* \in J^*$ with $w^*u^* = 0$ so that, since $w^* \in Re \setminus Jac(Re)$, $u^* = 0$, a contradiction.

(2) ⇒ (1) Suppose $R \notin \mathcal{P}$. Then there are $e \in \mathcal{E}$, $I < eR$ and $J < Re$ such that for all $v^* \in I^*$ and $w^* \in J^*$, $w^*v^* \neq 0$ and such that there are $r \in I$, $r^* \in I^*$, $s \in J$ and $s^* \in J^*$ with $s^*r^* = s^*r + sr^*$.

By Lemma 3.1.7 there exist $f, g \in \mathcal{E}$, $v \in Jac(fRf)$ and $w \in Jac(gRg)$ with $rf = r^*v \in I^*$ and $gs = ws^* \in J^*$. If we denote the indecomposable finitely presented R-module gR/gs^*r^*fR by M, then w acts by left multiplication as an endomorphism of M, because $gs^*r^*f = gs^*r^*v + ws^*r^*f$, $w \in gRg$ and $v \in fRf$. But then $w \in Jac(gRg)$ implies that $g - w$ acts (by left multiplication) as an automorphism of M so that (with $(g-w)^{-1}$ the left inverse of $g - w$ in the ring gRg) $(g-w)^{-1}s^*r^*f$ must be zero in M, i.e. $(g-w)^{-1}s^*r^*f = gs^*r^*ft$ for some $t \in fRf$. This implies that $gs^*r^*f = (g-w)(g-w)^{-1}s^*r^*f = (g-w)s^*r^*ft$ so that, since $(g-w)s^*r^*f = gs^*r^*v$, $gs^*r^*f = gs^*r^*vt$ and therefore $gs^*r^*(f - vt) = 0$. But since $t \in fRf$ and $v \in Jac(fRf)$ this means that $gs^*r^*f = 0$, a contradiction. □

We conclude this section with a definition and an easy remark.

Definition 3.1.10 Let $R \in \mathcal{P}$ and let $e \in \mathcal{E}$. For $I < eR$, we put $\mathcal{J}(I) = \bigcup_{r^* \in I^*} lann_{Re}(r^*) \leq Jac(R)e$ (where $lann_{Re}(r^*) = \{s \in Re : sr^* = 0\}$); for $J < Re$, we put $\mathcal{I}(J) = \bigcup_{s^* \in J^*} rann_{eR}(s^*) \leq eJac(R)$ (where $rann_{eR}(s^*) = \{r \in eR : s^*r = 0\}$).

Remark 3.1.11 Let $R \in \mathcal{P}$. It is easy to see that $\langle I, \mathcal{J}(I)\rangle$ is an e-pair whenever $I < eR$, and $\langle \mathcal{I}(J), J\rangle$ is an e-pair whenever $J < Re$. Also with,

$I < eR$ and $J < Re$, $\langle I, J\rangle$ is an e-pair iff $\mathfrak{I}(J) \leq I$ iff $\mathfrak{J}(I) \leq J$. So, in particular, if $I < eR$ ($J < Re$), then $\langle I, Jac(R)e\rangle$ ($\langle eJac(R), J\rangle$) is an e-pair.

3.2 Basic Opens and Isolated Points in Ziegler Spectra over Serial Rings

In this section we work out a description of the basic open sets in Zg_R and use this to characterise the points in Zg_R which are isolated (by a T-minimal pair) in Zg_R^T, where T is any complete theory for R-modules. We will make use of the fact that *if $e \in \mathcal{E}$, then for every indecomposable pure-injective R-module U the interval $[0, (e|x)(U)]$ in $pp_R^1(U)$ is linearly ordered* (see [5] (2.4) & (1.9)).

Lemma 3.2.1 *Let R be a serial ring and let U be a point in Zg_R. Then for every e-pair $\langle I, J\rangle$ of U, the set*

$$\{(xr = 0 \wedge s|x / xt = 0 + v|x) : r \in I, s \in J^*, t \in I^*, v \in J\}$$

forms a neighbourhood basis for U.

Proof. Let $\langle I, J\rangle$ be an e-pair of U and let O be an open containing U. Then, by Lemma 1.2.1, there exist $\varphi(x)/\psi(x) \in J^*/I(x)$ such that $U \in (\varphi(x)/\psi(x)) \subseteq O$ and $\psi(x) \leq \varphi(x) \leq e|x$. Say $\varphi(x) \equiv \sum_{i=1}^n xr_i = 0 \wedge s_i|x$ and $\psi(x) \equiv \sum_{j=1}^m xt_j = 0 \wedge v_j|x$.
For all $1 \leq i \leq n$, $xr_i = 0 \wedge s_i|x \leq \varphi(x) \leq e|x$. So $\varphi(U) = (xr = 0 \wedge s|x)(U)$ where $r = r_i$ and $s = s_i$ for some $1 \leq i \leq n$. Therefore $xr = 0 \wedge s|x \in J^*/I(x)$ and $U \in O' = (xr = 0 \wedge s|x / \sum_{j=1}^m xt_j = 0 \wedge v_j|x) \subseteq (\varphi(x)/\psi(x)) \subseteq O$. Note

that we can assume that $r \in eR$, $s \in Re$ and $t_j \in eR$, $v_j \in Re$ for all $1 \leq j \leq m$. So, since $\neg\psi(x) \in J^*/I(x)$, we have that for all $1 \leq j \leq m$, $t_j \in I^*$ or $v_j \in J$. Collect the $t_j \in I^*$ and rename them to form $T = \{t'_1, \ldots, t'_k\} \subseteq I^*$ (some $k \leq m$ and possibly $T = \emptyset$) and collect the $v_j \in J$ and rename them to form $S = \{v'_1, \ldots, v'_l\} \subseteq J$ (some $l \leq m$ and possibly $S = \emptyset$). (Of course $k + l \geq m$.)

Then for all $j = 1, \ldots, m$, $t_j \in T$ or $v_j \in S$, and it is easy to see that

$$\sum_{j=1}^{m} xt_j = 0 \wedge v_j|x \leq \sum_{j=1}^{k} xt'_j = 0 + \sum_{j=1}^{l} v'_j|x$$

Therefore, $U \in O'' := (xr = 0 \wedge s|x/\sum_{j=1}^{k}(xt'_j = 0 \wedge e|x) + \sum_{j=1}^{l} v'_j|x) \subseteq O' \subseteq O$. If $T \neq \emptyset$, renumber the elements of T such that

$$t'_1R \leq t'_2R \leq \cdots \leq t'_kR$$

(note that we can do this because $T \subseteq I^* \subset eR$ and eR is a uniserial right R-module).

Then $T_R \models \sum_{j=1}^{k} xt'_j = 0 \leftrightarrow xt'_1 = 0$.

If $S \neq \emptyset$, renumber the elements of S to get

$$Rv'_1 \leq Rv'_2 \leq \cdots \leq Rv'_l$$

(note that we can do this because $S \subseteq J \subset Re$ and Re is a uniserial left R-module).

Then $T_R \models \sum_{j=1}^{l} v'_j|x \leftrightarrow v'_l|x$. So $(\sum_{j=1}^{m} xt_j = 0 \wedge v_j|x) \wedge e|x \leq (xt'_1 = 0 + v'_l|x) \wedge e|x \equiv (xt'_1 = 0 \wedge e|x) + v'_l|x$.

Conclusion: $U \in (xr = 0 \wedge s|x/(xt'_1 = 0 \wedge e|x) + v'_l|x) \subseteq O'' \subseteq O$. Finally, note that $(xr = 0 \wedge s|x/(xt' = 0 \wedge e|x) + v'|x) = (xr = 0 \wedge s|x/xt' = 0 + v'|x)$.$\square$

Corollary 3.2.2 *If R is a serial ring, then the set of opens in Zg_R of the form*

$$(xr = 0 \wedge s|x/xt = 0 + v|x)$$

with $r, t \in eR$ and $s, v \in Re$ and $e \in \mathcal{E}$, forms a basis of the opens in Zg_R.

Note that if $U \in Zg_R$, $e \in \mathcal{E}$, $r, t \in eR$ and $s, v \in Re$, then $U \in (xr = 0 \wedge s|x/xt = 0 + v|x)$ iff U has an e-pair $\langle I, J\rangle$ with $I \in [rR, tR)$ and $J \in [Rv, Rs)$.
We will use this fact and the above corollary without explicitly referring to them.

Proposition 3.2.3 *Let R be a serial ring and let T be a complete theory of R-modules. If U be a point in Zg_R^T, then U is isolated in Zg_R^T iff for every (or, equivalently, for some) e-pair $\langle I, J\rangle$ of U there exist principal right R-ideals $I', I'' \leq eR$ and there exist principal left R-ideals $J', J'' \leq Re$ with $I \in [I'', I')$ and $J \in [J'', J')$ such that if $\langle K, L\rangle$ is an e-pair with $K \in [I'', I')$ and $L \in [J'', J')$, then either $H(L^*/I(x)) \notin Zg_R^T$ or $\langle K, L\rangle \sim \langle I, J\rangle$.*

Proof. ($\Rightarrow$) Let $\langle I, J\rangle$ be an e-pair of U. By Lemma 3.2.1 we have that U is isolated in Zg_R^T by an open

$$O = (xr = 0 \wedge s|x/(xt = 0 \wedge e|x) + v|x)$$

where $r \in I$, $t \in I^*$, $s \in J^*$ and $v \in J$.
Put $I' = tR$, $J' = Rs$, $I'' = rR$ and $J'' = Rv$. It is easy to see that these ideals satisfy the required condition.

($\Leftarrow$) Let $\langle I, J\rangle$ and $I' = tR$, $I'' = rR$, $J' = Rs$, $J'' = Rv$ be as in the statement. Then it is easy to see that $U \in O := (xr = 0 \wedge s|x/(xt = 0 \wedge e|x) + v|x)$. Moreover, O isolates U in Zg_R^T. For if V is a point in Zg_R^T which is contained in O, then (since $s|x \leq e|x$) there exists an e-pair $\langle K, L\rangle$ of V with $K \in [I'', I')$ and $L \in [J'', J')$. And therefore $V \cong U$. $\square$

In the proof of the following proposition we will make use of the fact that for a complete theory T for R-modules, if $\psi <_T \varphi$, where $\psi, \varphi \in pp_R^1$, then there is some $U \in Zg_R^T$ with $\psi(U) < \varphi(U)$; here R is an arbitrary ring (this is a result by Ziegler; see [18](4.35) or [37](4.8)).
We will use the fact that for a complete theory T for R-modules, if p is a pp-type and $\varphi/\psi \in p$ is a T-minimal pair, then for any $\theta \in p^+$, also $(\varphi \wedge \theta)/(\psi \wedge \theta) \in p$ and is a T-minimal pair (see [18](9.1)).

Proposition 3.2.4 *Let R be a serial ring and let T be a complete theory for R-modules. If U be a point in Zg_R^T, then U is isolated in Zg_R^T by a T-minimal pair iff*
for every (or, equivalently, for some) e-pair $\langle I, J\rangle$ of U there exist principal right R-ideals $I', I'' \leq eR$ and there exist principal left R-ideals $J', J'' \leq Re$ with $I \in [I'', I')$ and $J \in [J'', J')$ and such that if $\langle K, L\rangle$ is an e-pair with $K \in [I'', I')$ and $L \in [J'', J')$, then either $H(L^*/K(x)) \notin Zg_R^T$ or $K = I$ and $L = J$.

Proof ($\Leftarrow$) Let $\langle I, J\rangle$ and $I' = tR$, $I'' = rR$, $J' = Rs$, $J'' = Rv$ be as in the statement. Then we have that U is isolated in Zg_R^T by

$$(xr = 0 \wedge s|x/(xt = 0 \wedge e|x) + v|x)$$

for some $e \in \mathcal{E}$.

Call $(xr = 0 \wedge s|x) \equiv \varphi(x)$ and $[\varphi(x) \wedge ((xt = 0 \wedge e|x) + v|x)] \equiv \psi(x)$. We'll show that φ/ψ is a minimal pair.

So suppose that there exists a $\sigma(x) \in pp_R^1$ such that $\psi(x) <_T \sigma(x) <_T \varphi(x)$. Then, by [18](4.35), there exists an indecomposable pure-injective $V \in Zg_R^T$ such that $\sigma(V) < \varphi(V)$, i.e. $V \in (\varphi(x)/\sigma(x)) \cap Zg_R^T \subseteq (\varphi(x)/\psi(x)) \cap Zg_R^T$. So $V \cong U$. And, again by [18](4.35), there exists an indecomposable pure-injective $W \in Zg_R^T$ such that $\psi(W) < \sigma(W)$, i.e. $W \in (\sigma(x)/\psi(x)) \cap Zg_R^T \subseteq (\varphi(x)/\psi(x)) \cap Zg_R^T$. So $W \cong U$. Therefore $\psi(U) < \sigma(U) < \varphi(U)$, and in particular $\sigma(U) \neq 0$.

Take an $a \in \sigma(U) \setminus \psi(U)$. Then $a \in (\varphi(x)/\psi(x))(U)$. So there exists an e-pair $\langle K, L\rangle$ of U such that $T_R \models tp_U^{\pm}(a) \leftrightarrow L^*/K(x)$, where $r \in K$, $v \in L$, $t \in K^*$ and $s \in L^*$, i.e. $I'' \leq K < I'$ and $J'' \leq L < J'$. And by the hypothesis we then have that $K = I$ and $L = J$, i.e. $T_R \models tp_U^{\pm}(a) \leftrightarrow J^*/I(x)$.

But we can also take a $b \in \varphi(U) \setminus \sigma(U)$. Then $b \in \varphi(U)\setminus\psi(U)$, i.e. $b \in (\varphi(x)/\psi(x))(U)$. By a similar argument as above we have that $T_R \models tp_u^{\pm}(b) \leftrightarrow J^*/I(x) \leftrightarrow tp_U^{\pm}(a)$. But this contradicts $a \in \sigma(U)$ and $b \notin \sigma(U)$.

Conclusion: φ/ψ is a T-minimal pair isolating U in Zg_R^T.

$(\Rightarrow)$ Let U be isolated in Zg_R^T by the T-minimal pair φ/ψ. Then there exists an e-pair $\langle I, J\rangle$ of U (some $e \in \mathcal{E}$) with $r \in I$, $s \in J^*$, $t \in I^*$ and $v \in J$ such that

$$(xr = 0 \wedge s|x/(xt = 0 \wedge e|x) + v|x)$$

isolates U in Zg_R^T and where $(xr = 0 \wedge s|x) \leq_T \varphi(x)$ and $\psi(x) \leq_T ((xt = 0 \wedge e|x) + v|x)$. We will show that $\langle I, J\rangle$ satisfies the necessary conditions; together with Lemma 1.2.2 the proof will then be complete.

Now, $(xr = 0 \wedge s|x) \in J^*/I(x)^+$ implies that $\varphi(x) \in J^*/I(x)^+$ and therefore, by [18](9.1), $(\varphi(x) \wedge (xr = 0 \wedge s|x))/(\psi(x) \wedge (xr = 0 \wedge s|x))$ is a T-minimal pair in $J^*/I(x)$.

Let $a \in U$ be such that $T_R \models tp^{\pm}_U(a) \leftrightarrow J^*/I(x)$. Call $\theta(x) \equiv (xr = 0 \wedge s|x)$ and $\sigma(x) \equiv [((xt = 0 \wedge e|x) + v|x) \wedge \theta(x)]$. Then $\theta(x) \in J^*/I(x)$ and so , by [18](9.1), $\theta/\psi \wedge \theta$ is a T-minimal pair in $J^*/I(x)$. And therefore θ/σ is a T-minimal pair in $J^*/I(x)$: $\psi(x) \wedge \theta(x) \leq \sigma(x) \leq \theta(x)$ and θ/σ does not collapse ($a \in \theta(U) \setminus \sigma(U)$). So (θ/σ) isolates U in Zg^T_R. Call $I' = tR$, $I'' = rR$, $J' = Rs$ and $J'' = Rv$. Then $I'' \leq I \leq I'$ and $J'' \leq J \leq J'$.

Let $\langle K, L \rangle$ be an e-pair of a point in Zg^T_R and such that $I'' \leq K < I'$ and $J'' \leq L < J'$. We'll show that $K = I$ and $L = J$.

Let $b \in U$ be such that $T_R \models tp^{\pm}_U(b) \leftrightarrow L^*/K(x)$.

<u>Case 1</u>: $K \leq I$.

Suppose that $K < I$. Take a $u \in I \setminus K$. Then $I'' \leq uR$, so $xu = 0 \leq xr = 0$ and $(xu = 0 \wedge s|x)(U) \leq \theta(U)$. But $b \notin (xu = 0 \wedge s|x)(U)$ (since $u \in K^*$) and $b \in \theta(U)$ (since $r \in K$ and $s \in L^*$) and therefore $(xu = 0 \wedge s|x)(U) < \theta(U)$. And because θ/σ is a T-minimal pair and $\theta(x) \leq e|x$ we must have that $(xu = 0 \wedge s|x)(U) \leq \sigma(U)$. But $a \notin \sigma(U)$ and $a \in (xu = 0 \wedge s|x)(U)$ (since $u \in I$ and $s \in J^*$). Contradiction.

Therefore $K = I$.

Suppose that $J < L$. Take a $w \in L \setminus J$. Then $J'' < Rw$, so $v|x \leq w|x$ and $\sigma(U) \leq ((xt = 0 \wedge e|x) + w|x)(U) \cap \theta(U) \leq \theta(U)$. But $b \notin ((xt = 0 \wedge e|x) + w|x)(U)$ (since $t \in I^* = K^*$ and $w \in L$) and $b \in \theta(U)$ (since $r \in I = K$ and $s \in L^*$). This implies that, because θ/σ is a T-minimal pair, $\sigma(U) = ((xt = 0 \wedge e|x) + w|x)(U) \cap \theta(U)$. But $a \notin \sigma(U)$, $a \in (w|x)(U)$ (since

$w \in J^*$) and $a \in \theta(U)$. Contradiction.

Suppose that $L < J$. Take a $w \in J \setminus L$. Then $J'' < Rw$, so $\sigma(U) \leq ((xt = 0 \wedge e|x) + w|x)(U) \cap \theta(U)$. But $a \notin ((xt = 0 \wedge e|x) + w|x)(U)$ (since $t \in I^*$ and $w \in J$) and $a \in \theta(U)$. This implies that, since θ/σ is a T-minimal pair, $\sigma(U) = ((xt = 0 \wedge e|x) + w|x)(U) \cap \theta(U)$. But $b \notin \sigma(U)$ (since $t \in K^*$ and $v \in L$), $b \in \theta(U)$ and $b \in (w|x)(U)$ (since $w \in L^*$). Contradiction.

Therefore $K = I$ and $L = J$ if $K \leq I$.

Case 2: $I \leq K$.

A similar argument shows that $K = I$ and $L = J$ if $I \leq K$. $\square$

Chapter 4

Ziegler Spectra over Serial Rings with Krull Dimension

4.1 Examples

All the information about the Ziegler spectra presented below will follow from the results from the subsequent sections, but can also be easily obtained using the results from the previous chapter. We leave the details to the reader and hope that the examples will clarify the techniques which we use in the subsequent sections.

Example 1. The ring $R = \begin{pmatrix} \mathbb{Z}_{(p)} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is a right (but not left) Noetherian serial ring. The set $\mathcal{E} = \{e, f\}$ with $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is a complete set of orthogonal primitive idempotents in R. $eR = \begin{pmatrix} \mathbb{Z}_{(p)} & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$, $Re = \begin{pmatrix} \mathbb{Z}_{(p)} & 0 \\ 0 & 0 \end{pmatrix}$, $fR = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Q} \end{pmatrix}$, $Rf = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$.

The chain of the right R-ideals contained in eR is:

$$I_0 < I_q < \cdots < I_n < I_{n-1} < \cdots < I_2 < I_1 < eR$$

where $I_q = \left(\begin{smallmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right)$, $I_n = \left(\begin{smallmatrix} p^n\mathbb{Z}_{(p)} & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right)$ for all $n \geq 1$ and I_0 is the zero ideal $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.
The chain of the left R-ideals contained in Re is:

$$J_0 < \cdots < J_m < J_{m-1} < \cdots < J_2 < J_1 < Re$$

where $J_m = \left(\begin{smallmatrix} p^m\mathbb{Z}_{(p)} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for all $m \geq 1$ and J_0 is the zero ideal.
One can check that for every $I < eR$ and for every $J < Re$, $\langle I, J\rangle$ is an e-pair.
The only right R-ideal properly contained in fR is the zero ideal which we will denote by K_0.
The chain of left R-ideals contained in Rf is:

$$L^0 < \cdots < L_n < L_{n-1} < \cdots < L_1 < L_0 < L_{-1} < L_{-2} < \cdots < L_{-n} < \cdots < L_q < Rf$$

where $L_q = \left(\begin{smallmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right)$, $L_z = \left(\begin{smallmatrix} 0 & p^z\mathbb{Z}_{(p)} \\ 0 & 0 \end{smallmatrix}\right)$ for all $z \in \mathbb{Z}$ and L^0 is the zero ideal.
One can check that for every $K < fR$ and for every $L < Rf$, $\langle K, L\rangle$ is an f-pair.
The $\sim$-equivalence classes in the set of pairs are the following:
$\{\langle I_0, J_0\rangle, \langle K_0, L^0\rangle\}$,
$\{\langle I_0, J_m\rangle : m \geq 1\} \cup \{\langle K_0, L_z\rangle : z \in \mathbb{Z}\}$,
$\{\langle I_n, J_0\rangle : n \geq 1\}$,
$\{\langle I_n, J_m\rangle : n, m \geq 1 \text{ and } n + m = k\}$ for $k > 1$,
$\{\langle I_q, J_m\rangle : m \geq 1\}$ and

$\{\langle I_q, J_0\rangle\}$.

So $Zg_R = \{H(J_1^*/I_n(x)), n \geq 1; H(J_1^*/I_0(x)); H(J_0^*/I_0(x)); H(J_0^*/I_1(x)); H(J_1^*/I_q(x)); H(J_0^*/I_q(x)); H(L_q^*/K_0(x))\}$.

One can easily verify that $H(J_1^*/I_n(x))$ with $n \geq 1$ and $H(J_1^*/I_0(x))$ are the points of CB-rank 0 (i.e. the isolated points); in fact, the isolated points are isolated by a minimal pair:

for $n, m \geq 1$, the e-type $J_m^*/I_n(x)$ contains the minimal pair $(xr_n = 0 \wedge e|x)/[(xt_n = 0 + s|x) \wedge xr_n = 0 \wedge e|x]$ where $r_n = \left(\begin{smallmatrix} p^{n-1} & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $t_n = r_{n-1}$ and $v = p^m e$;

and for $m \geq 1$, the e-pair $J_m^*/I_0(x)$ contains the minimal pair $e|x/[(xt = 0 + v_m|x) \wedge e|x]$ where $t = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $v_m = p^m e$.

$H(J_0^*/I_0(x))$, $H(J_0^*/I_1(x))$, $H(J_1^*/I_q(x))$ and $H(L_q^*/K_0(x))$ are the points of CB-rank 1. $H(J_0^*/I_q(x))$ is of CB-rank 2.

The closed points are $H(J_0^*/I_0(x)); H(J_1^*/I_n(x)), n \geq 1; H(J_0^*/I_q(x))$ and $H(L_q^*/K_0(x))$.

And $Cl(H(J_1^*/I_q(x))) = \{H(J_1^*/I_q(x)), H(J_0^*/I_q(x))\}$,

$Cl(H(J_0^*/I_1(x))) = \{H(J_0^*/I_1(x)), H(J_0^*/I_q(x))\}$,

$Cl(H(J_1^*/I_0(x))) = \{H(J_1^*/I_0(x)), H(J_0^*/I_0(x)), H(L_q^*/K_0(x))\}$.

The injective points are $H(J_0^*/I_0(x)), H(J_0^*/I_1(x))$ and $H(J_0^*/I_q(x))$. None of the injective points is isolated.

The isolated points are dense: $H(J_1^*/I_0(x))$ is isolated and $Cl(H(J_1^*/I_0(x))) = \{H(J_1^*/I_0(x)), H(J_0^*/I_0(x)), H(L_q^*/K_0(x))\}$; every open containing $H(J_0^*/I_1(x))$ or $H(J_1^*/I_q(x))$ or $H(J_0^*/I_q(x))$ contains $H(J_1^*/I_n(x))$ for some $n \geq 1$.

Zg_R is a T_0-space but not a T_1-space (not every point is closed).

Example 2. The ring $R = \left(\begin{smallmatrix} \mathbb{Z}_{(p)} & p\mathbb{Z}_{(p)} \\ \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} \end{smallmatrix}\right)$ is a two-sided Noetherian serial ring. The set $\mathcal{E} = \{e, f\}$ with $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $f = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is a complete set of orthogonal primitive idempotents in R. $eR = \left(\begin{smallmatrix} \mathbb{Z}_{(p)} & p\mathbb{Z}_{(p)} \\ 0 & 0 \end{smallmatrix}\right)$, $Re = \left(\begin{smallmatrix} \mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_{(p)} & 0 \end{smallmatrix}\right)$, $fR = \left(\begin{smallmatrix} 0 & 0 \\ \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} \end{smallmatrix}\right)$, $Rf = \left(\begin{smallmatrix} 0 & p\mathbb{Z}_{(p)} \\ 0 & \mathbb{Z}_{(p)} \end{smallmatrix}\right)$.

The chain of right R-ideals contained in eR is:

$$I_0 < \cdots < I'_n < I_n < I'_{n-1} < I_{n-1} < \cdots < I'_1 < I_1 < eR$$

where $I_n = \left(\begin{smallmatrix} p^n\mathbb{Z}_{(p)} & p^n\mathbb{Z}_{(p)} \\ 0 & 0 \end{smallmatrix}\right)$ for all $n \geq 1$, $I'_n = \left(\begin{smallmatrix} p^n\mathbb{Z}_{(p)} & p^{n+1} \\ 0 & 0 \end{smallmatrix}\right)$ for all $n \geq 1$ and I_0 is the zero ideal $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

The chain of left R-ideals contained in Re is:

$$J_0 < \cdots < J'_m < J_m < J'_{m-1} < J_{m-1} < \cdots < J'_1 < J_1 < Re$$

where $J_m = \left(\begin{smallmatrix} p^m\mathbb{Z}_{(p)} & 0 \\ p^m\mathbb{Z}_{(p)} & 0 \end{smallmatrix}\right)$ for all $m \geq 1$, $J'_m = \left(\begin{smallmatrix} p^{m+1}\mathbb{Z}_{(p)} & 0 \\ p^m\mathbb{Z}_{(p)} & 0 \end{smallmatrix}\right)$ for all $m \geq 1$ and J_0 is the zero ideal.

One can check that for every $I < eR$ and for every $J < Re$, $\langle I, J\rangle$ is an e-pair.

The chain of right R-ideals contained in fR is:

$$K_0 < \cdots < K'_n < K_n < K'_{n-1} < K_{n-1} < \cdots < K'_1 < K_1 < fR$$

where $K_n = \left(\begin{smallmatrix} 0 & 0 \\ p^n\mathbb{Z}_{(p)} & p^n\mathbb{Z}_{(p)} \end{smallmatrix}\right)$ for all $n \geq 1$, $K'_n = \left(\begin{smallmatrix} 0 & 0 \\ p^n\mathbb{Z}_{(p)} & p^{n+1}\mathbb{Z}_{(p)} \end{smallmatrix}\right)$ for all $n \geq 1$ and K_0 is the zero ideal.

The chain of left R-ideals contained in Rf is:

$$L_0 < \cdots L'_m < L_m < L'_{m-1} < L_{m-1} < \cdots < L'_1 < L_1 < Rf$$

where $L_m = \left(\begin{smallmatrix} 0 & p^m\mathbb{Z}_{(p)} \\ 0 & p^m\mathbb{Z}_{(p)} \end{smallmatrix}\right)$ for all $m \geq 1$, $L'_m = \left(\begin{smallmatrix} 0 & p^{m+1}\mathbb{Z}_{(p)} \\ 0 & p^m\mathbb{Z}_{(p)} \end{smallmatrix}\right)$ for all $m \geq 1$ and L_0 is the zero ideal.

One can check that for every $K < fR$ and for every $L < Rf$, $\langle K, L\rangle$ is an f-pair.

The $\sim$-equivalence classes in the set of pairs are the following:

$\{\langle I_0, J_0\rangle, \langle K_0, L_0\rangle\}$,

$\{\langle I_0, J_m\rangle : m \geq 1\} \cup \{\langle K_0, L_m\rangle : m \geq 1\}$,

$\{\langle I_0, J'_m\rangle : m \geq 1\} \cup \{\langle K_0, L'_m\rangle : m \geq 1\}$,

$\{\langle I_n, J_0\rangle : n \geq 1\} \cup \{\langle K_n, L_0\rangle : n \geq 1\}$,

$\{\langle I'_n, J_0\rangle : n \geq 1\} \cup \{\langle K'_n, L_0\rangle : n \geq 1\}$,

$\{\langle I_n, J_m\rangle : n, m \geq 1 \text{ and } n+m = k\} \cup \{\langle K_n, L_m\rangle : n, m \geq 1 \text{ and } n+m = k\}$ for $k \geq 1$,

$\{\langle I'_n, J_m\rangle : n, m \geq 1 \text{ and } n+m = k\} \cup \{\langle K'_n, L_m\rangle : n, m \geq 1 \text{ and } n+m = k\}$ for $k \geq 1$,

$\{\langle I_n, J'_m\rangle : n, m \geq 1 \text{ and } n+m = k\} \cup \{\langle K_n, L'_m\rangle : n, m \geq 1 \text{ and } n+m = k\}$ for $k \geq 1$ and

$\{\langle I'_n, J'_m\rangle : n, m \geq 1 \text{ and } n+m = k\} \cup \{\langle K'_n, L'_m\rangle : n, m \geq 1 \text{ and } n+m = k\}$ for $k \geq 1$.

So $Zg_R = \{H(J_1^*/I_n(x)), n \geq 1; H(J_1'^*/I_n(x)), n \geq 1; H(J_1^*/I'_n(x)), n \geq 1; H(J_1'^*/I'_n(x)), n \geq 1; H(J_0^*/I_1(x)); H(J_0^*/I'_1(x)); H(J_0^*/I_0(x)); H(J_1^*/I_0(x)); H(J_1'^*/I_0(x))\}$.

The points of CB-rank 0 (i.e. the isolated points) are $H(J_1^*/I_n(x)), n \geq 1; H(J_1'^*/I_n(x)), n \geq 1$ and $H(J_1^*/I'_n(x)), n \geq 1$. One can check that the isolated points are isolated by a minimal pair. $H(J_0^*/I_0(x))$ is the unique point of CB-rank 2.

The remaining points are of CB-rank 1.

The closed points are $H(J_1^*/I_n(x)), n \geq 1; H(J_1'^*/I_n(x)), n \geq 1; H(J_1^*/I'_n(x)), n \geq$

1 and $H(J_0^*/I_0(x))$. So every isolated point is closed, i.e. the clopen points are precisely the isolated points.
$Cl(H(J_0^*/I_1(x))) = \{H(J_0^*/I_1(x)), H(J_0^*/I_0(x))\}$,
$Cl(H(J_0^*/I_1'(x))) = \{H(J_0^*/I_1'(x)), H(J_0^*/I_0(x))\}$,
$Cl(H(J_1'^*/I_0(x))) = \{H(J_1'^*/I_0(x)), H(J_0^*/I_0(x))\}$.
The injective points are $H(J_0^*/I_0(x)), H(J_0^*/I_1(x))$ and $H(J_0^*/I_1'(x))$.

4.2 Isolated Points in Ziegler Spectra over Serial Rings with Krull Dimension

In this section we show that if R is a serial ring with Krull dimension, then for every complete theory T for R-modules, a point in Zg_R^T is isolated in Zg_R^T iff it is isolated in Zg_R^T by a T-minimal pair.
We need a definition and an easy lemma.

Definition 4.2.1 Let R be serial, $e \in \mathcal{E}$, $I \leq eR$ and $A \subseteq [0, eR]$.
I is called a *right A-accumulation point* if $I = \bigcap_{\lambda\in\Lambda} I_\lambda$ for some family $\{I_\lambda\}_\Lambda$ of right R-ideals with $I < I_\lambda \in A$ for all $\lambda \in \Lambda$.
I is called a *left A-accumulation point* if $I = \bigcup_{\lambda\in\Lambda} I_\lambda$ $(= \sum_{\lambda\in\Lambda} I_\lambda)$ for some family $\{I_\lambda\}_\Lambda$ of right R-ideals with $I > I_\lambda \in A$ for all $\lambda \in \Lambda$.
I is called an *A-accumulation point* if it is a right or left A-accumulation point.
If I is not a right A-accumulation point, then there is some $eR \geq K > I$ for which $A \cap [I, K) = \{I\}$ or $\emptyset$.
If I is not a left A-accumulation point, then there is some $K < I$ for which $A \cap (K, I] = \{I\}$ or $\emptyset$.

We say that I is a *right (left) accumulation point* if it is a right (left) A-accumulation point for some subset A of $[0, eR]$.
Similar definitions may be formulated for $J < Re$ and A a subset of $[0, Re]$.

Lemma 4.2.2 *If R is a serial ring with Krull dimension and if A is a nonempty subset of $[0, eR]$ or $[0, Re]$, then there is an element of A which is not an A-accumulation point.*

Proof. If $Kdim(R) = 0$ (i.e. R is Artinian) or A is finite, then the statement holds trivially. So we assume that $Kdim(R) > 0$ and that A is infinite.
We proceed by transfinite induction on $Kdim(A)(\leq Kdim(R))$.
Let $Kdim(A) = \alpha > 0$.
Let $I_1 > I_2 > \cdots$ be an infinite descending chain in A. We can assume that $A \cap (I_{i+1}, I_i)$ is nonempty for every $i \geq 1$. There is some $i \geq 1$ with $Kdim(A \cap (I_{i+1}, I_i)) < \alpha$ so that, by the induction hypothesis, there is some $I \in A \cap (I_{i+1}, I_i)$ which is not a $A \cap (I_{i+1}, I_i)$-accumulation point and therefore not an A-accumulation point. □

Theorem 4.2.3 *Suppose R is a serial ring with Krull dimension and let T be a complete theory of R-modules. Then the points of Zg_R^T which are isolated by a T-minimal pair form a dense subset of Zg_R^T. Therefore, every point in Zg_R^T which is isolated in Zg_R^T is isolated by a T-minimal pair (i.e.* **T** *satisfies* Condition ($\wedge$) *from* [18]*).*

Proof. Let O be a basic open in Zg_R with $O \cap Zg_R^T \neq \emptyset$. Then there are $e \in \mathcal{E}$, $r, t \in eR$ and $s, v \in Re$ with $O = (xr = 0 \wedge s|x/xt = 0 + v|x)$, and there is an e-pair $\langle I, J \rangle$ with $rR \leq I < tR$ and $Rs \leq J < Rv$.

Put

$A = \{I \in [rR, tR) : \exists J \in [Rs, Rv) \text{ with } \langle I, J\rangle \text{ an } e\text{-pair and } H(J^*/I(x)) \in Zg_R^T\}.$

Then $A \neq \emptyset$ and, by Lemma 4.2.2, A has an element which is not an A-accumulation point. Let I be such an element.
Put

$$B = \{J \in [Rs, Rv) : \langle I, J\rangle \text{ is an } e\text{-pair and } H(J^*/I(x)) \in Zg_R^T\}.$$

Then $B \neq \emptyset$ and, by Lemma 4.2.2, B has an element which is not a B-accumulation point. Let J be such an element.
If I is maximal in A, put $t_1 = t$; otherwise take $t_1 \in I' \backslash I$, where $I < I' \leq eR$ is such that $A \cap [I, I') = \{I\}$.
If I is minimal in A, put $r_1 = r$; otherwise take $r_1 \in I \backslash I''$, where $I'' < I$ is such that $A \cap (I'', I] = \{I\}$.
If J is maximal in B, put $s_1 = s$; otherwise take $s_1 \in J' \backslash J$, where $J < J' \leq Re$ is such that $B \cap [J, J') = \{J\}$.
If J is minimal in B, put $v_1 = v$; otherwise take $v_1 \in J \backslash J''$, where $J'' < J$ is such that $B \cap (J'', J] = \{J\}$.
Then $(xr_1 = 0 \wedge s_1|x / xt_1 = 0 + xv_1|x) \subseteq O$ and $xr_1 = 0 \wedge s_1|x / xt_1 = 0 + xv_1|x$ is a T-minimal pair in $J^*/I(x)$. $\square$

The following corollary is an immediate consequence of Theorem 4.2.3 and [18](10.19), which says that if R is a ring satisfying Condition $(\wedge)$, then for every complete theory T for R-modules, if $U \in Zg_R^T$, then $m\text{-}dim(U) = CB(U)$ in Zg_R^T.

Corollary 4.2.4 *If R is a serial ring with Krull dimension, then for every $U \in Zg_R$, $CB(U) = m\text{-}dim(U)$ and, in particular, for every complete theory T of R-modules, $m\text{-}dim(T) = CB(Zg_R^T)$.*

4.3 CB-analysis of Ziegler Spectra over Serial Rings with Krull Dimension

Since Krull dimension is a measure for the complexity of the lattice of ideals in a ring and CB-rank is a measure for the complexity of the Ziegler spectrum of a ring, it is not unreasonable to expect, for a serial ring R, some connection between Kdim(R) and $CB(Zg_R)$.

In this section we will show that if R is a serial ring with Krull dimension α, then Zg_R has CB-rank and $CB(Zg_R) \leq 2\alpha$. We also characterise the serial rings in $\mathcal{P}$ with Krull dimension in terms of CB-rank.

We need a definition and a few lemmas.

Definition 4.3.1 Let R be a serial ring and $e \in \mathcal{E}$. For $I < eR$ we define $d(I)$ as follows:

$d(I) = 0$ if I is not an accumulation point;

If α is an ordinal and $d(I) \not< \alpha$, then $d(I) = \alpha$ provided there are principal right ideals $I', I'' \leq eR$ with $I \in [I'', I')$ such that if $K \in [I'', I')$ and $K \neq I$, then $d(K) < \alpha$.

A similar definition can be given for $J < Re$.

Given $I < eR$ (with R serial and $e \in \mathcal{E}$), $d(I)$ needn't be defined. We will however see that if R has Krull dimension, then $d(I)$ is defined for all $I < eR$, and $d(I) \leq Kdim(R)$ (and the same holds for all $J < Re$).
Note that for $I < eR$, $d(I) = 0$ iff I is principal and I is not a right accumulation point.

Lemma 4.3.2 *Let R be a serial ring, M a uniserial module over R and $a \in M$. If β is an ordinal with $aJac(\beta)^{k+1} = aJac(\beta)^k$, then $aJac(\gamma) = aJac(\beta)^k = aJac(\beta+1)$ for all $\gamma > \beta$.*

Proof. Clearly $aJac(\beta)^k = aJac(\beta)^{k+1}$ implies that $aJac(\beta+1) = aJac(\beta)^k$. Assume that $aJac(\lambda) = aJac(\beta+1)$ for some $\lambda > \beta$ and suppose that $aJac(\lambda)^2 < aJac(\lambda)$ (i.e. $aJac(\beta+1)Jac(\lambda) < aJac(\beta+1)$). For every $b \in aJac(\beta+1) \setminus aJac(\beta+1)Jac(\lambda)$ Lemma 2.2.3 gives $bJac(\lambda) = aJac(\beta+1)Jac(\lambda)$. But $bJac(\lambda) \leq aJac(\lambda)^2$ whenever $b \in aJac(\beta+1)$. Therefore $aJac(\lambda)^2 = aJac(\lambda)$, a contradiction. We conclude that if $\lambda \geq \beta$ and $aJac(\lambda) = aJac(\beta+1)$, then $aJac(\lambda)^2 = aJac(\lambda)$ and therefore $aJac(\lambda)^i = aJac(\lambda) = aJac(\beta+1)$ for all $i \geq 1$.
An induction argument now shows that $aJac(\gamma) = aJac(\beta+1)$ for all $\gamma > \beta$.
$\square$

Lemma 4.3.3 ([16] (5.1))
Let M be a uniserial module over a serial ring R, and α an ordinal. If $MJac(\alpha) < N < M$, then there is some ordinal $\delta < \alpha$ and some $w \in M$ such that $wJac(\delta)^2 < wJac(\delta) = N < wR$.

Proof. Since $uJac(\alpha) < N$ for every $u \in M \backslash N$, we can pick an ordinal $\beta \leq \alpha$ which is minimal with the property that $vJac(\beta) < N$ for some $v \in M \backslash N$. Suppose β is a limit ordinal. Then 2.2.3 implies that $N > vJac(\beta) = \bigcap_{\gamma<\beta} vJac(\gamma)$ so that $N > vJac(\gamma)$ for some $\gamma < \beta$, which contradicts the minimality of β. We conclude that β is a nonlimit ordinal, say $\beta = \delta + 1$. So $N > vJac(\beta) = \bigcap_{i\geq 1} vJac(\delta)^i$ which implies that $vJac(\delta)^k \leq N$ for some $k \geq 1$. Let k be minimal with this property. Then $vJac(\delta) \leq N < vJac(\delta)^{k-1}$. Take $w \in vJac(\delta)^{k-1} \backslash N$. Then $wJac(\delta) \leq N$. But $wJac(\delta) < N$ would contradict the minimality of β, and therefore, $wJac(\delta) = N$.
Suppose that $wJac(\delta) = wJac(\delta)^2$. Then Lemma 4.3.2 implies that $N = wJac(\lambda)$ for all $\lambda \geq \delta$. In particular, $N = wJac(\delta) \leq MJac(\alpha) < N$, a contradiction. We conclude that $wJac(\delta)^2 < wJac(\delta) = N$. $\square$

Lemma 4.3.4 *Let R be a serial ring, $e \in \mathcal{E}$ and α some ordinal.*
If $rJac(\alpha)^2 < rJac(\alpha) = I < rR$ for some $r \in eR$, then $d(I) \leq \alpha$.
And, similarly, if $Jac(\alpha)^2 r < Jac(\alpha)r = J < Rr$ for some $r \in Re$, then $d(J) \leq \alpha$.

Proof. We give a proof by induction on α.
If $rJac(0)^2 < rJac(0) = I < rR$ for some $r \in eR$, then both rR/I and $I/rJac(0)^2$ are simple and nonzero so that I is not an accumulation point and therefore $d(I) = 0$.
Let $rJac(\alpha)^2 < rJac(\alpha) = I < rR$ with $r \in eR$.
If $rJac(\alpha)^2 < K < rJac(\alpha)$, then, by Lemma 4.3.3, there is some $s \in rJac(\alpha)$

and some ordinal $\delta < \alpha$ with $sJac(\delta)^2 < sJac(\delta) = K < sR$, so that by the induction hypothesis, $d(K) \leq \delta < \alpha$.

The proof is complete if we show that for $I < K < rJac(0)$, $d(K) < \alpha$ (since then, with $s \in rJac(\alpha) \setminus rJac(\alpha)^2$ and $t \in rJac(0) \setminus rJac(\alpha)$, $I \in [sR, tR)$ and for every $K \in [sR, tR)$, if $K \neq I$, then $d(K) < \alpha$.

If $I < K < rJac(0)$, then there is some $\beta < \alpha$ and some $i \geq 1$ with $rJac(\beta)^i < K$ (because r is local so that $I = rJac(\alpha) = \bigcap_{\beta<\alpha, i\geq 1} rJac(\beta)^i$ by Lemma 2.2.3). Let γ be minimal with $rJac(\gamma)^j < K$ for some $j \geq 1$. And let $k \geq 1$ be minimal with $rJac(\gamma)^k < K$. Then $rJac(\gamma)^k < K \leq rJac(\gamma)^{k-1}$.

If $K < rJac(\gamma)^{k-1}$, then Lemma 4.3.3 together with the induction hypothesis yield $d(K) \leq \delta$ for some ordinal $\delta < \gamma < \alpha$.

If $rJac(\gamma)^k < K = rJac(\gamma)^{k-1}$, then, with $k-1 \geq 1$ (for otherwise $K = rR$, contradicting $K < rJac(0)$) and $k-1$ must be the minimal $n \geq 0$ with $rJac(\gamma)^n = K$ (for otherwise $K = rJac(\gamma)^{k-1} = rJac(\gamma)^k$).

Therefore $rJac(\gamma)^{k-1} = K < rJac(\gamma)^{k-2}$ so that if $rJac(\gamma)^k < L < K$, then Lemma 4.3.3 and the induction hypothesis imply that $d(L) \leq \delta$ for some ordinal $\delta < \gamma$; and, similarly, if $K < L < rJac(\gamma)^{k-2}$, then $d(L) \leq \lambda$ for some ordinal $\lambda < \gamma$. Therefore, $d(K) \leq \gamma < \alpha$. $\square$

Lemma 4.3.5 *Let R be a serial ring with $Kdim(R) = \alpha < \infty$ and $e \in \mathcal{E}$. If $I < eR$ ($J < Re$), then $d(I) \leq \alpha$ ($d(J) \leq \alpha$), and therefore, $d(I)$ (($d(J)$) is defined.*

Proof. Since $Kdim(R) = \alpha$, $Jac(\alpha)$ is nilpotent. Let $k \geq 1$ be minimal with $eJac(\alpha)^k = 0$.

Suppose $I = 0$. Then $I = eJac(\alpha)^k < eJac(\alpha)^{k-1}$ and if $I < K < eR$, then

Lemma 4.3.3 and Lemma 4.3.4 imply that $d(K) \leq \delta$ for some ordinal $\delta < \alpha$. Therefore $d(I) \leq \alpha$.
Suppose $0 < I$. Then $eJac(\alpha)^k < I$. Let γ be minimal with $eJac(\gamma)^i < I$ for some $i \geq 1$. Let $k \geq 1$ be minimal with $eJac(\gamma)^k < I$. Then $eJac(\gamma)^k < I \leq eJac(\gamma)^{k-1}$ and an argument identical to one in the proof of Lemma 4.3.4 gives that $d(I) \leq \gamma \leq \alpha$. □

The above lemma says that if R is a serial ring with Krull dimension, then d is defined everywhere and bounded. The converse is true as well:

Lemma 4.3.6 *Let R be a serial ring. If there is an ordinal α such that for every $e \in \mathcal{E}$, $d(I)$ is defined and $d(I) \leq \alpha$, whenever $I < eR$, then R has Krull dimension and $Kdim(R) \leq \alpha + 1$.*

Proof. It suffices to show that for all $e \in \mathcal{E}$, eR has Krull dimension and $Kdim(eR) \leq \alpha + 1$. For $I_1, I_2 < eR$ we define $d([I_1, I_2])$ as the least ordinal β for which $d(K) < \beta$ whenever $K \in [I_1, I_2]$; we prove by induction on $d([I_1, I_2])$ that $Kdim([I_1, I_2]) < d([I_1, I_2])$ (so that $Kdim([I_1, I_2]) < \alpha + 1$).
If $d([I_1, I_2]) = 1$, then $K \in [I_1, I_2]$ implies $d(K) = 0$, i.e. K is not an accumulation point. Therefore there exists no infinite descending chain in $[I_1, I_2]$, which implies that $Kdim([I_1, I_2]) = 0 < 1$.
Let $d([I_1, I_2]) = \beta$ ($\leq \alpha+1$). If $K_0 > K_1 > K_2 > \cdots$ is an infinite descending chain in $[I_1, I_2]$, then $\bigcap_{i \geq 0} K_i \in [I_1, I_2]$ so that $d(\bigcap_{i \geq 0} K_i) < \beta$, which implies that there is an ordinal $\lambda < \beta$ and some $k \geq 0$ for which $d(I) < \lambda < \beta$ whenever $I \in (\bigcap_{i \geq 0} K_i, K_k]$. Therefore $d([K_{k+1}, K_k]) \leq \lambda < \beta$ so that, by the induction hypothesis, $Kdim([K_{k+1}, K_k]) < \lambda < \beta$. Therefore $[I_1, I_2]$ has

Krull dimension and $Kdim([I_1, I_2]) < \beta$. $\square$

For ordinals α and β, $\alpha \oplus \beta$ denotes their Cantor sum, which is obtained by representing both ordinals in their Cantor Normal Form (see [6]), say $\alpha = n_1\omega^{\gamma_1} + \cdots + n_k\omega^{\gamma_k}$ and $\beta = m_1\omega^{\delta_1} + \cdots + m_l\omega^{\delta_l}$ (where the n_i and the m_i are integers and $\gamma_1 > \cdots > \gamma_k$ and $\delta_1 > \cdots \delta_l$) and then summing the resulting ω-polynomials in the same way as one would add elements of $\mathbb{Z}[X]$ for example. The Cantor sum has the usual properties of addition (but does not admit substraction).

Proposition 4.3.7 *Let R be a serial ring with Krull dimension. Then every point of Zg_R has CB-rank, and if $\langle I, J\rangle$ is an e-pair, then $CB(H(J^*/I(x))) \leq d(I) \oplus d(J)$.*
So, for every $U \in Zg_R$, $CB(U) \leq inf\{d(I) \oplus d(J) : \langle I, J\rangle$ is an e-pair of $U\}$.

Proof. We give a proof by induction on $d(I) \oplus d(J)$.
Put $U = H(J^*/I(x))$. If $d(I) \oplus d(J) = 0$, then $d(I) = d(J) = 0$ so that U is isolated (by Proposition 3.2.3), i.e. $CB(U) = 0$.
Let $d(I) \oplus d(J) = \beta$. Because of Lemma 4.3.5 there are principal right ideals $I', I'' \leq eR$ and principal left ideals $J', J'' \leq Re$ with $I \in [I'', I')$ and $J \in [J'', J')$ and such that if $K \in [I'', I')$ and $L \in [J'', J')$, then $K = I$ or $d(K) < d(I)$, and $L = J$ or $d(L) < d(J)$. Say $I' = tR$, $I'' = rR$, $J' = Rs$ and $J'' = Rv$. Put $O = (xr = 0 \wedge s|x/xt = 0 + v|x)$. Then $U \in O$ and if $\langle K, L\rangle$ is an e-pair with $H(K^*/L(x)) \in O$, then $K = I$ and $L = J$ or $d(K) \oplus d(L) < d(I) \oplus d(J)$. So, if $V \in O$ and $V \not\cong U$, then $CB(V) < \beta$, by the induction hypothesis, and therefore $V \notin Zg_R^{(\beta)}$. So if $U \in Zg_R^{(\beta)}$, then

$CB(U) = \beta$, and otherwise $CB(U) < \beta$. □

A nonzero module is said to be *superdecomposable* (or *continuous*) if it has no indecomposable direct summands. It is known that a pure-injective module with m-dimension is not superdecomposable (for example, see [18](10.10) or [37](7.8)).

Corollary 4.3.8 *Let R be a serial ring with $Kdim(R) = \alpha$, then $CB(Zg_R) \leq 2\alpha$ so that there is no superdecomposable pure-injective R-module.*

Proof. The first part follows immediately from Proposition 4.3.7. The second part follows from Corollary 4.2.4. □

Lemma 4.3.9 *Let R be a serial ring in $\mathcal{P}$ with the following properties:*
(1) Zg_R has CB-rank;
(2) For every $e \in \mathcal{E}$, if $I < eR$ and if $H((Jac(0)e)^*/I(x))$ is isolated in $Zg_R^{(\beta)}$ ($\beta \leq CB(Zg_R)$), then $H(Jac(0)e)^*/I(x)$ is isolated by a $\mathbf{T}^{(\beta)}$-minimal pair.
Then R has Krull dimension and $Kdim(R) \leq CB(Zg_R) + 1$.

Proof. By Theorem 2.2.5 it suffices to show that for every $e \in \mathcal{E}$, eR has Krull dimension.
Let $e \in \mathcal{E}$. Put $J = Jac(0)e$. For $I < eR$ we prove by induction on $CB(H(J^*/I(x)))$ that $d(I)$ exists and $d(I) \leq CB(H(J^*/I(x))) \leq CB(Zg_R)$. The result then follows from Lemma 4.3.6.
If $CB(H(J^*/I(x))) = 0$, then $H(J^*/I(x))$ is isolated in Zg_R so that $J^*/I(x)$ contains a minimal pair, say $xr = 0 \wedge s|x/xt = 0 + v|x$. Therefore, if $K \in [rR, tR)$, then $K = I$, since $\langle K, J\rangle$ is an e-pair. So $d(I) = 0$.

Let $CB(H(J^*/I(x))) = \beta$. Then $J^*/I(x)$ contains a $\mathbf{T}^{(\beta)}$-minimal pair, say $xr = 0 \wedge s|x/xt = 0 + v|x$. Therefore, if $K \in [rR, tR)$, then either $K = I$ or $CB(H(J^*/K(x))) < \beta$. If $CB(H(J^*/K(x))) < \beta$, then the induction hypothesis implies that $d(K) \leq CB(H(J^*/K(x))) < \beta$. Therefore, $d(I)$ is defined and $d(I) \leq \beta$. $\square$

Let R be a commutative serial ring. Then $R \in \mathcal{P}$ (this was noted before) and R satisfies *(2)* of the above lemma:

If $\langle I, Jac(0)e \rangle$ and $\langle K, Jac(0)e \rangle$ are equivalent e-pairs then there is some $u \in (Jac(0)e)^*$ for which $uK = I$ (so $I \leq K$) or $K = uI$ (so $K \leq I$). But if $u \notin Jac(0)e$, then $tu = e$ for some $t \in R$ and therefore, $uK = I$ ($K = uI$) implies $K = tI$ ($tK = I$) so that $K \leq I$ ($I \leq K$). This shows that (in case of a commutative serial ring) if $\langle I, Jac(0)e \rangle$ and $\langle K, Jac(0)e \rangle$ are equivalent e-pairs, then $I = K$ (and therefore, by Proposition 3.2.4, *(2)* is satisfied). Therefore, *A commutative serial ring has Krull dimension iff its Ziegler spectrum has CB-rank.*

Corollary 4.3.10 *For a ring $R \in \mathcal{P}$ the following are equivalent:*

(1) R has Krull dimension;

(2) Zg_R has CB-rank and for every $U \in Zg_R$, if $CB(U) = \beta$, then U is isolated in $Zg_R^{(\beta)}$ by a $\mathbf{T}^{(\beta)}$-minimal pair;

(3) $\mathbf{T}$ has m-dimension.

Proof. *(1)* $\Leftrightarrow$ *(2)* follows from Corollary 4.3.8, Theorem 4.2.3 and Remark 3.1.11.

(3) $\Rightarrow$ *(2)* follows from [18](10.16) (which says that if $\mathbf{T}$ has m-dimension,

then $\mathbf{T}$ satisfies Condition ($\wedge$)) and from [18](10.19) (see previous section).
(1) $\Rightarrow$ *(3)* follows from Corollary 4.3.8 and Corollary 4.2.4. $\square$

Clearly, *(2)* in the above Corollary can be replaced by "Zg_R has CB-rank and for every $U \in Zg_R$, if U is isolated in Zg_R^T, with T a complete theory of R-modules, then U is isolated in Zg_R^T by a T-minimal pair".

Lemma 4.3.11 *Let R be a serial ring with the following properties:*
(1) R has Krull dimension;
(2) For every $e \in \mathcal{E}$, if $\langle I, J\rangle$ and $\langle K, L\rangle$ are equivalent e-pairs, then $d(I) \oplus d(J) = d(K) \oplus d(L)$;
(3) For every e-pair $\langle I, J\rangle$ there are principal $I', I'' < eR, J', J'' < Re$ with $I \in [I'', I')$, $J \in [J'', J')$ and such that $\langle K, J\rangle$ is an e-pair for every $K \in [I'', I')$ and $\langle I, L\rangle$ is an e-pair for every $L \in [J'', J')$.
Then for every e-pair $\langle I, J\rangle$, $CB(H(J^*/I(x))) = d(I) \oplus d(J)$. Therefore, for $U \in Zg_R$ and $e, f \in \mathcal{E}$, if $\langle I, J\rangle$ is an e-pair of U and $\langle K, L\rangle$ is an f-pair of U, then $d(I) \oplus d(J) = d(K) \oplus d(L)$.

Proof. We give a proof by induction on $d(I) \oplus d(J)$.
If $d(I) \oplus d(J) = 0$, then $d(I) = d(J) = 0$ so that $H(J^*/I(x))$ is isolated in Zg_R, i.e. $CB(U) = 0$.
Let $d(I) \oplus d(J) = \beta > 0$. By Proposition 4.3.7 we know that $CB(H(J^*/I(x))) \leq \beta$. Suppose that $CB(H(J^*/I(x))) < \beta$. Then $H(J^*/I(x))$ is isolated in $Zg_R^{(\gamma)}$ for some ordinal $\gamma < \beta$. Together with Lemma 1.2.1 this implies that there exist $r, t \in eR$ and $s, v \in Re$ with $I \in [rR, tR)$, $J \in [Rv, Rs)$ and such that if $K \in [rR, tR)$ and $L \in [Rv, Rs)$, then, provided $\langle K, L\rangle$ is an

e-pair, either $H(L^*/K(x)) \notin Zg_R^{(\gamma)}$ or $\langle K, L\rangle \sim \langle I, J\rangle$, in which case (by *(2)*) $d(K) = d(I)$ and $d(L) = d(J)$. If $\langle K, L\rangle$ is an e-pair and $H(L^*/K(x)) \notin Zg_R^{(\gamma)}$, then $CB(H(L^*/K(x))) < \gamma$ and therefore, by the induction hypothesis, $d(K) \oplus d(L) < \gamma$.
Because of *(3)* we can assume that $[rR, tR)$ and $[Rv, Rs)$ are such that $K \in [rR, tR)$ and $L \in [Rv, Rs)$ imply that $\langle K, J\rangle$ and $\langle I, L\rangle$ are e-pairs. So, if $K \in [rR, tR)$, then either $d(K) \oplus d(J) < \gamma$ or $d(K) = d(I)$, and if $L \in [Rv, Rs)$, then either $d(I) \oplus d(L) < \gamma$ or $d(L) = d(K)$. We also can assume that $[rR, tR)$ and $[Rv, Rs)$ are such that for every $K \in [rR, tR)$ either $K = I$ or $d(K) < d(I)$, and for every $L \in [Rv, Rs)$ either $L = J$ or $d(L) = d(J)$. Therefore, if $K \in [rR, tR)$, then either $K = I$ or $d(K) \oplus d(J) < \gamma < \beta = d(I) \oplus d(J)$, i.e. either $K = I$ or there is an ordinal λ with $d(K) < \lambda < d(I)$. This implies that $[rR, tR) = I$, i.e. $d(I) = 0$. Similarly, $d(J) = 0$ so that $\beta = 0$, contradicting $\beta > 0$. $\square$

We conclude this section with an easy remark.

Remark 4.3.12 Let R be a serial ring such that for every e-pair $\langle I, J\rangle$, $d(I)$ and $d(J)$ are defined and $CB(H(J^*/I(x))) = d(I) \oplus d(J)$. Then for every $U \in Zg_R$, $CB(U) = \beta$ iff $U \in Zg_R^{(\beta)}$ and every e-pair $\langle I, J\rangle$ of U is such that $J^*/I(x)$ contains a $\mathbf{T}^{(\beta)}$-minimal pair.

Proof. ($\Rightarrow$) Let $CB(U) = \beta$. Let $\langle I, J\rangle$ be an e-pair of U. Then $d(I) \oplus d(J) = \beta$ and there are $r, t \in eR$, $s, v \in Re$ with $I \in [rR, tR)$, $J \in [Rv, Rs)$ such that if $\langle K, L\rangle$ is an e-pair with $K \in [rR, tR)$ and $L \in [Rv, Rs)$, then either $K = I$ and $L = J$, or $d(K) \oplus d(L) < \beta$. Since $d(K) \oplus d(L) < \beta$

implies $CB(H(L^*/K(x))) < \beta$, this proves that $xr = 0 \wedge s|x/xt = 0 + v|x$ is a $\mathbf{T}^\beta$-minimal pair in $J^*/I(x)$.

($\Leftarrow$) This is trivial. $\Box$

4.4 Ziegler Spectra over Noetherian Serial Rings

In this section, we investigate the Ziegler spectra over two-sided Noetherian serial rings. We show that the Ziegler spectrum over a two-sided Noetherian serial ring R has CB-rank either 0 or 2 and that it has CB-rank 0 iff R is (two-sided) Artinian. We also show that the Ziegler spectrum over a two-sided Noetherian serial ring R is a T_0-space (i.e. for all $U, V \in Zg_R$, there is an open containing only one of U, V).

A nonzero module (over any ring) is called *uniform* if any two nonzero submodules have nonzero intersection; in other words, a nonzero module is uniform if every nonzero submodule is an essential submodule. Clearly, every essential extension of a uniform module is uniform, from which it follows that the injective hull of a uniform module is uniform, and therefore indecomposable.

Clearly, nonzero uniserial modules are uniform, so that the injective hull of a nonzero uniserial module is indecomposable.

Let R be a serial ring in $\mathcal{P}$.

Let E be an injective indecomposable R-module and $0 \neq a \in Ee$ ($e \in \mathcal{E}$). Put $I = ann_{eR}(a)$. We then have that if $s^* \in \mathcal{J}(I)^*$, then $s^*r^* \neq 0$ for all

$r^* \in I^*$ so that we can define a monomorphism $R/I \longrightarrow R : 1 + I \mapsto s^*$ for every $s^* \in J(I)^*$. Using that E is injective one now easily shows that $E \models s^*|a$ for every $s^* \in \mathcal{J}(I)^*$. This implies that $tp_E^{\pm}(a) \supseteq \mathcal{J}(I)^*/I(x)$, and therefore $tp_E^{\pm}(a) = \mathcal{J}(I)^*/I(x)$ by Theorem 3.1.2, so that $E \cong H(\mathcal{J}(I)^*/I(x))$. Conversely, if $I < eR$ ($e \in \mathcal{E}$), then $H(\mathcal{J}(I)/I(x))$ is injective: $E(eR/I)$ is indecomposable (since eR/I is uniserial), so that $E(eR/I)$, by the above, $H(\mathcal{J}(I)/I(x)) \cong E(eR/I)$.
We conclude:

Remark 4.4.1 Let R be a serial ring in $\mathcal{P}$. Then the injective points of Zg_R are precisely the points of the form $E(eR/I) \cong H(\mathcal{J}(I)^*/I(x))$ for some $I < eR$ and $e \in \mathcal{E}$ (in particular, if $\langle K, L \rangle \sim \langle I, \mathcal{J}(I) \rangle$ then $L = \mathcal{J}(K)$).

A ring R is said to be *of finite representation type* (FRT for short) if every (right) R-module is a direct sum of indecomposable submodules (i.e. R is *right pure-semisimple*) and if there are, up to isomorphism, only finitely many indecomposable (right) R-modules.The commutative rings of FRT are precisely the Artinian principal ideal rings ([34](1.2)). It is known that a right Artinian ring is of FRT just in case its Ziegler spectrum is finite ([18](11.38)). Also, every right pure-semisimple ring is right Artinian ([18](11.10)).

Proposition 4.4.2 *Suppose that $R \in \mathcal{P}$ and that R has Krull dimension. If the injective points of Zg_R are isolated, then R is right Artinian (and therefore two-sided Artinian) so that Zg_R is discrete (i.e. $CB(Zg_R) = 0$) and R is of FRT.*

Proof. Suppose that R is not right Artinian. Then there exists an $e \in \mathcal{E}$

and an infinite descending chain

$$eR > I_1 > I_2 > I_3 > \cdots$$

So $I := \bigcap_{i \geq 1} I_i$ has no successor. By Theorem 4.2.3 and Proposition 3.2.4, this implies that the injective point $H(\mathcal{J}(I)^*/I(x))$ is not isolated, which contradicts the hypothesis.

If R is an Artinian serial ring, then Proposition 3.2.4 implies that Zg_R is discrete, and therefore finite (since Zg_R is quasi-compact) so that R is of FRT by [18](11.38). $\square$

It is known that *if R is a two-sided Noetherian serial ring, then for every $e \in \mathcal{E}$, eR is Artinian iff Re is Artinian* ([33](5.9)).

Let R be a two-sided Noetherian serial ring. Then Proposition 2.1.6 implies that, if eR (or, equivalently Re) is not Artinian, the chain of right ideals contained in eR looks like

$$0 = eJac(1) < \cdots < eJac(0)^i < eJac(0)^{i-1} < \cdots < eJac(0)^2 < eJac(0) < eR$$

where $eJac(0)^i/eJac(0)^{i+1}$ is simple (by Lemma 2.1.2) and nonzero for every $i \geq 0$, so that $d(eJac(0)^i) = 0$ for every $i \geq 0$ and $d(eJac(1)) = 1$; if eR is Artinian and nonzero, then the above chain reduces to

$$0 = eJac(0)^k < eJac(0)^{k-1} < \cdots < eJac(0) < eR$$

for some $k > 1$, in which case $d(I) = 0$ for every $I < eR$.

(Similarly for Re.)

It is easy to see that if R is two-sided Noetherian serial, then a point $U \in Zg_R$

is isolated (in Zg_R) iff for every e-pair $\langle I, J\rangle$ of U, $d(I) = d(J) = 0$. (This follows from Proposition 3.2.4, Theorem 4.2.3 and Remark 3.1.11.)

Lemma 4.4.3 *Let R be a two-sided Noetherian serial ring. Let $e \in \mathcal{E}$. If $\langle I, J\rangle$ and $\langle K, L\rangle$ are equivalent e-pairs, then $d(I) + d(J) = d(K) + d(L)$.*

Proof.

Let $\langle I, J\rangle$ and $\langle K, L\rangle$ be equivalent e-pairs. By the above we can assume that eR (and Re) is not Artininan and that at least one of $d(I), d(J)$ is nonzero.

Case 1: $d(I) = d(J) = 1$.

In this case $H(J^*/I(x))$ is not isolated, $I = eJac(1) = 0$ and $J = Jac(1)e = 0$. Therefore $d(K) + d(L) \neq 0$ (otherwise $H(J^*/I(x))$ would be isolated). Suppose $d(K) = 0$. Then $K = eJac(0)^i$ for some $i \geq 1$ and $K \neq 0$ (otherwise Re would be Artinian). Therefore $\langle K, L\rangle \sim \langle I, J\rangle$ implies that $uK = I = 0$ for some $u \in eRe\backslash\{0\}$, i.e. $uJac(0)^i = 0$ which implies that uR is Artinian. But, since $u \neq 0$, $uR \neq 0$, so that $uR = eJac(0)^j$ for some $j \geq 0$. Therefore eR would be Artinian, a contradiction. We conclude that $d(K) \neq 0$, i.e. $d(K) = 1$. And, similarly, $d(L) = 1$.

Case 2: $d(I) = 1$ and $d(J) = 0$.

In this case $H(J^*/I(x))$ is not isolated, $I = eJac(1) = 0$, $J = Jac(0)^i$ for some $i \geq 1$ and $eJac(0)^j \neq 0$ for all $j \geq 1$. As in Case 1, $d(K) + d(L) \neq 0$. Suppose $d(L) = 1$. Then $0 = L = Jac(1)e$ and, since $J \neq 0$, $\langle K, L\rangle \sim \langle I, J\rangle$ implies that $Ju = L = 0$ for some $u \in eRe\backslash\{0\}$, i.e. $Jac(0)^i u = 0$ which implies that Ru is Artinian. But, since $u \neq 0$, $Ru \neq 0$, so that $Ru = eJac(0)^j$ for some $j \geq 1$. Therefore Re is Artinian, a contradiction. We conclude that $d(L) = 0 = d(J)$ and therefore $d(K) = 1 = d(I)$.

Case 3: $d(I) = 0$ and $d(J) = 1$.

This case is similar to the previous one. □

As noted before, if R is an Artinian serial ring, then $CB(Zg_R) = 0$. We know that if R is a two-sided Noetherian serial ring, then $Kdim(R_R) \leq 1$ so that $CB(Zg_R) \leq 2$. In [33](Section 5) it is shown that *every two-sided Noetherian serial ring decomposes as a direct sum of an Artinian serial ring and a prime serial ring (which is non-singular).* If R is a serial ring in $\mathcal{P}$, then for every $e \in \mathcal{E}$, $\langle 0, 0 \rangle$ is an e-pair. The following is now easy to show.

Corollary 4.4.4 *Let R be a two-sided Noetherian serial ring. Then the following hold:*
(1) For every e-pair $\langle I, J \rangle$, $CB(H(J^*/I(x))) = d(I) + d(J)$;
(2) $CB(Zg_R) = 2$ iff R is not Artinian (i.e. $CB(Zg_R) = 0$ or 2).

To conclude this section, we show that the Ziegler spectrum over a two-sided Noetherian serial ring is a T_0-space.

Proposition 4.4.5 *Let R be a two-sided Noetherian serial ring. Let U and V be topologically indistinguishable points in Zg_R, i.e. $U \in Cl(V)$ and $V \in Cl(U)$.*
Then U and V coincide in Zg_R, i.e. $U \cong V$.

Proof. Assume that $U \ncong V$. Let $\langle I, J \rangle$ be an e-pair of U. Then I and J are principal; say $I = rR$ and $J = Rv$. So $U \in (xr = 0 \wedge e|x/v|x)$, which implies that, since $U \in Cl(V)$, we can find an e-pair $\langle K, L \rangle$ with $H(L^*/K(x) \cong V$ and such that $r \in K$ and $v \in L$, i.e. $I \leq K$ and $J \leq L$. And $I < K$ or

$J < L$. Again K and L are principal; say $K = tR$ and $L = Rs$ (so $t \notin I$ or $s \notin J$).

Since $V \in (xt = 0 \wedge e|x/s|x)$ and $V \in Cl(U)$ we can find an e-pair $\langle I_1 . J_1 \rangle$ in R with $\langle I_1, J_1 \rangle \sim \langle I, J \rangle$ and such that $t \in I_1$ and $s \in J_1$, i.e. $I \leq K \leq I_1$ and $J \leq L \leq J_1$. And $K < I_1$ or $L < J_1$.

Again I_1 and J_1 are principal; say $I_1 = r_1 R$ and $J_1 = Rv_1$ (so $r_1 \notin K$ or $v_1 \notin L$).

Since $U \in (xr_1 = 0 \wedge e|x/v_1|x)$ and $U \in Cl(V)$ we can find an e-pair $\langle K_1, L_1 \rangle$ in R with $\langle K_1, L_1 \rangle \sim \langle K, L \rangle$ and such that $r_1 \in K_1$ and $v_1 \in L_1$; i.e. $I \leq K \leq I_1 \leq K_1$ and $J \leq L \leq J_1 \leq L_1$. And $I_1 < K_1$ or $J_1 < L_1$. Since $U \not\cong V$ (i.e. at each step we have at least one strict inclusion) we can continue this process and obtain an ascending chain

$$I_0 = I \leq K_0 = K \leq I_1 \leq K_1 \leq \cdots$$

of right R-ideals and an ascending chain

$$J_0 = J \leq L_0 = L \leq J_1 \leq L_1 \leq \cdots$$

of left R-ideals. And since R is right Noetherian the first chain must be stationary which implies that (since $U \not\cong V$) the second chain is not stationary, and this is in contradiction with R being left Noetherian. $\square$

4.5 CB analysis of Ziegler Spectra over Valuation Domains with Krull Dimension

In this Section we will see that if R is a commutative valuation domain with $Kdim(R) = \alpha$, then $CB(Zg_R) = 2\alpha$. This result has already been shown by

Puninski in [24]. We follow the same route as in [24] but use Definition 4.3.1 (rather than Puninski's definition of "coarsity") and previously obtained results, avoiding commutativity as much as we can. It will turn out that also for a valuation domain R which belongs to $\mathcal{P}$ we have that $Kdim(R) = \alpha$ implies $CB(Zg_R) = 2\alpha$.

In [24] it is also shown that a commutative valuation domain has Krull dimension just in case its Ziegler spectrum has CB-rank. We will see that this result follows from Lemma 4.3.11 and Lemma 4.3.9.

For a valuation domain R, $c(R)$ is defined as the least ordinal α for which $Jac(\alpha)^2 = Jac(\alpha)$. Clearly, if R is a valuation domain with Krull dimension, then $Kdim(R) = c(R)$. (In [24] $c(R)$ was introduced for commutative valuation domains.)

Lemma 4.5.1 *Let R be a valuation domain with $c(R) = \alpha$. If $\beta < \alpha$ and $r \in R\backslash\{0\}$, then $rJac(\beta)^{k+1} < rJac(\beta)^k$ for all $k \geq 1$.*

Proof. If $rJac(\beta)^{k+1} = rJac(\beta)^k$ then, since R is a domain, $Jac(\beta)^{k+1} = Jac(\beta)^k = Jac(\beta+1)$. $\beta < \alpha$ and $c(R) = \alpha$ imply $Jac(\beta)^2 < Jac(\beta)$ so that $Jac(\beta+1) < Jac(\beta)$. Take $t \in Jac(\beta)\backslash Jac(\beta+1)$. Then $t^k \in Jac(\beta)^k = Jac(\beta+1)$ so that, since (by Lemma 2.2.6) $Jac(\beta+1)$ is prime, $t \in Jac(\beta+1)$, a contradiction. $\square$

Lemma 4.5.2 (see also [24] Lemma 3.2)

Let R be a valuation domain with $c(R) = \alpha$. Then for every $\beta < \alpha$ and for every $r \in R\backslash\{0\}$, $d(rJac(\beta)) = \beta = d(Jac(\beta)r)$. If $Kdim(R) = \alpha$, then for every $\beta \leq \alpha$ and for every $r \in R\backslash\{0\}$, $d(rJac(\beta)) = \beta = d(Jac(\beta)r)$.

Proof. Let $\alpha > 0$ and $r \in R\backslash\{0\}$. Then, by Lemma 4.5.1, $rJac(0)^2 < rJac(0) < rR$ so that both $rJac(0)/rJac(0)^2$ and $rR/rJac(0)$ are simple and nonzero and therefore $d(rJac(0)) = 0$.

If $\beta \leq \alpha$ is a limit ordinal, then $rJac(\beta) = \bigcap_{\gamma<\beta} rJac(\gamma)$ so that $d(rJac(\gamma)) = \gamma$ for every $\gamma < \beta$(which follows from the induction hypothesis) implies that for every $K \in (rJac(\beta), rJac(0)]$ and for every $\gamma < \beta$ there is some $I \in (rJac(\beta), K]$ with $d(I) \geq \gamma$. Therefore $d(rJac(\beta)) \geq \beta$. If $\beta < \alpha$, then $c(R) = \alpha$ implies that $rJac(\beta)^2 < rJac(\beta) < rR$ so that by Lemma 4.3.4 and the induction hypothesis, $d(rJac(\beta)) \leq \beta$. Therefore $d(rJac(\beta)) = \beta$ if $\beta < \alpha$.

If $0 < \beta \leq \alpha$ is not a limit ordinal, then, with $\beta = \lambda + 1$ (and so $\lambda < \beta \leq \alpha$), $rJac(\beta) = \bigcap_{i\geq 1} rJac(\lambda)^i$.

We claim that for all $i \geq 1$ there is some $I \in (rJac(\beta), rJac(\gamma)^i]$ with $d(I) = \lambda$. It then follows that $d(rJac(\beta)) \geq \beta$. As before, we also have that $d(rJac(\beta)) \leq \beta$ if $\beta < \alpha$ so that $d(rJac(\beta)) = \beta$ if $\beta < \alpha$.

We prove the claim by induction on $i \geq 1$.

If $i = 1$, take $I = rJac(\lambda)$.

We have that, since $\lambda < \alpha$, if $i \geq 1$, then $rJac(\beta) < rJac(\lambda)^{i+1} < rJac(\lambda)^i$ (by Lemma 4.5.1) so that, by the induction hypothesis (from the induction argument on $i \geq 1$), there is some $I \in (rJac(\beta), rJac(\lambda)^i]$ with $d(I) = \lambda$. Now, if $rJac(\lambda)^{i+1} < I < rJac(\lambda)^i$, then (by Lemma 4.3.3) $I = sJac(\delta)$ for some $\delta < \lambda$ and some $s \in rJac(\lambda)^i\backslash\{0\}$ so that the induction hypothesis (from the induction argument on β) implies that $d(I) = \delta < \lambda$, contradicting $d(I) = \lambda$. Therefore, $I \in (rJac(\beta), rJac(\lambda)^{i+1}]$. This proves the claim.

Now we suppose that $Kdim(R) = \alpha$.

By Lemma 4.3.5 we know that $d(rJac(\alpha)) \leq \alpha$ and the above shows that $d(rJac(\alpha)) \geq \alpha$.

A similar proof shows that $d(Jac(\beta)r) = \beta$ for all $\beta < \alpha$ and that $d(Jac(\alpha)r) = \alpha$ if $Kdim(R) = \alpha$. □

Lemma 4.5.3 *Let R be a valuation domain with $c(R) = \alpha > 0$. Then for every ordinal $\beta < \alpha$ and for $\gamma < \beta$, $Jac(\gamma)Jac(\beta) = Jac(\beta)$.*

Proof. For $\gamma < \beta < \alpha = c(R)$, Lemma 4.5.1 gives $Jac(\beta) < Jac(\gamma)$. By Lemma 2.2.3 we have that for all $r \in Jac(\gamma)\backslash Jac(\beta) \leq R\backslash Jac(\beta)$, $rJac(\beta) = Jac(\beta)$. So $Jac(\beta) \leq Jac(\gamma)Jac(\beta) \leq Jac(\beta)$ and therefore $Jac(\beta) = Jac(\gamma)Jac(\beta)$. □

Proposition 4.5.4 (see also [24] Lemma 3.3)
If R is a valuation domain with $c(R) = \alpha$, then for every right (left) ideal $I \in (Jac(\alpha), R)$ there exist some $r \notin I$ and a unique ordinal $\delta \leq \alpha$ with $I = rJac(\delta)$ ($I = Jac(\delta)r$).

Proof. If $\alpha = 0$, the statement is trivial. So we can assume that $\alpha > 0$.

We show that for every right ideal $I \in (Jac(\alpha), R)$ there exist some $r \in R$ and some ordinal $\delta \leq \alpha$ with $I = rJac(\delta)$ (so, since R is local, $r \notin I$). The uniqueness of the ordinal δ for a given $I \in (Jac(\alpha), R)$ then follows from Lemma 4.5.2.

Because of Lemma 4.3.3 it suffices to show that, given an ordinal $\beta < \alpha$, we have that for every $i \geq 1$, there exist some $r \in R$ and some ordinal $\delta \leq \beta$ with $Jac(\beta)^i = rJac(\delta)$.

Let $\beta < \alpha$. We proceed by induction on $i \geq 1$. For $i = 1$ there is nothing to prove.

If $i \geq 2$, then $0 < Jac(\beta)^{i+1} < Jac(\beta)^i < Jac(\beta)^{i-1}$ (by Lemma 4.5.1). By the induction hypothesis, $Jac(\beta)^{i-1} = tJac(\gamma)$ for some $\gamma \leq \beta$ and some $t \in R$. Therefore, with $I = Jac(\beta)^i$, $tJac(\gamma)Jac(\beta) = Jac(\beta)^{i-1}Jac(\beta) = I$. We claim that $tJac(\gamma)Jac(\beta+1) < I < tJac(\gamma)$. Lemma 4.3.3 then implies that $Jac(\beta)^i = I = rJac(\delta)$ for some $r \in R$ and some ordinal $\delta \leq \beta$.

To prove the claim, note that $tJac(\gamma)Jac(\beta+1) \leq I = tJac(\gamma)Jac(\beta) \leq tJac(\gamma)$. Since $\gamma \leq \beta$, Lemma 4.5.3 and Lemma 4.5.1 imply that $Jac(\gamma)Jac(\beta+1) = Jac(\beta+1) < Jac(\beta)^2 \leq Jac(\gamma)Jac(\beta)$ so that $tJac(\gamma)Jac(\beta+1) < tJac(\gamma)Jac(\beta) = I$. To show that $I < tJac(\gamma)$ we can, because of Lemma 4.5.1, assume that $\gamma < \beta$ and therefore, by Lemma 4.5.3, $Jac(\gamma)Jac(\beta) = Jac(\beta) < Jac(\gamma)$, which implies that $I < tJac(\gamma)$. □

From the above Lemma it follows that every proper right (left) ideal I in a valuation domain with Krull dimension is of the form $rJac(\beta)$ $(Jac(\beta)r)$ for some $r \notin I$ and and a unique $\beta \leq Kdim(R)$.

Corollary 4.5.5 (see also [24] Theorem 3.5)
Let R be a valuation domain with Krull dimension. If $\langle I, J\rangle$ and $\langle K, L\rangle$ are equivalent 1-pairs, then $d(I) = d(K)$ and $d(J) = d(L)$.

Proof. This follows from Proposition 4.5.4, Lemma 4.5.2. □

If R is a valuation domain in $\mathcal{P}$ (for example if R is commutative), then for every proper right ideal I and for every proper left ideal J, $\langle I, J\rangle$ is a 1-pair. We therefore have the following:

Proposition 4.5.6 (see also [24] Theorem 3.5)
Let R be a valuation domain in $\mathcal{P}$. If R also has Krull dimension, then for every proper right ideal I and for every proper left ideal J, $CB(H(J^*/I(x))) = d(I) \oplus d(J)$.

Proof. Follows from Corollary 4.5.5 and Proposition 4.3.11. □

Corollary 4.5.7 (see also [24] Corollary 3.6)
Let R be a valuation domain in $\mathcal{P}$ with $Kdim(R) = \alpha$, then $CB(Zg_R) = 2\alpha$ and $E(R)$ is the unique point in Zg_R of maximal CB-rank.

Proof. Since $Kdim(R) = \alpha$, Lemma 4.5.2 implies that $d(0) = \alpha$, so that, by Proposition 4.5.6 and by Corollary 4.3.8, $H(0^*/0(x)) = E(R)$ is the unique point in Zg_R of maximal CB-rank. □

Corollary 4.5.8 ([24] Theorem 3.7)
If R is a commutative valuation domain then the following are equivalent:
(1) R has Krull dimension;
(2) Zg_R has CB-rank;
(3) There is no superdecomposable pure-injective R-module.

Proof. *(1)* $\Rightarrow$ *(2)* follows from Corollary 4.3.8, *(2)* $\Rightarrow$ *(1)* follows from Lemma 4.3.9 and *(1)* $\Leftrightarrow$ *(3)* was shown in [23]. □

In [24] it is shown that if R is a commutative valuation domain, then the points in Zg_R which are isolated by a minimal pair are dense if $Jac(0)$ is principal, and if $Jac(0)$ is not principal, then there are no isolated points at all ([24] Lemma 3.1). With an argument which is essentially the same as in [24], the first part can be generalised to:

Lemma 4.5.9 *Let R be a valuation domain in $\mathcal{P}$. If $Jac(0)^2 < Jac(0)$, then the points in Zg_R which are isolated by a minimal pair are dense.*

Proof. Let $O = (xr = 0 \wedge s|x/xt = 0 + v|x)$ be a nonempty open in Zg_R. Then $rR < tR$ implies that $r = tu$ for some $u \in Jac(0)$ and $Rv < Rs$ implies that $v = ws$ for some $w \in Jac(0)$.
Since $Jac(0)^2 < Jac(0)$ we have that $Jac(0)$ is principal both as a right and a left ideal, say $z_1R = Rz_2 = Jac(0)$.
Then $tuR \leq tz_1R = tJac(0)$ implies that $(xtz_1 = 0) \leq (xtu = 0)$ and $Rws \leq Rz_2s = Jac(0)s$ implies that $(ws|x \leq z_2s|x)$. Therefore $O' = (xtz_1 = 0 \wedge s|x/xt = 0 + z_2s|x) \subseteq O$, and also $[tz_1R, tR) = \{tz_1R\}$ and $[Rz_2s, Rs) = \{Rz_2s\}$ so that (since R is a domain in $\mathcal{P}$), with $I = tz_1R, J = Rz_2s$, O' contains $H(J^*/I(x))$ and $xtz_1 = 0 \wedge s|x/xt = 0 + z_2s|x$ is a minimal pair in $J^*/I(x)$. $\square$

Part II

Ziegler Spectra over Coalgebras

Chapter 5

Preliminaries

5.1 Coalgebras and Comodules

Throughout, let $\mathbf{k}$ be a field. All tensor products are tensor products over $\mathbf{k}$. 1 denotes the unit of $\mathbf{k}$. By a *vectorspace* we mean a $\mathbf{k}$-vectorspace and by a *map* we mean a $\mathbf{k}$-linear map.

Let V and W be vectorspaces. The *twist map* $\tau : V \otimes W \longrightarrow W \otimes V$ is given by $\tau(v \otimes w) = w \otimes v$ (for $v \in V$ and $w \in W$).

The *dual space* of a vectorspace V is defined as $V^* = Hom_{\mathbf{k}}(V, \mathbf{k})$, the vectorspace of all $\mathbf{k}$-linear functionals on V. If $f \in V^*$ and $v \in V$, we often write $f(v)$ as $\langle f, v\rangle$. V is embedded in V^{**} (=$(V^*)^*$) so that V can be regarded as a subspace of V^{**} (i.e. $v(f) = \langle f, v\rangle$ for all $v \in V$,$f \in V^*$). If $f : V \longrightarrow W$ is a map between vectorspaces, then its *dual* map $f^* : W^* \longrightarrow V^*$ is defined by $\langle f^*(w^*), v\rangle = \langle w^*, f(v)\rangle$ for all $w^* \in W^*$,$v \in V$.

If W is a subspace of V, then $W^\perp$ is defined as $\{v^* \in V^* : \langle v^*, W\rangle = 0\}$ (which is a subspace of V^*), and if U is a subspace of V^*, then ${}^\perp U$ is defined

as $\{v \in V : \langle U, v\rangle = 0\}$ (which is a subspace of V). By regarding V as a subspace of V^{**}, we have that if W is a subspace of V^*, then $W^\perp \subseteq {}^\perp W$.
A subspace U of V^* is said to be *closed* if $U = W^\perp$ for some subspace W of V. A subspace W of V is said to be *cofinite* if V/W is finite dimensional. It is clear that a finite intersection of cofinite subspaces of a vectorspace is cofinite. The subspace of V spanned by a subset S of V will be denoted by $vecS$.

Lemma 5.1.1 ([26] and [31]) *Let V be a vectorspace.*
(1) For every subspace W of V, ${}^\perp(W^\perp) = W$.
(2) If W is a finite dimensional subspace of V, then $W^\perp$ is cofinite. Conversely, if W is a cofinite subspace of V^*, then ${}^\perp W$ is finite dimensional.
(3) If $f_1, \dots, f_n$ are independent functionals in V^*, then for every $g \in V^*$, $g \in vec\{f_1, \dots, f_n\}$, whenever $\bigcap_{i=1}^n Ker(f_i) \subseteq Ker(g)$. Therefore, every finite dimensional subspace of V^* is closed.
(4) A cofinite subspace W of V^* is closed iff $W^\perp = {}^\perp W (= \{f \in V^{**} : \langle f, W\rangle = 0\})$, where V is regarded as a subspace of V^{**}. Therefore, if W_1, W_2 are cofinite subspaces of V^* and if W_1 is closed, then $W_1 \subseteq W_2$ implies that W_2 is closed.
(5) If U and W are subspaces of a space V, then $(U \cap W)^\perp = U^\perp + W^\perp$, and therefore, if U and W are closed subspaces of V^*, then $U \cap W$ is a closed subspace of V^*.

Proof.
(1) Let $v \in V$ with $\langle W^\perp, v\rangle = 0$. We have to show that $v \in W$. Suppose not. Then $V = W \oplus \mathbf{k}v \oplus U$ for some subspace U of V. Define $f \in V^*$ by

$f(v) = 1$, $f(W) = f(U) = 0$. Then $f \in W^\perp$, but $f(v) \neq 0$, a contradiction.

(2) Note that if W is a subspace of V, then $W^* \cong V^*/W^\perp$. This implies the first part. For the converse, if W is a cofinite subspace of V^*, then $(V^*/W)^*$ is finite dimensional. Since $(V^*/W)^* \cong W^\perp$ and ${}^\perp W$ is a subspace of $W^\perp$ (V is regarded as a subspace of V^{**}), this implies that ${}^\perp W$ is finite dimensional.

(3) We show by induction on n that if $f_1, \ldots, f_n$ are independent functionals in V^*, then

(a) there are $v_1, \ldots, v_n \in V$ with $f_i(v_j) = \delta_{i,j}$ (for all $1 \leq i, j \leq n$), and

(b) for every $g \in V^*$, $g \in vec\{f_1, \ldots, f_n\}$ whenever $\bigcap_{i=1}^n Ker(f_i) \subseteq Ker(g)$.

<u>Basis</u>: If $0 \neq f, g \in V^*$ and $Ker(f) \subseteq Ker(g)$, then there is $v \in V$ with $f(v) = 1$. Let $g(v) = \alpha$. Suppose $g \notin vec\{f\}$. Then there is $w \in V$ with $g(w) \neq \alpha f(w)$. Let $f(w) = \beta$. Then $f(w - \beta v) = 0$ so that $g(w - \beta v) = 0$, i.e. $g(w) = \alpha f(w)$, a contradiction.

<u>Step</u>: Let $f_1, \ldots, f_n$ be independent functionals in V^* and $g \in V^*$ with $\bigcap_{i=1}^n Ker(f_i) \subseteq Ker(g)$.

(a) By the induction hypothesis, there are $w_1, \ldots, w_{n-1} \in V$ with $f_i(w_j) = \delta_{i,j}$ for all $1 \leq i, j \leq n-1$. Suppose that for all $v \in V$, if $f_n(v) \neq 0$, then $f_i(v) \neq 0$ for some $1 \leq i \leq n-1$. Then $\bigcap_{i=1}^{n-1} Ker(f_i) \subseteq f_n$, so that, by the induction hypothesis, $f_n \in vec\{f_1, \ldots, f_{n-1}\}$, a contradiction. Therefore, there is some $w \in V$ with $f_n(w) = 1$ and $f_i(w) = 0$ for all $1 \leq i \leq n-1$. Let, for $1 \leq i \leq n-1$, $f_n(w_i) = \alpha_i$. Put, for all $1 \leq i \leq n-1$, $v_i = w_i - \alpha w$, and put $v_n = w$. Then, for all $1 \leq i, j \leq n$, $f_i(v_j) = \delta_{i,j}$.

(b) Let $g(v_i) = \beta_i$ for $1 \leq i \leq n$. Suppose $g \notin vec\{f_1, \ldots, f_n\}$. Then there is $v \in V$ with $g(v) \neq \sum_{i=1}^n \beta_i f_i(v)$. Let $f_i(v) = \gamma_i$ for $1 \leq i \leq n$. Then for all $1 \leq i \leq n$, $f_i(v - \sum_{i=1}^n \gamma_i v_i) = 0$ so that $g(v - \sum_{i=1}^n \gamma_i v_i) = 0$, i.e.

$g(v) = \sum_{i=1}^{n} \beta_i f_i(v)$, a contradiction.
It now follows that if W is a finite dimensional subspace of V^*, then $W = ({}^{\perp}W)^{\perp}$, i.e. W is closed.
(4) ($\Rightarrow$) Let $W = U^{\perp}$ for some subspace U of V. Then, by *(1)*, $U = {}^{\perp}W$. But ${}^{\perp}W$ is embedded in $(V^*/W)^*$, so that, since W is cofinite, U is finite dimensional, say $U = vec\{v_1, \ldots, v_n\}$. Then $\bigcap_{i=1}^{n} Ker(v_i) \subseteq Ker(f)$ whenever $f \in W^{\perp}$ and therefore, by *(3)*, $W^{\perp} \subseteq U = {}^{\perp}W$. The other inclusion is trivial.
($\Leftarrow$) This follows from *(1)*.
(5) It is clear that $(U \cap W)^{\perp} \subseteq U^{\perp} + W^{\perp}$. For the other inclusion, let A, B, C be the subspaces of V for which $V = A \oplus (U+W)$, $U = B \oplus (U \cap W)$ and $W = C \oplus (U \cap W)$. Then $U+W = B \oplus (U \cap W) \oplus C$ and $V = A \oplus B \oplus (U \cap W) \oplus C$. Let $f \in (U \cap W)^{\perp}$. Then there are $g, h \in V^*$ with $f = g + h$, $g(C \oplus (U \cap W)) = 0$ and $h(A \oplus B) = 0$ and therefore $h(U \cap W) = 0$ so that $g \in W^{\perp}$ and $h \in U^{\perp}$. $\square$

It is clear that the standard definitions for ($\mathbf{k}$)-algebras, subalgebras and algebra morphisms can be formulated in terms of tensor products. The following three definitions are dualisations of these formulations, hence the terminology "coalgebra", "subcoalgebra" and "coalgebra morphism".
If V is a vectorspace, then the map $V \longrightarrow \mathbf{k} \otimes V : v \mapsto 1 \otimes v$ is denoted by $1 \otimes -$; the map $V \longrightarrow V \otimes \mathbf{k} : v \mapsto v \otimes 1$ is denoted by $- \otimes 1$.

Definition 5.1.2 Let C be a vectorspace. We say that (C, Δ, ϵ) is a $\mathbf{k}$-*coalgebra* if $\Delta : C \longrightarrow C \otimes C$ and $\epsilon : C \longrightarrow \mathbf{k}$ are maps satisfying the following two conditions:
(C1) $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ (*coassociativity* of Δ);

(C2) $(\epsilon \otimes id) \circ \Delta = 1 \otimes -$ and $(id \otimes \epsilon) \circ \Delta = - \otimes 1$ where $1 \otimes -$ $(- \otimes 1)$ is the natural isomorphism $C \longrightarrow \mathbf{k} \otimes C$ (resp. $C \longrightarrow C \otimes \mathbf{k}$) (*counitary property* of ϵ).

Δ is called the *comultiplication* and ϵ is called the *couint.*

By a *coalgebra* we mean a $\mathbf{k}$-coalgebra. And, if there is no danger for confusion, we will abbreviate (C, Δ, ϵ) by C.

An element c of a coalgebra C is called *group-like* if $\Delta(c) = c \otimes c$ and $\epsilon(c) = 1$, and it is called *primitive* if $\Delta(c) = c \otimes 1 + 1 \otimes c$.

Definition 5.1.3 Let (C, Δ, ϵ) be a coalgebra and D a subspace of C.
D is called a *subcoalgebra* of C if $\Delta D \subseteq D \otimes D$ (the comultiplication $D \longrightarrow D \otimes D$ is $\Delta \upharpoonright D$ and the counit $D \longrightarrow \mathbf{k}$ is $\epsilon \upharpoonright D$).

Definition 5.1.4 Let $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ be coalgebras.
A map $f : C \longrightarrow D$ is called a *coalgebra morphism* if $\Delta_D \circ f = (f \otimes f)\Delta_C$ and if $\epsilon_C = \epsilon_D \circ f$.

It is well-known (see [31]) that *if (C, Δ, ϵ) is a coalgebra, then C^* is a $\mathbf{k}$-algebra* with algebra structure given by $\langle fg, c \rangle = (f \otimes g)\Delta(c)$ for $f, g \in C^*$ and for $c \in C$. So we have that, by *(C2)*, $1_{C^*} = \epsilon$.
Also, if A is a finite dimensional $\mathbf{k}$-algebra, then its dual vectorspace, A^*, is a coalgebra. The reason is that if V is a finite dimensional vectorspace, then $V^* \otimes V^*$ is isomorphic to $(V \otimes V)^*$, so that the dual map of the multiplication map $m : A \otimes A \longrightarrow A$ gives a comultiplication on A^* and the dual map of the unit map $u : \mathbf{k} \longrightarrow A$ gives a counit on A^*.

If V is an infinite dimensional vectorspace, then $V^* \otimes V^*$ is properly embedded in $(V \otimes V)^*$. Consequently, if A is an infinite dimensional $\mathbf{k}$-algebra we can not construct a comultiplication on A^* in the same way as in the finite dimensional case. However, this problem can be dealt with by taking the so called *finite dual* of A, A° which is defined as $A^\circ = \{f \in A^* : f(I) = 0 \text{ for some cofinite ideal } I \text{ of } A\}$.
Dualisation of the definitions for modules over a ($\mathbf{k}$)-algebra, submodules and module morphisms (in terms of tensor products) leads to the following three definitions.

Definition 5.1.5 Let (C, Δ, ϵ) be a coalgebra. (M, ρ) is called a *right C-comodule* if M is a vectorspace and $\rho : M \longrightarrow M \otimes C$ is a map satisfying the following two conditions:
(M1) $(id \otimes \epsilon)\rho = - \otimes 1$;
(M2) $(\rho \otimes id)\rho = (id \otimes \Delta)\rho$.

Definition 5.1.6 Let C be a coalgebra and (M, ρ) a right C-comodule. A subspace N of M is called a *subcomodule* of M if $\rho(N) \subseteq N \otimes C$.

Definition 5.1.7 Let C be a coalgebra and let (M, ρ_M) and (N, ρ_N) be right C-comodules. A map $f : M \longrightarrow N$ is called a *comodule homomorphism* if $\rho_N \circ f = (f \otimes id) \circ \rho_M$. The notation $f : (M, \rho_M) \longrightarrow (N, \rho_N)$ expresses that f is a comodule homomorphism.

If there is no danger for confusion, (M, ρ) will be abbreviated by M.
Given a coalgebra C, the category of right C-comodules (with the right C-comodules as its objects and the comodule homomorphisms as its morphisms)

is denoted by $\mathcal{M}^C$.

Sigma Notation

Let (C, Δ, ϵ) be a coalgebra. For $c \in C$ we write $\Delta(c)$ as $\sum_{(c)} c_{(1)} \otimes c_{(2)}$. By the coassociativity of Δ we have that $\sum_{(c)} \Delta(c_{(1)}) \otimes c_{(2)} = \sum_{(c)} c_{(1)} \otimes \Delta(c_{(2)})$ which will be denoted by $\sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$. The couintary property of ϵ says that $\sum_{(c)} \epsilon(c_{(1)})c_{(2)} = \sum_{(c)} \epsilon(c_{(2)})c_{(1)} = c$.

If (M, ρ) is a right C-comodule, then for $a \in M$ we write $\rho(a)$ as $\sum_{(a)} a_{(0)} \otimes a_{(1)}$ (so the $a_{(0)}$ are in M and the $a_{(1)}$ are in C).

So *(M1)* says that $\sum_{(a)} \epsilon(a_{(1)})a_{(0)} = a$. And *(M2)* says that $\sum_{(a)} \rho(a_{(0)}) \otimes a_{(1)} = \sum_{(a)} a_{(0)} \otimes \Delta(a_{(1)})$.

Definition 5.1.8 Let H be an algebra with structure maps $m : H \otimes H \longrightarrow H$ (multiplication) and $u : \mathbf{k} \longrightarrow H$ (unit) which also has the structure of a coalgebra which structure maps $\Delta : H \longrightarrow H \otimes H$ and $\epsilon : H \longrightarrow \mathbf{k}$.

Then $(H, m, u, \Delta, \epsilon)$ is called a $\mathbf{k}$-*Hopfalgebra* if

(1) Δ and ϵ are algebra morphisms (or, equivalently, m and u are coalgebra morphisms), and

(2) there is some $S \in Hom_{\mathbf{k}}(H, H)$ which is an inverse to id_H under the convolution product $*$ (which is given by $(f * g)(c) = \sum f(c_{(1)})g(c_{(2)})$ for all $f, g \in Hom_{\mathbf{k}}(H, H)$).

S is called the *antipode* for the $\mathbf{k}$-Hopf algebra H.

By a *Hopfalgebra* we mean a $\mathbf{k}$-Hopfalgebra.

Example 5.1.9 Let G be a group. Then the group algebra $\mathbf{k}G$ can be given the structure of a Hopfalgebra as follows:

$\Delta : \mathbf{k}G \longrightarrow \mathbf{k}G \otimes \mathbf{k}G$ given by $\Delta(g) = g \otimes g$ for all $g \in G$, and $\epsilon : G \longrightarrow \mathbf{k}$ given by $\epsilon(g) = 1$ for all $g \in G$, give $\mathbf{k}G$ the structure of a coalgebra, and together with its algebra structure and the map $S \in Hom_{\mathbf{k}}(H, H)$ given by $S(g) = g^{-1}$ for all $g \in G$, H is a Hopfalgebra with antipode S.

Example 5.1.10 The set $\mathbf{k}^n = \underbrace{\mathbf{k} \times \cdots \times \mathbf{k}}_{n}$ is called an *affine n-space* and is denoted by $\mathbb{A}^n$. An *affine variety* is the set of common zeros in $\mathbb{A}^n$ (some n) of a collection of polynomials in $\mathbf{k}[X_1, \ldots, X_n]$. Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be affine varieties.

The cartesian product $V \times W \subseteq \mathbb{A}^{n+m}$ also is an affine variety: if $V = \{\overline{v} \in \mathbb{A}^n : f_i(\overline{v}) = 0 \text{ for all } i \in I\}$ and $W = \{\overline{w} \in \mathbb{A}^m : g_j(\overline{w}) = 0 \text{ for all } j \in J\}$, then $V \times W$ is equal to $\{(\overline{v}, \overline{w}) \in \mathbb{A}^{n+m} : f_i(\overline{v}) = 0 \text{ and } g_j(\overline{w}) = 0 \text{ for all } i \in I \text{ and } j \in J\}$.

By $\mathbf{k}[V]$ we denote the *affine algebra* of the affine variety V, which is defined as $\mathbf{k}[X_1, \ldots, X_n]/\mathfrak{I}(V)$, where $\mathfrak{I}(V)$ denotes the collection of polynomials in $\mathbf{k}[X_1, \ldots, X_n]$ vanishing on V (i.e. there is a one-to-one correspondence between the distinct polynomials on V and the elements of $\mathbf{k}[V]$).

A map $f : V \longrightarrow W$ is a *homomorphism of affine varieties* if there are $f_1, \ldots, f_m \in \mathbf{k}[V]$ such that for all $\overline{v} \in V$, $f(\overline{v}) = (f_1(\overline{v}), \ldots, f_m(\overline{v}))$.

An *affine algebraic group* is defined as an affine variety with the structure of a group such that its group-structure maps are homomorphisms of affine varieties.

Let $G \subseteq \mathbb{A}^n$ be an affine algebraic group. Then the affine algebra of G, $\mathbf{k}[G]$, has the structure of a Hopfalgebra as follows.

Its coalgebra structure is given by $\Delta : \mathbf{k}[G] \longrightarrow \mathbf{k}[G] \otimes \mathbf{k}[G]$ which takes $f \in \mathbf{k}[G]$ to $\sum_i g_i \otimes h_i$, where g_i and h_i are such that for all $x, y \in G$,

$f(xy) = \sum_i g_i(x)h_i(y)$, and by $\epsilon : \mathbf{k}[G] \longrightarrow \mathbf{k}$ taking $f \in \mathbf{k}[G]$ to $f(e_G)$. The antipode $S \in Hom_{\mathbf{k}}(\mathbf{k}[G], \mathbf{k}[G])$ is given by $S(f)(x) = f(x^{-1})$ for all $f \in \mathbf{k}[G]$ and all $x \in G$.

Example 5.1.11 For a group G, a *G-module* is defined as a vectorspace V with an action $G \times V \longrightarrow V : (x, v) \mapsto xv$ satisfying:
(1) $x(v_1 + v_2) = xv_1 + xv_2$;
(2) $x(\alpha v) = \alpha(xv)$;
(3) $xy(v) = x(yv)$;
(4) $e_G v = v$ (where e_G denotes the unit of G)
for all $x, y \in G$ and all $v, v_1, v_2 \in V$.
Let G be an affine algebraic group and V a G-module.
Then V is called a *rational G-module* if for any basis $\mathcal{B}$ of V and for every $v \in V$, there exists a finite subset $\{v_1, \dots, v_k\}$ of $\mathcal{B}$ such that the functions λ_i given by

$$xv = \sum_{i=1}^{k} \lambda_i(x) v_i \text{ for all } x \in G,$$

belong to $\mathbf{k}[G]$.
Let V be a rational G-module, then V can be given the structure of a $\mathbf{k}[G]$-comodule (with the coalgebra structure on $\mathbf{k}[G]$ as defined in Example 5.1.10) as follows. For $\{v_\gamma\}$ a basis of V, define a map $\rho : V \longrightarrow V \otimes \mathbf{k}[G]$ via $\rho(v_\gamma) := \sum_\delta v_\delta \otimes \lambda_{\gamma\delta}$ for all v_γ, where the $\lambda_{\gamma\delta}$ are defined by: $xv = \sum_\delta \lambda_{\gamma\delta}(x) v_\delta$ for all $x \in G$.
One checks that $\Delta(\lambda_{\gamma\delta}) = \sum_\eta \lambda_{\eta\delta} \otimes \lambda_{\gamma\eta}$ so that (V, ρ) is indeed a $\mathbf{k}[G]$-comodule.
Conversely, if (V, ρ) is a $\mathbf{k}[G]$-comodule, then V has the structure of a rational

G-module as follows. If $\{v_\gamma\}$ is a basis of V, define an action $G \times V \longrightarrow V$: $(x, v) \mapsto xv$ of G on V via $xv_\gamma := \sum_\delta \lambda_{\gamma\delta}(x)v_\delta$ for all $x \in G$ and all $\{v_\gamma\}$, where the $\lambda_{\gamma\delta}$ are defined by $\rho(v_\gamma) = \sum_\delta v_\delta \otimes \lambda_{\gamma\delta}$.
It follows that if G is an affine algebraic group, then *there is a one-to-one correspondence between the rational G-modules and the* $\mathbf{k}[G]$*-comodules.*

5.2 Rational Modules and Comodules

In this section we compare $\mathcal{M}^C$, the category of right comodules over a coalgebra C, with C^*-Mod, the category of left modules over the dual algebra C^*.

Throughout, let $(C, \triangle, \epsilon)$ (C for short) be an arbitrary coalgebra over a field $\mathbf{k}$. The set of the closed and cofinite left C^*-ideals will be denoted by $\mathcal{I}$. All C-comodules are right C-comodules and all C^*-modules are left C^*-modules. If $f : M \longrightarrow N$ is a morphism in C^*-Mod and $A \subseteq M$, then the image of A under f will be denoted by $(A)f$ and if g is a morphism in C^*-Mod with domain N, then the composition $g \circ f$ will be denoted by fg.
The C-comodules and C^*-modules are closely related. To specify their relation we need the notion of a rational C^*-module (see [31] and also [26]), which is very much like the notion of a rational G-module (for G an affine algebraic group).

Definition 5.2.1 A C^*-module M is called *rational* in case for every $a \in M$ there exist finitely many $d_1, \ldots, d_n \in C$ and $a_1, \ldots, a_n \in M$ such that for every $c^* \in C^*$, $c^*a = \sum_{i=1}^n c^*(d_i)a_i$.

The class of rational C^*-modules will be denoted by $\mathfrak{Rat}$ and the category of rational C^*-modules will be denoted by C^*-Rat (i.e. C^*-Rat is the full subcategory of C^*-Mod, the objects of which are the rational C^*-modules). It is clear that $\mathfrak{Rat}$ is a hereditary pretorsion class in C^*-Mod, i.e. $\mathfrak{Rat}$ is closed under submodules, direct sums and arbitrary quotients.

Any C-comodule (M,ρ) can be given the structure of a C^*-module in the following way: if $a \in M$ and $\rho(a) = \sum_{(a)} a_{(0)} \otimes a_{(1)}$, then define, for $c^* \in C^*$, $c^*a = \sum_{(a)} c^*(a_{(1)})a_{(0)}$. To see that this is well-defined, take a basis $\{c_i\}_{i\in I}$ of C. $\sum_{(a)} a_{(0)} \otimes a_{(1)}$ is a finite sum $\sum_{i=1}^n a_i \otimes d_i$ where the a_i are the (nonzero) elements of $\{a_{(0)}\}$ and the d_i are the (nonzero) elements of $\{a_{(1)}\}$. For each d_i there is a set $\{\alpha_{ij}\}_{j\in I} \subseteq \mathbf{k}$, with $\alpha_{ij} = 0$ for all but finitely many $j \in I$, such that $d_i = \sum_{j\in I} \alpha_{ij}c_j$. And there exists a set $\{b_j\}_{j\in I} \subseteq M$, with $b_j = 0$ for all but finitely many $j \in I$, such that $\rho(a) = \sum_{j\in I} b_j \otimes c_j = \sum_{i=1}^n a_i \otimes d_i$. It suffices to show that ,for all $k \in I$, $\sum_{i=1}^n c_k^*(d_i)a_i = \sum_{j\in I} c_k^*(c_j)b_j$. Comparing the terms in $\sum_{j\in I} b_j \otimes c_j$ and $\sum_{i=1}^n a_i \otimes d_i = \sum_{j\in J}\sum_{i=1}^n \alpha_{ij}a_i \otimes c_j$ yields that, for all $j \in I$, $b_j = \sum_{i=1}^n \alpha_{ij}a_i$. So, for all $k \in I$, $\sum_{j\in I} c_k^*(c_j)b_j = b_k = \sum_{i=1}^n \alpha_{ik}a_i = \sum_{i=1}^n c_k^*(d_i)a_i$.
One easily checks that the map $C^* \otimes M \longrightarrow M$ defined above gives M the structure of a C^*-module; it is clear that M is rational.

Conversely, every rational C^*-module can be given the structure of a C-comodule: if $a \in M$ and, for all $c^* \in C^*$, $c^*a = \sum_{i=1}^n c^*(d_i)a_i$, then put $\rho(a) = \sum_{i=1}^n a_i \otimes d_i$. Again, to see that this is well-defined, we take a basis $\{c_i\}_{i\in I}$ of C. For each d_i there is a set $\{\alpha_{ij}\}_{j\in I} \subseteq \mathbf{k}$, with $\alpha_{ij} = 0$ for all but finitely many $j \in I$, such that $d_i = \sum_{j\in I} \alpha_{ij}c_j$. So, for all $k \in I$,

$c_k^* a = \sum_{i=1}^n \alpha_{ik} a_i$, and therefore $\sum_{i=1}^n a_i \otimes d_i = \sum_{k\in I} \sum_{i=1}^n a_i \otimes \alpha_{ik} c_k = \sum_{k\in I} c_k^* a \otimes c_k$, i.e. $\sum_{i=1}^n a_i \otimes d_i = \sum_{k\in I} c_k^* a \otimes c_k$ for every basis $\{c_i\}_{i\in I}$ of C. We will include a proof of the fact that the map $\rho : M \longrightarrow M \otimes C$ defined above gives M the structure of a C-comodule.

Clearly, ρ is a $\mathbf{k}$-linear map. Let $\{c_i\}_{i\in I}$ be a basis of C. Note that the scalar multiplication $C^* \otimes M \longrightarrow M$ can be expressed in terms of ρ as follows: for all $c^* \in C^*$ and for all $a \in M$ we have that $c^* a = f(id \otimes \langle , \rangle)(\tau \otimes id)(id \otimes \rho)(c^* \otimes a)$, where f is the natural isomorphism $M \otimes \mathbf{k} \longrightarrow M$. So for all $c^* \in C^*$ and for all $a \in M$, $c^* a = \sum_{i\in I} c^*(c_i) c_i^* a$. Since $1_{C^*} = \epsilon$, this implies that for all $a \in M$, $a = \epsilon a = \sum_{i\in I} \epsilon(c_i) c_i^* a$, i.e. ρ satisfies the counitary property.

Remains to show that for all $a \in M$, $(\rho \otimes id)\rho(a) = (id \otimes \Delta)\rho(a)$. Let f' be the natural isomorphism $M \otimes \mathbf{k} \otimes \mathbf{k} \longrightarrow M$. We have that, for all $a \in M$

and for all $c^*, d^* \in C^*$,

$$\begin{aligned}
f'(id \otimes c^* \otimes d^*)(\rho \otimes id)\rho(a) &= f'(id \otimes c^* \otimes d^*)(\rho \otimes id)\sum_{i \in I} c_i^* a \otimes c_i \\
&= f'(id \otimes c^* \otimes d^*)\sum_{i,j \in I} c_j^*(c_i^* a) \otimes c_j \otimes c_i \\
&= f' \sum_{i,j \in I} c_j^*(c_i^* a) \otimes c^*(c_j) \otimes d^*(c_i) \\
&= \sum_{i,j \in I} c^*(c_j) d^*(c_i) c_j^*(c_i^* a) \\
&= c^*(d^* a) \\
&= (c^* d^*)a \text{ (since } M \text{ is a } C^*\text{-module)} \\
&= \sum_{i \in I} c^* d^*(c_i) c_i^* a \\
&= \sum_{i \in I} c_i^* a \sum_{(c_i)} c^*(c_{i_{(1)}}) d^*(c_{i_{(2)}}) \\
&\quad \text{(by definition of multiplication in } C^*\text{)} \\
&= f'(id \otimes c^* \otimes d^*)(id \otimes \Delta)\rho(a).
\end{aligned}$$

So, for all $a \in M$, $(id \otimes c^* \otimes d^*)(\rho \otimes id)\rho(a) - (id \otimes c^* \otimes d^*)(id \otimes \Delta)\rho(a) = 0$ for all $c^*, d^* \in C^*$. But one can check that $C^* \otimes C^*$ is dense in $(C \otimes C)^*$ (i.e. if $0 \neq x \in C \otimes C$, then there is a $y \in C^* \otimes C^*$ such that $y(x) \neq 0$; $C^* \otimes C^*$ is regarded as a subspace of $(C \otimes C)^*$ via $\langle c^* \otimes d^*, c \otimes d \rangle = c^*(c)d^*(d)$ for $c^*, d^* \in C^*$ and $c, d \in C$). Therefore, for all $a \in M$, $(\rho \otimes id)\rho(a) = (id \otimes \Delta)\rho(a)$.
We conclude that there is a 1-1 correspondence between the C-comodules and the C^*-modules (see [31]).

Proposition 5.2.2 ([31]) *The categories $\mathcal{M}^C$ and C^*-Rat are equivalent. So, in particular, if C is finite dimensional, then $\mathcal{M}^C$ and C^*-Mod are equivalent.*

Proof. Define the functor $\mathbf{R} : \mathcal{M}^C \longrightarrow C^*\text{-}Rat$ as follows. If (M, ρ) is a C-comodule, let $\mathbf{R}(M, \rho)$ equal M with the C^*-module structure defined above. If $f : (M, \rho) \longrightarrow (M', \rho')$ is a C-comodule morphism, let $\mathbf{R}f : \mathbf{R}(M, \rho) \longrightarrow \mathbf{R}(M', \rho')$ equal f as a $\mathbf{k}$-linear map. One can easily check that $\mathbf{R}f$ is in fact a C^*-module morphism.

Let $\mathbf{S} : C^*\text{-}Rat \longrightarrow \mathcal{M}^C$ be the functor defined as follows. If M is a rational C^*-module, let $\mathbf{S}M$ equal (M, ρ) where ρ is the C-comodule structure map $M \longrightarrow M \otimes C$ defined above. If $f : M \longrightarrow M'$ is a C^*-module morphism, let $\mathbf{S}f : \mathbf{S}M \longrightarrow \mathbf{S}M'$ equal f as a $\mathbf{k}$-linear map. One can easily check that $\mathbf{S}f$ is in fact a C-comodule morphism.

Clearly $\mathbf{SR} = id_{\mathcal{M}^C}$ and $\mathbf{RS} = id_{C^*-Rat}$. □

The following lemma is a useful characterisation for the rational C^*-modules. (See [26] for a more general setting and somewhat different proof.)

Lemma 5.2.3 ([26]) *A C^*-module M is rational iff for every $a \in M$, the left C^*-ideal $ann_{C^*}(a)$ (=$\{c^* \in C^* : c^*a = 0\}$) is cofinite and closed.*

Proof. ($\Rightarrow$) Let $a \in M$, $\rho(a) = \sum_{(a)} a_{(0)} \otimes a_{(1)}$ and let V be the finite dimensional subspace of C spanned by the $a_{(1)}$'s.

Then (by Lemma 5.1.1 *(2)*) $V^\perp$ is cofinite and closed. But for every $c^* \in V^\perp$, $c^*a = \sum_{(a)} c^*(a_{(1)})a_{(0)} = 0$ and therefore $V^\perp \subseteq ann_{C^*}(a)$, which implies that $ann_{C^*}(a)$ is cofinite and therefore (by Lemma 5.1.1 *(4)*), $ann_{C^*}(a)$ is closed.

($\Leftarrow$) Let $a \in M$ and $c_1^*, \ldots, c_n^* \in C^*$ such that $\{c_i^* + ann_{C^*}(a)\}_{1 \le i \le n}$ is a basis of $C^*/ann_{C^*}(a)$. Let V be the subspace of C for which $V^\perp = ann_{C^*}(a)$. Then the restrictions of the c_i^* to V are independent (for if there are $\alpha_1 \ldots, \alpha_n \in \mathbf{k}$

with $\sum_{i=1}^n \alpha_i c_i^*(v) = 0$ for all $v \in V$, then $\sum_{i=1}^n \alpha_i c_i^* \in V^\perp = ann_{C^*}(a)$, a contradiction). By (the proof of) Lemma 5.1.1 *(3)* there are $d_1, \dots, d_n \in V$ with $c_i^*(d_j) = \delta_{i,j}$ (for all $1 \leq i, j \leq n$).

Let $c^* \in C^*$. Then there are $\alpha_1, \dots, \alpha_n \in \mathbf{k}$ with $c^* - (\alpha_1 c_1^* + \cdots + \alpha_n c_n^*) \in ann_{C^*}(a)$ so that, for all $1 \leq i \leq n$, $c^*(d_i) = \alpha_i$ and $c^* a = \sum_{i=1}^n \alpha_i c_i^* a = \sum_{i=1}^n c^*(d_i) c_i^* a$. Therefore, with $a_i = c_i^* a$, we have that for every $c^* \in C^*$, $c^* a = \sum_{i=1}^n c^*(d_i) a_i$. $\square$

Corollary 5.2.4 (see also [31] and [26])

(1) Every C^*-module M contains a unique maximal rational submodule, namely $\{a \in M : ann_{C^*}(a)$ is cofinite and closed$\}$. This module will be denoted by M^{rat}.

(2) Every finitely generated rational C^*-module is finite dimensional.

Proof. *(1)* Let M be a C^*-module. By Lemma 5.2.3 it suffices to show that $\{a \in M : ann_{C^*}(a)$ is cofinite and closed$\}$ is a submodule of M. So, let $a, b \in M$ with both $ann_{C^*}(a)$ and $ann_{C^*}(b)$ cofinite and closed. Since $ann_{C^*}(a) \cap ann_{C^*}(b) \leq ann_{C^*}(a+b)$, Lemma 5.1.1*(5),(4)* imply that $ann_{C^*}(a+b)$ is cofinite and closed. Next we show that $C^* a$ is a rational submodule of M. Since $ann_{C^*}(a)$ is cofinite and closed we have that, by the proof of Lemma 5.2.3, there exist $a_1, \dots, a_n \in C^* a$ and $d_1, \dots, d_n \in C$ such that for all $c^* \in C^*$, $c^* a = \sum_{i=1}^n c^*(d_i) a_i$. Let $c^* \in C^*$. Then for all $e^* \in C^*$, $e^*(c^* a) = (e^* c^*) a = \sum_{i=1}^n e^* c^*(d_i) a_i$. The algebra structure of C^* is such that (for every $1 \leq i \leq n$) $e^* c^*(d_i) = \sum_{(d_i)} e^*(d_{i_{(1)}}) c^*(d_{i_{(2)}})$. Since for every $1 \leq i \leq n$, the $c^*(d_{i_{(2)}}) a_i$ are elements of $C^* a$ and the

$d_{i_{(1)}}$ are elements of C, it is clear from the proof of Lemma 5.2.3 that for every $c^* \in C^*$, $ann_{C^*}(c^*a)$ is cofinite and closed. We conclude that $\{a \in M : ann_{C^*}(a)$ is cofinite and closed$\}$ is the unique maximal rational submodule of M.

(2) It suffices to show that every cyclic rational C^*-module is finite dimensional. But this is clear, since every cyclic rational C^*-module is isomorphic to a C^*-module of the form C^*/I where, by Lemma 5.2.3, I is a cofinite and closed left C^*-ideal. $\square$

Note that if M is a C^*-module, M' a rational C^*-module and $f : M' \longrightarrow M$ a C^*-module morphism, then $Im(f)$ is a rational submodule of M.

Chapter 6

Ziegler Spectra over Coalgebras

6.1 The Category of Rational C^*-Modules

Since $\mathfrak{Rat}$ is closed under submodules and quotients, it easy to see that the full subcategory C^*-Rat of the Abelian category C^*-Mod is Abelian as well (in particular, the kernel, resp. cokernel, of any morphism in C^*-Rat is equal to its kernel, resp. cokernel, in C^*-Mod). From the fact that $\mathfrak{Rat}$ is closed under direct sums it then follows that $\mathfrak{Rat}$ is closed under colimits so that the category C^*-Rat clearly is cocomplete. It also follows that in the notation for the direct sum of rational modules we do not need to specify in which category (C^*-Rat or C^*-Mod) the direct sum is taken.
It is easy to see (for example with Corollary 5.2.4) that the class $\mathfrak{Rat}$, in general, is not closed under products. Nevertheless, we will see that the category C^*-Rat has products, so that, being an Abelian category, C^*-Rat is also complete. If M is a rational C^*-module, then M is an epimorphic image of $\oplus_{a\in M} C^*/ann_{C^*}(a)$ (where $\oplus$ denotes the direct sum in the category

C^*-*Rat*, or equivalently, in C^*-Mod). Therefore, the family

$$\mathcal{G} := \{C^*/I : I \text{ is a cofinitite and closed left } C^*\text{-ideal }\}$$

is a generating family for the category C^*-*Rat*. From the fact that C^*-Mod satisfies Condition *(Ab5)* it clearly follows that also C^*-*Rat* satisfies *(Ab5)* and therefore: *the category C^*-Rat is a Grothendieck category.*

An important tool for the following results is the functor ${}^{rat} : C^*\text{-}Mod \longrightarrow C^*\text{-}Rat$ which is defined as follows: ^{rat} takes a C^*-module M to its unique maximal rational submodule M^{rat} and a morphism $f : M \longrightarrow N$ in C^*-Mod is taken to its restriction $f \restriction M^{rat} : M^{rat} \longrightarrow N^{rat}$ which is denoted by f^{rat}. Clearly, the functor ^{rat} is left exact. It is however, in general, not right exact. A nontrivial class of coalgebras for which ^{rat} is exact is the class of cosemisimple coalgebras (see [28] for details; a coalgebra C is called *simple* if it has no proper subcoalgebras, and C is called *cosemisimple* if it is a direct sum of simple subcoalgebras).

The following lemma says that the category C^*-*Rat* has products and that the functor ^{rat} commutes with products.

Lemma 6.1.1 *If (M, π_i) is the product in C^*-Mod of a family $\{M_i\}_i$ of C^*-modules, then (M^{rat}, π_i^{rat}) is the product in C^*-Rat of $\{M_i^{rat}\}_i$.*

Proof. Let $\{M_i\}_i$ be a family of C^*-modules and let (M, π_i) be its product in C^*-Mod. Let N be a rational C^*-module with a morphism $f_i : N \longrightarrow M_i^{rat} \subseteq M_i$ for each i. Then there is a unique morphism $f : N \longrightarrow M$ in C^*-Mod with $f\pi_i = f_i$ for each i. Since N is rational we have that $Im(f) \leq M^{rat}$

and therefore $f^{rat} : N \longrightarrow M^{rat}$ is the unique morphism in C^*-*Rat* for which $f^{rat}\pi_i^{rat} = f_i$. $\square$

Since $\mathfrak{Rat}$ is closed under submodules and both C^*-*Rat* and C^*-*Mod* are Grothendieck categories, *a rational C^*-module is finitely generated in C^*-Mod iff it is finitely generated in C^*-Rat iff it is finitely presented in C^*-Rat.* The second equivalence follows from the first equivalence and from Corollary 5.2.4*(2)*. An immediate consequence is that *the category C^*-Rat is locally coherent.* (An additive category is called *locally coherent* if it is locally finitely presented Abelian and if every finitely generated subobject of a finitely presented object is finitely presented.)
It is now also clear that the generating family

$$\mathcal{G} = \{C^*/I : I \text{ is a cofinite and closed left } C^*\text{-ideal}\}$$

is a family of finitely presented objects in the category of rational C^*-modules, i.e. *the category C^*-Rat is locally finitely presented.*

Since C^*-*Rat* is a Grothendieck category, we know that every rational C^*-module has an injective hull in C^*-*Rat*. We will express the injective hull of a rational C^*-module in C^*-*Rat* in terms of its injective hull in C^*-*Mod*. We need some technical lemmas.
For $\mathcal{M}$ a class of monomorphisms in C^*-*Mod*, $\mathcal{M}^{rat}$ denotes the class $\{f \in \mathcal{M} : f \text{ is a monomorphism in } C^*\text{-}Rat\}$.

Lemma 6.1.2 *Let $\mathfrak{M}$ be a class of monomorphisms in C^*-Mod.*
If M is a C^*-module which is $\mathfrak{M}$-injective, then M^{rat} is $\mathfrak{M}^{rat}$-injective.

Proof. Let M be an $\mathfrak{A}$-injective C^*-module, let $f : K \longrightarrow L$ be a monomorphism in $\mathfrak{A}^{rat}$ and let $g : K \longrightarrow M^{rat}$ be a morphism (in C^*-*Rat*).
Then, since $f \in \mathfrak{A}^{rat} \subseteq \mathfrak{A}$, there is some $h : L \longrightarrow M$ with $fh = g$, which implies that $fh^{rat} = g$. $\square$

Lemma 6.1.3 *(1) If $\{M_i\}$ is a family of C^*-modules, then $(\oplus M_i)^{rat} = \oplus M_i^{rat}$, i.e. the functor ^{rat} commutes with direct sums.*
(2) If f is a monomorphism in C^*-Mod, then $Im(f)^{rat} = Im(f^{rat})$.
(3) If f is a split monomorphism in C^*-Mod, then f^{rat} is a split monomorphism in C^*-Rat.
(4) If $f : M \longrightarrow N$ is an essential monomorphism in C^*-Mod, then f^{rat} is an essential monomorphism in C^*-Rat.

Proof. *(1)* Let $\{M_i\}$ be a family of C^*-modules. Then $\oplus_i M_i^{rat}$ is a rational C^*-module so that clearly $\oplus_i M_i^{rat} \leq (\oplus_i M_i)^{rat}$. For the other inclusion, let $(a_i)i \in (\oplus_i M_i)^{rat}$. Then, for all j, $ann_{C^*}((a_i)_i) \leq ann_{C^*}(a_j)$ so that, for all j, $ann_{C^*}(a_j)$ is cofinite and closed (by Lemma 5.1.1 *(4)*), i.e. $a_i \in M_i^{rat}$ for all i so that $(a_i)_i \in \oplus_i M_i^{rat}$.
(2) Let $f : M \longrightarrow N$ be a monomorphism in C^*-Mod. Clearly, $Im(f^{rat}) \leq Im(f)^{rat}$. For the other inclusion, let $b \in Im(f)^{rat}$. Then $b = (a)f$ for some $a \in M$. Since f is a monomorphism, $ann_{C^*}(a) = ann_{C^*}(b)$ which is cofinite and closed, and therefore $a \in M^{rat}$, i.e. $b \in Im(f^{rat})$.
(3) This follows from *(1)*, *(2)* and the fact that for every morphism f in C^*-Mod, $Ker(f)^{rat} = Ker(f^{rat})$.
(4) This follows from *(2)* and the fact that if K is a rational C^*-module, then for every C^*-module L, $K \cap L = K \cap L^{rat}$. $\square$

Proposition 6.1.4 *Every rational C^*-module has an injective hull in C^*-Rat. More precisely, if M is a rational C^*-module, then $E(M)^{rat}$ is the injective hull of M in C^*-Rat.*

Proof. Let M be a rational C^*-module. By Lemma 6.1.2, $E(M)^{rat}$ is injective in C^*-*Rat*, and by Lemma 6.1.3*(4)*, M is essentially embedded in $E(M)^{rat}$. $\square$

An object in a Grothendieck category is called *Noetherian* (*Artinian*) if the lattice of its subobjects is Noetherian (Artinian). (In general it is not true that every locally Artinian Grothendieck category also is locally Noetherian.) It is known that the category of modules over a fixed ring is locally Noetherian (locally Artinian) iff the ring is Noetherian (Artinian)(see [30] V §5).
Clearly, by Corollary 5.2.4*(2)*, every finitely generated object in C^*-*Rat* is Noetherian and Artinian (clearly, in the category C^*-*Rat*, just like in the category C^*-*Mod*, every Artinian object is Noetherian). So the elements of the family $\mathcal{G}$ of generators (introduced above) are Artinian (and Noetherian), which means that the category C^*-*Rat* is locally Artinian and locally Noetherian.
A consequence of the category C^*-*Rat* being locally finitely presented and locally Noetherian is the following proposition (see [30] V (4.5)). This result, translated to the category of comodules over a fixed coalgebra, was shown in [8] and later in [3]. Both proofs are more direct, but rather long.

Proposition 6.1.5 ([8]) *In the category C^*-Rat, the class of injective objects is closed under direct limits, and every injective object is a direct sum of indecomposable injective subobjects.*

Summary 6.1.6 *The category C^*-Rat (and therefore also $\mathcal{M}^C$) is*

- *locally finitely presented Abelian (which is equivalent to "locally finitely presented and Grothendieck"; see [4](2.4))*
- *locally Noetherian and locally Artinian*
- *locally coherent*

6.2 A Language for the Category of Comodules over a Coalgebra

In the following example we meet a coalgebra (which is introduced in [31]) for which the class of the rational modules over its dual algebra is not closed under ultraproducts. So, in general, *the class $\mathfrak{Rat}$ is not axiomatisable by a theory in a finitary language for C^*-modules* (e.g. see [2]), i.e. the class $\mathfrak{Rat}$ is, in general, not elementary.

Nevertheless, C^*-Rat is locally finitely presented so that the category is finitarily axiomatisable. In fact, we will write down a finitary equational theory T^{rat} in $\mathcal{L}(\mathcal{G})_\omega$ axiomatising C^*-Rat.

Using Lemma 0.2.2 we will then be able to compare purity in C^*-Rat with purity in C^*-Mod.

Example 6.2.1 Let C be an infinite dimensional vectorspace with basis $\{c_i\}_{i\in\omega}$. Equipped with maps $\Delta : C \longrightarrow C \otimes C$ and $\epsilon : C \longrightarrow \mathbf{k}$ defined via $\Delta c_k = \sum_{i=0}^{k} c_i \otimes c_{k-i}$ (for all $k \in \omega$) and $\epsilon(c_k) = \delta_{o,k}$ (for all $k \in \omega$), C is a (cocommutative) coalgebra. For each $i \in \omega$ define X^i in C^* via $X^i(c_j) = \delta_{i,j}$

(for all $j \in \omega$). It is easy to see that (for all $i, j \in \omega$) $X^i X^j = X^{i+j}$ and that $X^0 = 1_{C^*}$. Therefore the (commutative) algebra C^* can be identified with the formal power series ring $\{\sum_{i\in\omega} \lambda_i X^i : \lambda_i \in \mathbf{k}\}$. Let (for each $i \in \omega$) $\langle X^i \rangle$ denote the ideal in C^* generated by X^i. Clearly, each $C^*/\langle X^i \rangle$ is a rational C^*-module. It is easy to see that the product (in C^*-Mod) of the family $\{C^*/\langle X^i \rangle\}_{i\in\omega}$ is not rational.
Let $\mathfrak{U}$ be an ultrafilter containing all cofinite subsets of ω (i.e. all subsets A of ω for which $\omega \setminus A$ is finite). (That such an ultrafilter exists follows from the well-known fact that any set of subsets of a set S with the finite intersection property is contained in an ultrafilter over S.) Then $\mathfrak{U}$ does not contain any finite set so that with $x = (x_i)_{i\in\omega} = (X^0 + \langle X^i \rangle)_{i\in\omega} \in \prod_{i\in\omega} M_i$ and with $[x]$ the equivalence class of x under the equivalence relation $\sim$ on $\prod_{i\in\omega} M_i$ defined by $(y_i)_{i\in\omega} \sim (z_i)_{i\in\omega}$ iff $\{i \in \omega : y_i = z_i\} \in \mathfrak{U}$, it is easy to see that $ann_{C^*}([x]) = 0$ from which it follows that the ultraproduct $\prod_{i\in\omega} M_i/\mathfrak{U}$ is not a rational C^*-module.

The above is an example of an infinite dimensional coalgebra for which the Jacobson radical of its (commutative) dual algebra is cofinite and finitely generated.
Note that *every finitely generated left C^*-ideal is closed* (see [11](1.3.1) for a different proof). Indeed, let I be the left C^*-ideal generated by d^*. If $c^* \in ({}^\perp I)^\perp$, then $Ker(d^*) \subseteq Ker(c^*)$ so that, by Lemma 5.1.1*(3)*, $c^* \in I$, and therefore, $I = ({}^\perp I)^\perp$ (the inclusion $I \subseteq ({}^\perp I)^\perp$ is trivial). This shows that every principal left C^*-ideal is closed and the rest follows from Lemma 5.1.1*(5)*.

Lemma 6.2.2 ([11](1.1.2) and [15](5.2.9))

(1) If A is an algebra and I is a two-sided A-ideal which is cofinite and finitely generated as a left A-ideal, then I^n is cofinite and finitely generated as a left A-ideal for each $n \in \omega$.

(2) Let C be a coalgebra. Then $Jac(C^*)$ is a closed (2-sided) C^*-ideal with $\bigcap_{n\in\omega} Jac(C^*)^n = 0$.

With the above lemma, the proof from Example 6.2.1 shows that *if C^* is an infinite dimensional algebra with cofinite Jacobson radical which is finitely generated as a left C^*-ideal (so that $Jac(C^*)^n \neq 0$ for all $n \in \omega$), then the class of rational C^*-modules is not axiomatisable in $\mathcal{L}(C^*)_\omega$* (let $\mathcal{U}$ be as in Example 6.2.1 and let $x = (x_i)_{i\in\omega} = (1_{C^*} + Jac(C^*)^i)_{i\in\omega} \in \prod_{i\ in\omega} C^*/Jac(C^*)^i$).

We remind the reader of the Nakayama Lemma. It says that if M is a finitely generated (left) module over a ring R, then $Jac(R)M$ is a superfluous submodule of M (i.e. if N is a submodule of M with $Jac(R)M + N = M$, then $N = M$), from which it follows that $Jac(R)M \leq Rad(M)$ (where $Rad(M)$ denotes the radical of M, which is defined as the intersection of all maximal submodules of M and is equal to the sum of all superfluous submodules of M).

Let A be an algebra with cofinite Jacobson radical which is finitely generated as a left A-ideal and let I be a cofinite left A-ideal. Put $Rad^0(A/I) = A/I$ and, for $n \geq 1$, $Rad^n(A/I) = Rad(Rad^{n-1}(A/I))$. Since I is cofinite (in A), we have that $Rad^n(A/I)$ is finite dimensional (and so finitely generated) for every $n \geq 0$, so that $Rad^{n+1}(A/I) < Rad^n(A/I)$, whenever $Rad^n(A/I) \neq 0$.

Therefore (again, since I is cofinite), $Rad^k(A/I) = 0$ for some $k \geq 1$ so that, by the Nakayama Lemma, $Jac(A)^k(A/I) = 0$, i.e. $Jac(A)^k \leq I$, which implies that I must be finitely generated (since, by Lemma 6.2.2, $Jac(A)^k$ and $I/Jac(a)^k$ are finitely generated). We conclude that *if A is an algebra with cofinite Jacobson radical, then $Jac(A)$ is finitely generated as a left A-ideal iff every cofinite left A-ideal is finitely generated*(see [11]). An algebra satisfying the latter property is called a *left almost Noetherian algebra*.

Note that *the class $\mathfrak{Rat}$ is axiomatisable by a theory in the infinitary language $\mathcal{L}(C^*)_{\infty\omega}$ for C^*-modules* in which we allow infinite conjunctions (and disjunctions), and let $\mathfrak{I}$ denote the class of cofinite and closed left C^*-ideals. Then the axioms for (left) C^*-modules in the smaller language $\mathcal{L}(C^*)_\omega$ (which is also denoted by ${}_{C^*}\mathcal{L}$), together with the sentence

$$\forall x \bigvee_{I \in \mathfrak{I}} \bigwedge_{c^* \in I} c^* x = 0$$

axiomatise the class $\mathfrak{Rat}$ in the infinitary language $\mathcal{L}(C^*)_{\infty\omega}$ for (left) C^*-modules.

Even though the class $\mathfrak{Rat}$ is, in general, not axiomatisable in $\mathcal{L}(C^*)_\omega$, the category C^*-*Rat*, being locally finitely presented (and Abelian), is equivalent to a category which is axiomatisable in a (multi-sorted) finitary language.
With $\mathcal{G} = \{C^*/I : I \in \mathfrak{I}\}$, we adopt the following notational conventions for the finitary multi-sorted signature $\mathcal{L}(\mathcal{G})$:
• the set of sorts in $\mathcal{L}(\mathcal{G})$ is denoted by $S = \{s_I : I \in \mathfrak{I}\}$;
• for each sort s_I there is an operation symbol $+_I$ of arity $s_I \times s_I \longrightarrow s_I$, an operation symbol $-_I$ of arity $s_I \longrightarrow s_I$ and a constant symbol 0_I (later on

the index I will be dropped);

• for each morphism $\alpha : C^*/I \longrightarrow C^*/J$ in the full subcategory $\mathcal{G}$ of C^*-*Rat* there is a operation symbol of arity $s_J \longrightarrow s_I$, and for convenience, the symbol of the morphism is taken as the corresponding operation symbol in $\mathcal{L}(\mathcal{G})$. As usual, $x, y, x_1, x_2, \ldots$ denote variables in the language and the sort over which a variable x ranges is denoted by $s(x)$. Given a sort s_I, x_I denotes a variable in $\mathcal{L}(\mathcal{G})$ which ranges over the sort s_I.

Let $Diag(\mathcal{G})$ be the axioms in $\mathcal{L}(\mathcal{G})_\omega$ expressing the positive atomic diagram of the category $\mathcal{G}$, i.e. $Diag(\mathcal{G})$ the set of sentences in $\mathcal{L}(\mathcal{G})_\omega$ which express the "relations between the morphisms in $\mathcal{G}$" (bearing in mind that a morphism $\alpha : C^*/I \longrightarrow C^*/J$ in $\mathcal{G}$ corresponds to an operation symbol α in $\mathcal{L}(\mathcal{G})$ of arity $s_J \longrightarrow s_I$). For example, if $\gamma = \beta \circ \alpha$ in $\mathcal{G}$, where $\alpha : C^*/I \longrightarrow C^*/J$ and $\beta : C^*/J \longrightarrow C^*/K$, then "$\forall x_K(\gamma(x_K) = \alpha(\beta(x_K)))$" $\in Diag(\mathcal{G})$; also, if $\alpha, \beta, \gamma : C^*/I \longrightarrow C^*/K \in \mathcal{G}$ with $\gamma = \alpha + \beta$, then "$\forall x_K(\gamma(x_K) = \alpha(x_K) +_I \beta(x_K))$" $\in Diag(\mathcal{G})$. Clearly, $Diag(\mathcal{G})$ is an equational theory in $\mathcal{L}(\mathcal{G})_\omega$.

Let T^{rat} be the theory $Diag(\mathcal{G})$, together with the following axiom schemes:

(1) $\{\forall x_I(x_I +_I 0_I = x_I)\}_{I\in\mathcal{J}}$,

(2) $\{\forall x_I, y_I(x_I +_I y_I = y_I +_I x_I)\}_{I\in\mathcal{J}}$,

(3) $\{\forall x_I(x_I +_I (-_I(x_I)) = 0_I)\}_{I\in\mathcal{J}}$,

(4) $\{\forall x_I, y_I, z_I((x_I +_I y_I) +_I z_I = x_I +_I (y_I +_I z_I))\}_{I\in\mathcal{J}}$,

(5) $\{\forall x_I, y_I(\alpha(x_I +_I y_I) = \alpha(x_I) +_J \alpha(y_I))\}_{I,J\in\mathcal{J};\alpha:C^*/J\longrightarrow C^*/I\in\mathcal{G}}$,

(6) $\{\forall x_I(\alpha(x_I) = x_I)\}_{I\in\mathcal{J};\alpha:C^*/I\longrightarrow C^*/I:1+I\mapsto 1+I}$.

T^{rat} is an equational theory, and we will show that the categories $Mod(T^{rat})$

and C^*-*Rat* are equivalent.

For $M \in C^*$-*Rat*, let $\mathbf{H}(M)$ be the following $\mathcal{L}(\mathcal{G})$-structure:

• for each $I \in \mathcal{I}$, the elements of sort s_I in $\mathbf{H}(M)$ are the elements of $Hom(C^*/I, M)$;

• for each $I \in \mathcal{I}$, the operation symbol $+_I$ is interpreted in $\mathbf{H}(M)$ as the addition in the Abelian additive group $Hom(C^*/I, M)$ and 0_I is interpreted in $\mathbf{H}(M)$ as the zero element in $Hom(C^*/I, M)$;

• an operation $\alpha : s_J \longrightarrow s_I$ in $\mathcal{L}(\mathcal{G})$ is interpreted in $\mathbf{H}(M)$ as $Hom(\alpha, M)$: $Hom(C^*/J, M) \longrightarrow Hom(C^*/I, M)$ taking $f \in Hom(C^*/J, M)$ to $f \circ \alpha = \alpha f \in Hom(C^*/I, M)$ (where, with our notational convention, $\alpha : C^*/I \longrightarrow C^*/J$ is the morphism in $\mathcal{G}$ corresponding to the operation symbol α in $\mathcal{L}(\mathcal{G})$).

One easily checks that for $M \in C^*$-*Rat*, $\mathbf{H}(M)$ is a model of T^{rat}.

If $a \in M$, then $ann_{C^*}(a) \in \mathcal{I}$ and for every $I \in \mathcal{I}$ with $I \leq ann_{C^*}(a)$, a can be regarded as an element of $Hom(C^*/I, M)$ which takes $1 + I$ to a, and this morphism will be denoted by a_I. So the elements of $\mathbf{H}(M)$ of sort s_I are the elements of the form a_I where $a \in M$, $I \in \mathcal{I}$ and $I \leq ann_{C^*}(a)$. (If more convenient, we will use $f, g, h, \ldots$ to denote elements of $\mathbf{H}(M)$.)

Conversely, every model of T^{rat} can be given the structure of a rational C^*-module. For $M \in Mod(T^{rat})$, let $\mathbf{G}(M)$ be the set $M/\sim$ where $\sim$ is the equivalence relation on M defined as follows: if $a, b \in M$ with $s(a) = s_I$ and $s(b) = s_J$, then $a \sim b$ iff $\forall K \leq I \cap J$ with $K \in \mathcal{I}$, if

$$\alpha : C^*/K \longrightarrow C^*/I : 1 + K \mapsto 1 + I$$

and

$$\beta : C^*/K \longrightarrow C^*/J : 1 + K \mapsto 1 + J$$

then $\alpha(a) = \alpha(b)$.

The $\sim$-equivalence class of $a \in M$ is denoted by $[a]$.

$\mathbf{G}(M)$ can be given the structure of a left C^*-module as follows:

• for every $a, b \in M$ with $s(a) = s_I$, $s(b) = s_J$ and $K = I \cap J$, put $[a] + [b] = [\alpha(a) +_K \beta(b)]$, where

$$\alpha : C^*/K \longrightarrow C^*/I : 1 + K \mapsto 1 + I$$

and

$$\beta : C^*/K \longrightarrow C^*/J : 1 + K \mapsto 1 + J$$

(one easily checks that this is well-defined);

• for every $a \in M$ with $s(a) = s_I$ and for each $c^* \in C^*$, put $c^*[a] = [\alpha(a)]$, where

$$\alpha : C^*/I \longrightarrow C^*/I : 1 + I \mapsto c^* + I$$

(again, one easily checks that this is well-defined).

The axioms T^{rat} now say that, with the above structure, $\mathbf{G}(M)$ is a C^*-module. Moreover, $\mathbf{G}(M)$ is a rational C^*-module, for if $a \in M$ and $s(a) = s_I$, then $c^*[a] = 0$ for all $c^* \in I$.

For $f : M \longrightarrow N$ a morphism in C^*-*Rat*, define $\mathbf{H}(f) : \mathbf{H}(M) \longrightarrow \mathbf{H}(N)$ by putting, for every $a \in M$ and for every $I \leq ann_{C^*}(a)$, $\mathbf{H}(f)(a_I) = (f(a))_I$.

Clearly, this is a good definition and $\mathbf{H}(f)$ is a morphism in the category $Mod(T^{rat})$.

For $f : M \longrightarrow N$ a morphism in $Mod(T^{rat})$, define $\mathbf{G}(f) : \mathbf{G}(M) \longrightarrow \mathbf{G}(N)$ by putting, for every $a \in M$, $\mathbf{G}(f)([a]) = [f(a)]$. It is easy to see that this is well-defined and that $\mathbf{G}(f)$ is a morphism in C^*-*Rat*.

We now have a functor $\mathbf{H} : C^*\text{-}Rat \longrightarrow Mod(T^{rat})$ and a functor $\mathbf{G} : Mod(T^{rat}) \longrightarrow C^*\text{-}Rat$.

Let $M \in C^*\text{-}Rat$ and $a, b \in M$ with $I = ann_{C^*}(a)$ and $J = ann_{C^*}(b)$. One easily checks that $[a_I] = [b_J]$ iff $a = b$. We therefore can identify $\mathbf{GH}(M)$ with M as sets via $[a_I] \longleftrightarrow a$. Under this identification, it is easy to show that for all $c^* \in C^*$, $c^*[a_I] = c^*a$ and $[a_I] + [b_J] = a + b$, from which it follows that $\mathbf{GH}(M)$ and M can be identified as (rational) C^*-modules.

Let $M \in Mod(T^{rat})$ and $a, b \in M$. It is easy to show that, for $I, J \in \mathcal{I}$ with $I \leq ann_{C^*}([a])$ and $J \leq ann_{C^*}([b])$, $[a]_I = [b]_J$ iff $a = b$ (use axiom scheme *(6)*). So we can identify $\mathbf{HG}(M)$ and M as S-sorted sets via $[a]_I \longleftrightarrow a$, where $I = ann_{C^*}([a])$. One shows that, under this identification, $\alpha([a]_I) = \alpha(a)$ for every $\alpha : C^*/K \longrightarrow C^*/I$ in $\mathcal{G}$ and that, for a' any other element of M of sort s_I, $[a]_I +_I [a']_I = a +_I a'$, from which it follows that $\mathbf{HG}(M)$ and M can be identified as $\mathcal{L}(\mathcal{G})$-structures (models of T^{rat}).

It is easy to see that, with the above identifications, $\mathbf{GH}(f) = f$ for every morphism f in C^*-*Rat*, and $\mathbf{HG}(f) = f$ for every morphism f in $Mod(T^{rat})$. We conclude that *the categories C^*-Rat ($\mathcal{M}^C$) and $Mod(T^{rat})$ are equivalent, i.e. the (finitary) equational theory T^{rat} in $\mathcal{L}(\mathcal{G})_\omega$ axiomatises the category* *of rational C^*-modules (the category of C-comodules) $Str(\mathfrak{L}(\mathcal{G}))$.*

Remark 6.2.3 In a similar way one shows that the category C^*-*Rat* is equivalent to the category of contravariant functors $\mathbf{F}$ from $\mathcal{G}$ to *Ab* (the category of Abelian groups) which satisfy the condition that for every $I \in \mathcal{I}$, if $\alpha : C^*/I \longrightarrow C^*/I : 1 + I \mapsto 1 + I$, then $\mathbf{F}(\alpha)(a) = a$ for every a in the Abelian group $\mathbf{F}(C^*/I)$.

For M a rational C^*-module, the $\mathfrak{L}(\mathcal{G})$-structure $\mathbf{H}(M)$ will be denoted by ${}_{\mathcal{G}}M$. From now on, we will leave out the subscripts ${}_I$ ($I \in \mathcal{I}$) in the language $\mathfrak{L}_\omega$.

The *pp*-formula in $\mathfrak{L}(\mathcal{G})_\omega$ are the formulae of the form

$$\exists y_1, \dots, y_k \bigwedge_{i=1}^{m} (\sum_{j=1}^{n} \alpha_{ji}(x_j) + \sum_{l=1}^{k} \beta_{li}(y_l) = 0)$$

where, for all $i = 1, \dots m$, the α_{ji} are operation symbols of arity $s(x_j) \longrightarrow s_I$ and the β_{li} are operation symbols of arity $s(y_l) \longrightarrow s_I$ for some $I \in \mathcal{I}$.

Lemma 6.2.4 *If N is a pure submodule of M in C^*-Rat, then N is a pure submodule of M in C^*-Mod.*

Proof. Let N be a pure submodule of M in C^*-*Rat*. To show that N is a pure submodule of M in C^*-*Mod* it suffices to show that, as $\mathfrak{L}(C^*)$-structures, N is a pure substructure of M in $Str(\mathfrak{L}(C^*))$ (by Lemma 0.2.2).
Let $\varphi(\overline{x})$ be a *pp*-formula in $\mathfrak{L}(C^*)_\omega$ (also denoted by ${}_{C^*}\mathfrak{L}$) and let $\overline{a}$ =

$(a_1, \dots, a_n)$ be a tuple in N with $M \models \varphi(\overline{a})$. It suffices to show that $N \models \varphi(\overline{a})$. Say

$$\varphi(\overline{x}) \equiv \exists y_1, \dots, y_k \bigwedge_{i=1}^{m} (\sum_{j=1}^{n} c_{ji}^* x_j + \sum_{l=1}^{k} d_{li}^* y_l = 0)$$

and let $\overline{b} = (b_1, \dots, b_k)$ be a tuple in M with $M \models \psi(\overline{a}, \overline{b})$, where

$$\psi(\overline{x}, \overline{y}) \equiv \bigwedge_{i=1}^{m} (\sum_{j=1}^{n} c_{ji}^* x_j + \sum_{l=1}^{k} d_{li}^* y_l = 0)$$

Put $I_j = ann_{C^*}(a_j)$ $(j = 1, \dots, n)$, $J_l = ann_{C^*}(b_l)$ $(l = 1, \dots, k)$ and $K = \bigcap_{j=1}^{n} I_j \cap \bigcap_{l=1}^{k} J_l$.

Let α_{ji} be the morphism $C^*/K \longrightarrow C^*/I_j$ in $\mathcal{G}$ taking $1 + K$ to c_{ji}^* $(j = 1, \dots, n,\ i = 1, \dots, m)$ and let β_{li} be the morphism $C^*/K \longrightarrow C^*/J_l$ taking $1 + K$ to d_{li}^* $(l = 1, \dots, k,\ i = 1, \dots, m)$.

Then, with our notational convention,

$${}_{\mathcal{G}}M \models \exists z_1, \dots, z_k \bigwedge_{i=1}^{m} (\sum_{j=1}^{n} \alpha_{ji} a_{I_j} + \sum_{l=1}^{k} \beta_{li} z_l = 0)$$

where $s(z_l) = s_{J_l}$ $(l = 1, \dots, k)$.

Since N is a pure submodule of M in C^*-Rat, i.e. ${}_{\mathcal{G}}N$ is a pure substructure of ${}_{\mathcal{G}}M$ in $Mod(T^{rat})$, Lemma 0.2.2 and Proposition 0.2.1 imply that ${}_{\mathcal{G}}N \models \exists z_1, \dots, z_k \overline{\psi}(a_{I_1}, \dots, a_{I_n}, z_1, \dots, z_k)$ where

$$\overline{\psi}(t_1, \dots, t_n, z_1, \dots, z_k) \equiv \bigwedge_{i=1}^{m} (\sum_{j=1}^{n} \alpha_{ji} t_j + \sum_{l=1}^{k} \beta_{li} z_l = 0)$$

with $s(t_j) = s_{I_j}$ $(j = 1, \dots, n)$. Let $f_1, \dots, f_k$ be a tuple in ${}_{\mathcal{G}}N$ with ${}_{\mathcal{G}}N \models \overline{\psi}(a_{I_1}, \dots, a_{I_n}, f_1, \dots, f_k)$.

Then clearly, with $(1 + J_l) f_l = b_l' \in N$ $(j = 1, \dots, k)$, we have that $N \models$

$\psi(a_1,\dots,a_n,b'_1,\dots,b'_k)$, i.e. $N \models \varphi(\overline{a})$. $\square$

We can show that, under an extra condition on the algebra C^*, also the converse of Lemma 6.2.4 is true:

Lemma 6.2.5 *Let C^* be such that every cofinite and closed left C^*-ideal is finitely generated. If N is a submodule of the rational C^*-module M, then N is a pure submodule of M in C^*-Rat iff N is a pure submodule of M in C^*-Mod.*

Proof. $(\Rightarrow)$ This is true in general (see Lemma 6.2.4).
$(\Leftarrow)$ Let N and M be rational C^*-modules and let N be a pure submodule of M in C^*-Mod.
Let $\varphi(\overline{x})$ be the *pp*-formula

$$\exists y_1,\dots,y_k \bigwedge_{i=1}^{m}\Big(\sum_{j=1}^{n}\alpha_{ji}(x_j)+\sum_{l=1}^{k}\beta_{li}(y_l)=0\Big)$$

in $\mathcal{L}(\mathcal{G})_\omega$ and let $\overline{f}=(f_1,\dots,f_n)$ be a tuple in ${}_{\mathcal{G}}N$ with ${}_{\mathcal{G}}M \models \varphi(\overline{f})$. To show that N is a pure submodule of M in C^*-*Rat*, it suffices to show that ${}_{\mathcal{G}}N \models \varphi(\overline{f})$.
Let $\overline{g}=(g_1,\dots,g_k)$ be a tuple in ${}_{\mathcal{G}}M$ with ${}_{\mathcal{G}}M \models \psi(\overline{f},\overline{g})$ where

$$\psi(\overline{x},\overline{y}) \equiv \bigwedge_{i=1}^{m}\Big(\sum_{j=1}^{n}\alpha_{ji}(x_j)+\sum_{l=1}^{k}\beta_{li}(y_l)=0\Big).$$

Let $s(x_j)=s_{I_j}$ $(j=1,\dots,n)$ and $s(y_l)=s_{J_l}$ $(l=1,\dots,k)$.
With $\alpha_{ji}: C^*/K \longrightarrow C^*/I_j$ and $\beta_{li}: C^*/K \longrightarrow C^*/J_l$, let c^*_{ji} be a representative of $(1+K)\alpha_{ji}$, let d^*_{li} be a representative of $(1+K)\beta_{li}$ (for all i,j,l)

and let $\underline{\psi}(\overline{x}, \overline{y})$ be the *pp*-formula

$$\bigwedge_{i=1}^{m}(\sum_{j=1}^{n} c_{ji}^{*}x_j + \sum_{l=1}^{k} d_{li}^{*}y_l = 0)$$

in ${}_{C^*}\mathfrak{L}$. Then, with $a_j = (1 + I_j)f_j$ and $b_l = (1 + J_l)g_l$ (for all j, l), we have that $M \models \underline{\psi}((a_j)_j, (b_l)_l)$. In fact, since $J_l b_l = 0$ for all l, we have that

$$M \models \exists z_1, \dots, z_k(\underline{\psi}((a_j)_j, \overline{z}) \wedge \bigwedge_{l=1}^{k} J_l z_l = 0)$$

so that, since N is a pure submodule of M in C^*-Mod,

$$N \models \exists z_1, \dots, z_k(\underline{\psi}((a_j)_j, \overline{z}) \wedge \bigwedge_{l=1}^{k} J_l z_l = 0)$$

An element of N which is annihilated by J_l can be regarded as an element of $Hom(C^*/J_l, N)$ so that the above implies that ${}_{\mathfrak{g}}N \models \varphi(\overline{f})$, as required. $\square$

Recall that under the condition on C^* in the above lemma, the class $\mathfrak{Rat}$ is not axiomatisable in a finitary language for C^*-modules. The same condition will appear again in the next section where it will enable us to compare $Zg(C^*\text{-}Rat)$ with ${}_{C^*}Zg$.

6.3 Ziegler Spectra over Coalgebras

By the *Ziegler spectrum over a coalgebra* C, we mean the Ziegler spectrum of the category of right C-comodules, i.e. the Ziegler spectrum of the category of rational left C^*-modules.

In this section we compare $Zg(C^*\text{-}Rat)$ with ${}_{C^*}Zg$, under a condition on C^*

which we have met in the previous section, namely: every cofinite and closed left C^*-ideal is finitely generated. We show that, under this condition on C^*, $Zg(C^*\text{-}Rat)$ is embedded in ${}_{C^*}Zg$ (as a topological space).

With the following result

Theorem 6.3.1 (see [13](7.32) and [4](3.5)Theorem 1)
Let $\mathcal{C}$ be a locally finitely presented category with products. An object $A \in \mathcal{C}$ is pure-injective iff for every small index set Γ, the summation map $\oplus_\Gamma A \longrightarrow A$ factors through the pure monomorphism $\oplus_\Gamma A \longrightarrow \prod_\Gamma A$.

and with the fact that the functor ^{rat} commutes with direct sums and products (see Lemmas 6.1.3 and 6.1.1) it is easy to prove that

Proposition 6.3.2 *If M is a C^*-module which is pure-injective in C^*-Mod, then its rational submodule M^{rat} is pure-injective in C^*-Rat.*

It is known that for R-modules M and N, $PE(M \oplus N) = PE(M) \oplus PE(N)$. Note that *if M is a rational C^*-module, then $PE(M) = PE(PE(M)^{rat})$, so in particular, $PE(M)^{rat}$ is a pure submodule of $PE(M)$ in C^*-Mod.* Indeed, M is a pure submodule of $PE(M)^{rat}$ in C^*-Mod and $PE(M)^{rat}$ is a pure submodule of the pure-injective $PE(PE(M)^{rat})$ in C^*-Mod, so that there is a split monomorphism $PE(M) \longrightarrow PE(PE(M)^{rat})$, and therefore there is a pure monomorphism $PE(M)^{rat} \longrightarrow PE(M) \oplus K = PE(PE(M)^{rat})$ in C^*-Mod (with $PE(M)^{rat} \leq PE(M)$) from which it follows that $PE(PE(M)^{rat}) = PE(M)$.

FROM NOW ON WE ASSUME THAT C^* IS SUCH THAT EVERY COFINITE AND CLOSED LEFT C^*-IDEAL IS FINITELY GENERATED.

Lemma 6.3.3 *Let C^* be such that every cofinite and closed left C^*-ideal is finitely generated.*
If M is pure-injective in C^*-Rat, then $PE(M)^{rat} = M$.

Proof. Let M be pure-injective in C^*-*Rat*. Since M is rational, M is a pure submodule of $PE(M)^{rat}$ in C^*-Mod and therefore, by Lemma 6.2.5, in C^*-*Rat*. Since M is pure-injective in C^*-*Rat*, this implies that M is a direct summand of $PE(M)^{rat}$, say $M \oplus K = PE(M)^{rat}$. Thus, $PE(M) \oplus PE(K) = PE(PE(M)^{rat}) = PE(M)$ (by the remark above) so that $K = 0$, i.e. $PE(M)^{rat} = M$. □

Corollary 6.3.4 *Let C^* be such that every cofinite and closed left C^*-ideal is finitely generated.*
If M is a rational C^*-module, then $PE(M)^{rat}$ is the pure-injective hull of M in C^*-Rat.

Proof. Let M be a rational C^*-module. Then M is a pure submodule of $PE(M)^{rat}$ in C^*-Mod and therefore (by Lemma 6.2.5) in C^*-*Rat*. By Lemma 6.3.2, $PE(M)^{rat}$ is pure-injective in C^*-*Rat*.
Let Q be pure-injective in C^*-*Rat* with M purely embedded in Q in C^*-*Rat*. Then M is purely embedded in Q in C^*-Mod so that M is purely embedded in $PE(Q)$ in C^*-Mod. Since $PE(Q)$ is pure injective in C^*-Mod, this implies that there is a split monomorphism $h : PE(M) \longrightarrow PE(Q)$. Then, by Lemma 6.1.3, $h^{rat} : PE(M)^{rat} \longrightarrow PE(Q)^{rat}$ is a split monomorphism in C^*-*Rat*, with $PE(Q)^{rat} = Q$ by Lemma 6.3.3. □

Lemma 6.3.5 *Let C^* be such that every cofinite and closed left C^*-ideal is finitely generated.*
If M is indecomposable and pure-injective in C^*-Rat, then also $PE(M)$ is indecomposable.

Proof. Let M be indecomposable pure-injective in C^*-*Rat*. Suppose $PE(M) = M_1 \oplus M_2$ with $M_1, M_2 \neq 0$. Then, by Lemma 6.3.4 and Lemma 6.1.3, $M = M_1^{rat} \oplus M_2^{rat}$ so that we can assume that $M_1^{rat} = M$ and therefore $M \leq M_1$ from which it follows that $PE(M) = M_1$ and $M_2 = 0$, a contradiction. $\square$

Corollary 6.3.6 *If C^* is such that every cofinite and closed left C^*-ideal is finitely generated, then $Zg(C^*\text{-}Rat)$ is embedded in ${}_{C^*}Zg$ via the function taking a point ${}_{\mathcal{G}}M$ of $Zg(C^*\text{-}Rat)$ to the point $PE(M)$ of ${}_{C^*}Zg$.*

Proof. This follows immediately from Lemma 6.3.3 and Lemma 6.3.5. $\square$

Notation

By ${}_{C^*}Zg^{\mathbf{p}}$ we denote the subspace $\{PE(M) : M \in Zg(C^*\text{-}Rat)\}$ of ${}_{C^*}Zg$.
${}_{C^*}Zg^{\mathbf{0}}$ denotes $\{N \in {}_{C^*}Zg : N^{rat} = 0\}$ and ${}_{C^*}Zg^{\mathbf{np}}$ denotes $\{N \in {}_{C^*}Zg : N^{rat} \neq 0$ and N^{rat} is not a pure submodule of $N\}$.

Lemma 6.3.7 *Let C^* be such that every cofinite and closed left C^*-ideal is finitely generated.*
Then ${}_{C^*}Zg^{\mathbf{p}} = \{N \in {}_{C^*}Zg : N^{rat} \neq 0$ and N^{rat} is a pure submodule of $N\}$.

Proof. If $N \in {}_{C^*}Zg^p$, then $N = PE(M)$ for some $M \in Zg(C^*\text{-}Rat)$ and $N^{rat} = M$ (by Corollary 6.3.4) which is a nonzero and pure submodule of

$PE(M) = N$.

For the converse inclusion, let $N \in {}_{C^*}Zg$ with N^{rat} a nonzero pure submodule of N. Then there is a split embedding $f : PE(N^{rat}) \longrightarrow N$ so that, since $N^{rat} \neq 0$ and N is indecomposable, $N = PE(N^{rat})$. By Proposition 6.3.2 N^{rat} is pure-injective in C^*-Rat so that it remains to show that N^{rat} is indecomposable. If $N^{rat} = N_1 \oplus N_2$, then $N = PE(N^{rat}) = PE(N_1) \oplus PE(N_2)$ so that $N_1 = 0$ or $N_2 = 0$, since N is indecomposable. $\square$

Remark 6.3.8 *Let C^* be such that every cofinite and closed left C^*-ideal is finitely generated.*

Then ${}_{C^*}Zg^{\mathbf{0}}$ is a closed subspace of ${}_{C^*}Zg$.

Proof. Let $N \in {}_{C^*}Zg^{\mathbf{np}} \cup {}_{C^*}Zg^{\mathbf{p}}$. Take $0 \neq a \in N^{rat}$. Then $I_a := ann_{C^*}(a)$ is finitely generated (so that $I_a x = 0$ is a *pp*-formula in $\mathcal{L}(C^*)_\omega$) and $N \in (I_a x = 0/x = 0)$. The result follows from the fact that for every $M \in {}_{C^*}Zg$, $M \in (I_a x = 0/x = 0)$ implies $M^{rat} \neq 0$. $\square$

Notation

The function $Zg(C^*\text{-}Rat) \longrightarrow {}_{C^*}Zg^{\mathbf{p}} : {}_{\mathcal{S}}M \mapsto PE(M)$ will be denoted by $\mathbf{P}$. Its inverse function is $\mathbf{R} : {}_{C^*}Zg^{\mathbf{p}} \longrightarrow Zg(C^*\text{-}Rat) : M \longrightarrow {}_{\mathcal{S}}(M^{rat})$ (see Lemma 6.3.7 and Corollary 6.3.6).

Lemma 6.3.9 *Let C^* be such that every cofinite and closed left C^*-ideal is finitely generated.*

Then the function $\mathbf{R} : {}_{C^*}Zg^{\mathbf{p}} \longrightarrow Zg(C^*\text{-}Rat)$ is continuous.

Proof. Let $(\varphi(x)/\varphi'(x))$ be a nonempty basic open in $Zg(C^*\text{-}Rat)$. Say

$$\varphi(x) \equiv \exists y_1, \dots, y_m \bigwedge_{i=1}^{n} (\alpha_i(x) + \sum_{j=1}^{m} \beta_{ji}(y_j) = 0)$$

where $s(x) = s_I$, $s(y_j) = s_{I_j}$, $\alpha_i : C^*/K \longrightarrow C^*/I$ and $\beta_{ji} : C^*/K \longrightarrow C^*/I_j$ (for some $K \in \mathfrak{I}$ and $j = 1, \dots, m$ and $i = 1, \dots, n$), and

$$\varphi'(x) \equiv \exists y_1, \dots, k \bigwedge_{i=1}^{l} (\alpha'_i(x) + \sum_{j=1}^{k} \beta'_{ji}(y_j) = 0)$$

where $s(x) = I$, $s(y_j) = s_{J_j}$, $\alpha'_i : C^*/K' \longrightarrow C^*/J_j$ and $\beta'_{ji} : C^*/K' \longrightarrow C^*/I$ (for some $K' \in \mathfrak{I}$ and $j = 1, \dots, k$ and $i = 1, \dots, l$).
Put

$$\psi(t) \equiv (It = 0) \wedge \exists z_1, \dots, z_m \{\bigwedge_{i=1}^{n} (c_i^* t + \sum_{j=1}^{m} d_{ji}^* z_j = 0) \wedge \bigwedge_{j=1}^{m} (I_j z_j = 0)\}$$

where (for all i and j) c_i^* is a representative of $(1+K)\alpha_i$ and d_{ji}^* is a representative of $(1+k)\beta_{ji}$, and put

$$\psi'(t) \equiv (It = 0) \wedge \exists z_1, \dots, z_k \{\bigwedge_{i=1}^{l} (c_i'^* t + \sum_{j=1}^{k} d_{ji}'^* z_j = 0) \wedge \bigwedge_{j=1}^{k} (J_j z_j = 0)\}$$

where (for all i and j) $c_i'^*$ is a representative of $(1+K')\alpha'_i$ and $d_{ji}'^*$ is a representative of $(1+K')\beta'_{ji}$.
Then $O := (\psi(t)/\psi'(t)) \cap {}_{C^*}Zg^{\mathbf{P}}$ is open in ${}_{C^*}Zg^{\mathbf{P}}$ and from the fact that for every $M \in_{C^*} Zg^{\mathbf{P}}$ we have that M^{rat} is a pure submodule of M, it clearly follows that $O = \mathbf{R}^{-1}((\varphi(x)/\varphi'(x)) = \{M \in {}_{C^*}Zg^{\mathbf{P}} : \mathfrak{g}(M^{rat}) \in (\varphi(x)/\varphi'(x))\}$.
$\square$

We can also show that $\mathbf{P}$ is continuous. For this we introduce a new signature for left C^*-modules.

First note that, since we assume that every $I \in \mathcal{I}$ is finitely generated, every C^*/I in $\mathcal{G}$ is finitely presented in C^*-Mod. So $\mathcal{G} \subseteq fp(C^*\text{-}Mod)$ from which it follows that $\mathcal{L}(\mathcal{G}) \subseteq \mathcal{L}(fp(C^*\text{-}Mod))$.

Let ${}_{fp}T$ denote the theory in the finitary language $\mathcal{L}(fp(C^*\text{-}Mod))_\omega$ (where the operation symbols are written on the left-hand side of the variables) for which the category $Mod({}_{fp}T)$ is equivalent to C^*-Mod. We identify the $\mathcal{L}(fp(C^*\text{-}Mod))$-structures (which are models of ${}_{fp}T$) with the ${}_{C^*}\mathcal{L}$-structures (which are models of ${}_{C^*}T$). Then for every $M \in Mod({}_{fp}T)$ we have that $M^{rat} = \{x \in |M| : s(x) \in \{s_I : I \in \mathcal{G}\}\}$.

Now, let O be a basic open in ${}_{C^*}Zg^{\mathbf{P}}$. Then $O = {}_{C^*}Zg^{\mathbf{P}} \cap (\varphi/\varphi')$ for some *pp*-formulae (in one free variable) $\varphi/\varphi' \in \mathcal{L}(fp(C^*\text{-}Mod))_\omega$. Since N^{rat} is a pure substructure of N for every indecomposable pure-injective model N of ${}_{fp}T$, it is clear that $O = {}_{C^*}Zg^{\mathbf{P}} \cap (\theta/\theta')$ for some *pp*-formulae (in one free variable) $\theta, \theta' \in \mathcal{L}(\mathcal{G})_\omega$ ($\subseteq \mathcal{L}(fp(C^*\text{-}Mod))_\omega$), and therefore $(\theta/\theta') = \mathbf{P}^{-1}(O)$. We conclude:

Proposition 6.3.10 *Let C^* be such that every cofinite and closed left C^*-ideal is finitely generated.*
Then the topological spaces $Zg(C^*\text{-}Rat)$ and ${}_{C^*}Zg^{\mathbf{P}}$ are homeomorphic.

Note that Lemma 6.3.9 can also be proven by working in the language $\mathcal{L}(fp(C^*\text{-}Mod))_\omega$.

Chapter 7

Rational Modules as Discrete Linearly Topological Modules

We will see how, for a coalgebra C, the rational C^*-modules can be viewed as discrete linearly topological C^*-modules. For certain linearly topological rings R, the techniques and results from the previous chapter can be generalised to relate the Ziegler spectrum of the category of discrete linearly topological (left, resp. right) R-modules with the (left, resp. right) Ziegler spectrum over R.

The functor $^{rat} : C^*\text{-}Mod \longrightarrow C^*\text{-}Rat$ can, of course, be viewed as a subfunctor of the identity functor on $C^*\text{-}Mod$, i.e. as a *preradical* of $C^*\text{-}Mod$ (in the terminology of [30]). Clearly, for every C^*-module M, $(M^{rat})^{rat} = M$, and we say that ^{rat} is an *idempotent* preradical.

A class of objects in an Abelian category is called a *pretorsion class* if it is

closed under quotient objects and direct sums (or coproducts).
So, for example, the class $\mathfrak{Rat}$ is a pretorsion class of C^*-Mod. $\mathfrak{Rat}$ is also closed under subobjects, i.e. $\mathfrak{Rat}$ is a *hereditary* pretorsion class of C^*-Mod. We have noted before that the functor ^{rat} is left exact. This also follows from some general facts about hereditary pretorsion classes (see [30] IV §1).

An additive Abelian group G is called a *topological group* if G is equipped with a topology for which the functions

$$+ : G \times G \longrightarrow G : (a, b) \longmapsto a + b$$

$$- : G \longrightarrow G : a \longmapsto -a$$

are continuous.
A ring R is a *topological ring* if it is equipped with a topology making $(R, +)$ into a topological group and such that the function

$$\times : R \times R \longrightarrow R : (r, s) \longmapsto rs$$

is continuous.
If R is a topological ring, then a module M over R is called a *topological R-module* if it is equipped with a topology making $(M, +)$ into a topological group and such that the function

$$\lambda : R \times M \longrightarrow M : (r, a) \longmapsto ra$$

is continuous.
A topological ring R is said to be *left linearly topological* if there is a fundamental system of neighbourhoods of 0 consisting of left R-ideals. (A system

$\mathcal{N}$ of neighbourhoods of a point in a topological space is called *fundamental* every neighbourhood of that point is contained in a neighbourhood from $\mathcal{N}$.) If R is a linearly topological ring, then a topological R-module is said to be a *linearly topological R-module* in case it has a fundamental system of neighbourhoods of 0 consisting of submodules.

Given a ring R, there is a bijective correspondence between the linear topologies on R and hereditary pretorsion classes of R-modules as follows:
if R is a linearly topological ring, then the class

$$\{M \in R - Mod : ann_R(a) \text{ is open for all } a \in M\}$$

is a hereditary pretorsion class of R-modules and, conversely, if $\mathcal{C}$ is a hereditary pretorsion class of R-modules, then, with

$$\mathcal{F} = \{I : I \text{ is a left } R\text{-ideal with } R/I \in \mathcal{C}\}$$

there is a unique topology on R making R into a linearly topological ring for which $\mathcal{F}$ is a fundamental system of neighbourhoods of 0, called the *$\mathcal{F}$-topology on R* and the set

$$\{O \subset R : \forall r \in O \quad \exists I \in \mathcal{F} \text{ with } I + r \subseteq O\}$$

is the set of the opens in the $\mathcal{F}$-topology on R, and note that $\mathcal{F}$ is the set of the open left R-ideals for this topology (see [30] and [36] for details).

Every module M over a linearly topological ring R can be equipped with a topology making M into a linearly topological R-module. Indeed, if $\mathcal{F}$ denotes the family of the open left ideals of a linearly topological ring R, and if M is an R-module, then there is a unique topology on M making M into a linearly topological R-module and with

$$\mathcal{F}(M) := \{N \subseteq M : (N : a) \in \mathcal{F} \text{ for every } a \in M\}$$

the set of its open submodules. This topology on M is called the *$\mathcal{F}$-topology on M* (see [30] and [36] for details). It is easy to see that the $\mathcal{F}$-topology on a module M over a linearly topological ring R is discrete (in which case we say that M is *$\mathcal{F}$-discrete*) iff for every $a \in M$, $ann_R(a) \in \mathcal{F}$.

So, in particular, C^* is a linearly topological ring with the family of its cofinite and closed left ideals, denoted by $\mathcal{I}$, as a fundamental system of neighbourhoods of 0. And the rational C^*-modules are precisely the C^*-modules for which their $\mathcal{I}$-topology is discrete.
We know that the category of rational C^*-modules is locally finitely presented, locally Noetherian and locally Artinian.
For R a linearly topological ring with $\mathcal{F}$ the family of its open ideals, the category of $\mathcal{F}$-discrete R-modules is denoted by $R\text{-}_{\mathcal{F}}Dis$.

Proposition 7.0.11 ([27](§4))
Let R be a linearly topological ring with $\mathcal{F}$ the family of its open left ideals. Then
(1) The category of $\mathcal{F}$-discrete R-modules, denoted by $R\text{-}_{\mathcal{F}}Dis$, is a Grothendieck Abelian category;
(2) $R\text{-}_{\mathcal{F}}Dis$ is locally finitely presented iff R has a fundamental system $\mathcal{N}$ of neighbourhoods of 0 consisting of left ideals such that for every $I \in \mathcal{N}$ and for every finite set $\{I_1, \dots, I_n\}$ of open left ideals, $Ker(\oplus_{i=1}^{n} R/I_i \longrightarrow R/I)$

is finitely generated

(it suffices to suppose $I_1 = \cdots = I_n \in \mathcal{N}$*);*

(3) R-$_{\mathfrak{F}}$*Dis is locally Artinian (Noetherian) iff for every open left R-ideal* I*,*

R/I *is Artinian (Noetherian)*

Bibliography

[1] F.W. Anderson and K.R. Fuller: *Rings and Categories of Modules*, Graduate Texts in Mathematics, Springer-Verlag (1992; 2nd ed.)

[2] J. Adámek & J. Rosický: *Locally Presentable and Accessible Categories*, LMS Lecture Notes Series 189, Cambridge University Press (1994)

[3] M. Batchelor: *A Decomposition Theorem for Comodules*

[4] W. Crawley-Boevey: *Locally Finitely Presented Additive Categories*, Communications in Algebra, 22(5), 1641-1674 (1994)

[5] P.C. Eklof and I. Herzog: *Model Theory of Modules over a Serial Ring*, Annals of Pure and Applied Logic 72 (1995) 145-176

[6] H.B Enderton: *Elements of Set Theory*, Academic Press, 1977

[7] R.Gordon and J.C. Robson: *Krull Dimension*, Memoirs of the AMS, no. 133, 1973

[8] J.A. Green: *Locally Finite Representations*, Journal of Algebra 41(1), 137-171 (1976)

[9] I. Herzog: *The Auslander-Reiten Translate*, Contemporary Mathematics, 130, 153-165 (1992)

[10] W, Hodges:*A Shorter Model Theory*, Cambridge University Press, 1997

[11] R.G Heyneman and D.E Radford: *Reflexivity and Coalgebras of Finite Type*, Journal of Algebra 28, 215-246 (1974)

[12] J.E. Humphreys: *Linear Algebraic Groups*, Graduate Texts in Mathematics, Springer-Verlag (1975)

[13] C.U. Jensen and H. Lenzing: *Model Theoretic Algebra*, Gordon and Breach, 1989

[14] G. Krause and T.H. Lenagan: *Transfinite Powers of the Jacobson Radical*, Communications in Algebra, 7(1), 1-8 (1979)

[15] S. Montgomery: *Hopf Algebras and Their Actions on Rings*, Regional Conference Series in Mathematics; no 82, AMS, 1992

[16] B.J. Müller and S. Singh: *Uniform Modules over Serial Rings*, Journal of Algebra 144, 94-109 (1991)

[17] N. Popescu: *Abelian Categories with Applications to Rings and Modules*, Academic Press, 1973

[18] M. Prest: *Model Theory and Modules*, LMS Lecture Note Series 130 (1988)

[19] M. Prest: *Some Model-Theoretic Properties of Torsion Theories in Abelian Categories*, Doctoral Thesis, University of Manchester, 1979

[20] M. Prest: *Elementary Torsion Theories and Locally Finitely Presented Categories*, Journal of Pure and Applied Algebra 18, 205-212 (1980)

[21] M. Prest: *The Zariski Spectrum of the Category of Finitely Presented Modules*, preprint

[22] G. Puninski: *Pure-injective Modules over Right Noetherian Serial Rings*, Communications in Algebra, 23(4), 1579-1591 (1995)

[23] G. Puninski: *Superdecomposable Pure-Injective Modules over Commutative Valuation Rings*, Algebra and Logic, 31(6), 377-386 (1992)

[24] G. Puninski: *Cantor-Bendixson Rank of the Ziegler Spectra over a Commutative Valuation Domain*, preprint

[25] G. Puninski: *Serial Rings* (in Russian), manuscript for book, Moscow State Social University

[26] D.E Radford: *Coreflexive Coalgebras*, Journal of Algebra 26, 512-535 (1973)

[27] J.-E. Roos: *Locally Noetherian Categories and Generalised Strictly Compact Rings. Applications*, Lecture Notes in Mathematics 92 (*Category Theory, Homology Theory and their Applications II*, Proceedings), Springer-Verlag, 1968

[28] T. Shudo: *A Note on Coalgebras and Rational Modules*, Hiroshima Journal of Mathematics, 6 (1976), 297-304

[29] S. Singh: *Serial Right Noetherian Rings*, Canadian Journal of Mathematics 36, 22-37 (1984)

[30] B. Stenström: Rings of Quotients (*An Introduction to Methods of Ring Theory*), Springer-Verlag, 1975

[31] M.E. Sweedler: *Hopf Algebras*, W.A. Benjamin, 1969

[32] S. Warner: *Topological Rings*, North-Holland Mathematics Studies 178 (Continuation of the Notas de Matemática), 1993.

[33] R.B. Warfield, Jr: *Serial Rings and Finitely Presented Modules*, Journal of Algebra 37, 187-222 (1975)

[34] R.B. Warfield, Jr: *Large Modules over Artinian Rings*, in: *Representation Theory of Algebras* (Proceedings), Lecture Notes in Pure and Applied Mathematics, 37, 1978

[35] R. Wisbauer; *Foundations of Module and Ring Theory*, Algebra, Logic and Applications Series, Vol.3, Gordon and Breach, Philadelphia (1991)

[36] M. Wright: *Krull Dimension in Serial Rings*, Journal of Algebra 124, 317-328 (1989)

[37] M. Ziegler: *Model Theory of Modules*, Annals of Pure and Applied Logic 26 (1984) 149-213