Equivalence of Some Homological Conditions for Ring Epimorphisms

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Conference in memory of Gena Puninski
Manchester, 6 April 2018

This talk is dedicated to Gena.

Three joint papers

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A. Facchini and G. Puninski, *Σ-pure-injective modules over serial rings,* in "Abelian Groups and Modules", A. Facchini and C. Menini Eds., Kluwer Academic Publishers, Dordrecht, 1995, pp. 145-162.

R. Camps, A. Facchini and G. Puninski, *Serial rings that are endomorphism rings of artinian modules,* in "Rings and radicals", B. J. Gardner, Liu Shaoxue and R. Wiegandt Eds., Pitman Research Notes in Math. Series, Longman, 1996.

A. Facchini and G. Puninski, *Classical localizations in serial rings,* Comm. Algebra 24 (11) (1996), 3537-3559.

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A module M_R over a ring R is *small* if for every family $\{ M_i \mid i \in I \}$ of R-modules and every homomorphism $\varphi \colon M_R \to \oplus_{i \in I} M_i$, there is a finite subset $F \subseteq I$ such that $\varphi(M) \subseteq \oplus_{i \in F} M_i$.

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M_R is *quasismall* if for every family $\{ M_i \mid i \in I \}$ of R-modules such that M_R is isomorphic to a direct summand of $\oplus_{i \in I} M_i$, there is a finite subset $F \subseteq I$ such that M_R is isomorphic to a direct summand of $\oplus_{i \in F} M_i$.

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For both problems: Gena's idea is to use chain rings R (i.e., rings R with both R_R and ${}_RR$ uniserial modules) which are nearly simple (are not Artinian and have only three two-sided ideals: 0, R and the Jacobson radical $J(R)$).

The theory (uniserial modules, direct summands of serial modules, non-quasismall uniserial modules,...) was further greatly developed in the following years by Pavel Příhoda.

Dubrovin-Puninski ring

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If R is a nearly simple chain domain, Gena proved that all modules R/rR ($r \in J(R)$, $r \neq 0$) are isomorphic. A *Dubrovin-Puninski ring* is a ring of the form $\mathrm{End}(R/rR)$.

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Joint work with Zahra Nazemian

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[F. and Nazemian, *Equivalence of Some Homological Conditions for Ring Epimorphisms*, submitted for publication, available in arXiv]

Fuchs and Salce

[L. Fuchs and L. Salce, *Almost perfect commutative rings*, J. Pure Appl. Algebra, Available online 9 March 2018.]

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Equivalence of nine conditions for modules over commutative rings R with perfect ring of quotients Q.

Fuchs and Salce, 2017

Theorem. If R is an order in a commutative perfect ring Q, then the following conditions are equivalent:
(i) R is an almost perfect ring (i.e., R/Rr is a perfect ring for every non-zero-divisors $r \in R$).
(ii) Flat R-modules are strongly flat.
(iii) Matlis-cotorsion R-modules are Enochs-cotorsion.
(iv) R-modules of w.d.≤ 1 are of p.d.≤ 1.
(v) The cotorsion pairs $(\mathcal{P}_1, \mathcal{D})$ and $(\mathcal{F}_1, \mathcal{WI})$ are equal ($\mathcal{P}_1$ R-modules of projective dimension ≤ 1 and $\mathcal{F}_1$ R-modules of weak dimension ≤ 1).
(vi) Divisible R-modules are weak-injective.
(vii) h-divisible R-modules are weak-injective.
($viii$) Homomorphic images of weak-injective R-modules are weak-injective.
(ix) R is h-local and Q/R is semi-artinian.

In our paper, we prove that seven of these nine conditions are equivalent for non-commutative rings, imposing a "hierarchy" of four sets of more and more strict conditions on the extension of rings $R \subseteq Q$.

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Set $K := Q/\varphi(R)$. The mapping φ is injective, and is a ring morphism, so that R can be viewed as a subring of Q via φ. We will always identify via φ the isomorphic rings R and $\varphi(R)$, so that φ will be always seen as an inclusion.

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We will denote by $t(M_R)$ the torsion submodule of any right R-module M_R in this torsion theory.

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A right R-module M_R is a right Q-module M_Q if and only if

$$\mathrm{Ext}^1_R(K_R, M_R) = 0 \quad \text{and} \quad \mathrm{Hom}(K_R, M_R) = 0$$

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As a consequence, if a right R-module M_R is a right Q-module M_Q, then its unique right Q-module structure is given by the canonical isomorphism $\mathrm{Hom}(Q_R, M_R) \to M_R$.

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The inclusion $R \hookrightarrow Q$ is an epimorphism $\Leftrightarrow$ the R-R-bimodule $Q \otimes_R Q$ is isomorphic to the R-R-bimodule Q via the canonical isomorphism induced by multiplication $\cdot\colon Q \times Q \to Q$ in the ring Q $\Leftrightarrow$ $Q \otimes_R K = 0$.

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Every right Q-module is a torsion-free R-module.

First set of conditions on the extension $R \subseteq Q$

Lemma

The following conditions are equivalent for a right R-module N_R:
(1) *Every homomorphism $R_R \to N_R$ extends to a right R-module morphism $Q \to N_R$.*
(2) *N_R is a homomorphic image of a right Q-module.*
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We say that a right R-module is *h-divisible* if it satisfies the equivalent conditions of this Lemma. Any direct sum of h-divisible right R-modules is h-divisible, homomorphic images of h-divisible modules are h-divisible, injective modules are h-divisible, and any right R-module B_R contains a unique largest h-divisible submodule $h(B_R)$ that contains every h-divisible submodule of B_R. We will say that B_R is *h-reduced* if $h(B_R) = 0$ (equivalently, if B_R has no nonzero h-divisible submodule, equivalently if $\mathrm{Hom}(Q_R, B_R) = 0$).

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Theorem

The right R-module $\mathrm{Hom}({}_R K_R, M_R)$ is torsion-free Matlis-cotorsion h-reduced for every right R-module M_R.

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$\varphi\colon R \to Q$ is the canonical homomorphism of R into its right localization $R_{\mathcal{F}}$, where $\mathcal{F} = \{\, I \mid I$ is a right ideal of R and $\varphi(I)Q = Q\,\}$ is a Gabriel topology consisting of dense right ideals.

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For every right R-module M_R, the kernel of the canonical right R-module morphism $M_R \to M \otimes_R Q$ is the torsion submodule $t(M_R)$ of M_R.

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For every right R-module M_R, the kernel of the canonical right R-module morphism $M_R \to M \otimes_R Q$ is the torsion submodule $t(M_R)$ of M_R.

The torsion submodule $t(M_R)$ of any right R-module M_R is isomorphic to $\mathrm{Tor}_1^R(M_R, {}_RK)$.

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A left R-module ${}_R D$ is *divisible* if $D = ID$ for every $I \in \mathcal{F}$ (equivalently, if $M \otimes_R D = 0$ for every torsion right R-module M_R).

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h-divisible left R-modules are divisible.

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Theorem

(1) *For every right R-module M_R, there is a short exact sequence of right R-modules*

$$0 \longrightarrow M_R/t(M_R) \longrightarrow M \otimes_R Q \longrightarrow M \otimes_R K \longrightarrow 0.$$

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$$0 \longrightarrow {}_RB/h({}_RB) \longrightarrow \mathrm{Ext}^1_R({}_RK_R, {}_RB) \longrightarrow \mathrm{Ext}^1_R({}_RQ_R, {}_RB) \longrightarrow 0.$$

Second set of conditions on the extension $R \subseteq Q$

Corollary

For every torsion right R-module M_R, the canonical mapping

$$\pi\colon \mathrm{Hom}({}_RK_R, M_R) \otimes_R K \to h(M_R)$$

defined by $\pi(f \otimes x) = f(x)$ for every $f \in \mathrm{Hom}(K_R, M_R)$ and $x \in K$ is a right R-module epimorphism.

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Second set of conditions on the extension $R \subseteq Q$

Notice that the concept of Enochs-cotorsion left R-module is an "absolute concept", in the sense that it depends only on the ring R, while the concept of Matlis-cotorsion left R-module is a "relative concept", in the sense that it also depends on the choice of the overring Q of R with ${}_RQ$ flat.

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Lemma

Weak-injective left R-modules are h-divisible and Matlis-cotorsion.

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(2) *If* $s \in S$ *and* $a \in R$*, then there are* $t \in S$ *and* $b \in R$ *such that* $sb = at$ *(i.e.,* S *satisfies the right Ore condition).*

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Thus, we now suppose that R is a ring and S is a multiplicatively closed subset of R satisfying:
(1) *If $a, b \in R$ and $ab \in S$, then $a \in S$.*
(2) *If $s \in S$ and $a \in R$, then there are $t \in S$ and $b \in R$ such that $sb = at$ (i.e., S satisfies the right Ore condition).*
(3) *The elements of S are not right zero-divisors.*

Third set of conditions: $\mathcal{F}$ is a 1-topology

We have already said that the Gabriel topology $\mathcal{F}$ always has a basis consisting of finitely generated right ideals. Now we will suppose that the Gabriel topology $\mathcal{F}$ is a 1-*topology*, that is, that $\mathcal{F}$ has a basis consisting of principal right ideals. Thus $\mathcal{F}$ is completely determined by the set $S := \{ s \in R \mid sR \in \mathcal{F} \}$.

Thus, we now suppose that R is a ring and S is a multiplicatively closed subset of R satisfying:
(1) *If $a, b \in R$ and $ab \in S$, then $a \in S$.*
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(3) *The elements of S are not right zero-divisors.*
(4) *$Q := R_{\mathcal{F}}$ is directly finite.*

Third set of conditions: $\mathcal{F}$ is a 1-topology

Lemma

Let $\mathcal{F}$ be the Gabriel topology consisting of all right ideals I of R such that $I \cap S \neq 0$, let $R_{\mathcal{F}}$ be the localization and $\varphi\colon R \to R_{\mathcal{F}}$ be the canonical mapping. Then $\mathcal{F}$ consists of dense right ideals, φ is a bimorphism and ${}_R R_{\mathcal{F}}$ is a flat left R-module.

Third set of conditions: $\mathcal{F}$ is a 1-topology

Proposition

For every torsion right R-module M_R, the canonical mapping

$$\pi\colon \mathrm{Hom}({}_RK_R, M_R) \otimes_R K \to h(M_R),$$

defined by $\pi(f \otimes x) = f(x)$ for every $f \in \mathrm{Hom}(K_R, M_R)$, is a right R-module isomorphism.

Third set of conditions: $\mathcal{F}$ is a 1-topology

Theorem

Let M_R be an h-reduced torsion-free right R-module. Then the canonical mapping $\lambda\colon M_R \to \mathrm{Hom}(K_R, M \otimes_R K)$ is injective and its cokernel is isomorphic to $\mathrm{Ext}^1_R(Q_R, M_R)$.

Third set of conditions: $\mathcal{F}$ is a 1-topology

Corollary

Let M_R be an h-reduced torsion-free Matlis-cotorsion right R-module. Then the canonical mapping

$$\lambda\colon M_R \to \mathrm{Hom}(K_R, M \otimes_R K)$$

is an isomorphism.

Third set of conditions: $\mathcal{F}$ is a 1-topology

In our setting we have the Matlis category equivalence:

Theorem

There is an equivalence of the category $\mathcal{C}$ of h-reduced torsion-free Matlis-cotorsion right R-modules with the category $\mathcal{T}$ of h-divisible torsion right R-modules, given by

$$- \otimes_R K \colon \mathcal{C} \to \mathcal{T} \qquad \textit{and} \qquad \mathrm{Hom}(K_R, -) \colon \mathcal{T} \to \mathcal{C}.$$

Fourth set of conditions: left and right flat bimorphisms, and 1-topologies

Now we assume that we have a *a ring R for which the set S of all its regular elements is both a right denominator set and a left denominator set (right and left Ore ring), $Q = R[S^{-1}] = [S^{-1}]R$ (the classical right and left ring of quotients of R) and* $\mathrm{F.dim}(Q_Q) = 0$.

Fourth set of conditions: left and right flat bimorphisms, and 1-topologies

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Related to [L. Angeleri, D. Herbera, J. Trlifaj, *Divisible modules and localization*, J. Algebra 294 (2005), 519–551].

$\operatorname{F.dim}(Q_Q) = 0$

The big finitistic dimension is defined by

$$\operatorname{F.dim}(Q_Q) := \sup\{\, \operatorname{p.d.}(M_Q) \mid M_Q \text{ any right } Q\text{-module} \\ \text{with } \operatorname{p.d.}(M_Q) < \infty \,\}.$$

Thus, for a ring Q, $\operatorname{F.dim}(Q_Q) = 0$ means that every right R-module has projective dimension 0 or ∞.

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Thus, for a ring Q, $\mathrm{F.dim}(Q_Q) = 0$ means that every right R-module has projective dimension 0 or ∞.

A commutative ring Q is perfect if and only if $\mathrm{F.dim}(Q_Q) = 0$.

For an arbitrary, not-necessarily commutative, ring Q, $\mathrm{F.dim}(Q_Q) = 0$ if and only if Q is right perfect and every simple right Q-module is a homomorphic image of an injective module (Bass).

Fourth set of conditions: left and right flat bimorphisms, and 1-topologies

Theorem

Assume that R is a right and left Ore ring and $\mathrm{F.dim}(Q_Q) = 0$, where $Q = R[S^{-1}] = [S^{-1}]R$. Then the following conditions are equivalent:

(i) Flat right R-modules are strongly flat.

(ii) Matlis-cotorsion right R-modules are Enochs-cotorsion.

(iii) h-divisible right R-modules are weak-injective.

(iv) Homomorphic images of weak-injective right R-modules are weak-injective.

(v) Homomorphic images of injective right R-modules are weak-injective.

(vi) Right R-modules of $\mathrm{w.d.} \leq 1$ are of $\mathrm{p.d.} \leq 1$.

(vii) The cotorsion pairs $(\mathcal{P}_1, \mathcal{D})$ and $(\mathcal{F}_1, \mathcal{WI})$ coincide.

(viii) Divisible right R-modules are weak-injective.