Math 42041/62041 Noncommutative Algebra—Exercises 4

I will go through parts of this on Tuesday 13th November.

1) [Left over from last week.] Let $\theta : M \to M$ be an endomorphism of a left R-module M. If M is noetherian and θ is surjective, prove that θ is an isomorphism. [Hint: Consider the kernels $Ker(\theta^n)$.]

2) (a) Read up the course notes from Corollary 3.9 through to the "simplifying trick" before Exercise 3.11.
(b) Use the ideas there to prove that the following rings are left noetherian. Which if any is left Artinian?

$$R_1 = \begin{pmatrix} A_1 & A_1 \\ 0 & A_1 \end{pmatrix} \qquad R_2 = \begin{pmatrix} \mathbb{Q}[x,y] & \mathbb{Q}[x,y] \\ (x,y) & \mathbb{Q}[x,y] \end{pmatrix} \qquad R_3 = R_2/I \quad \text{for } I = \begin{pmatrix} (x,y)^3 & (x,y)^3 \\ (x,y)^3 & (x,y)^3 \end{pmatrix}.$$

You may assume they indeed are all rings. (This amounts to checking that R_1 and R_2 are closed under multiplication and that I is an ideal of R_2. The last assertion is easy—I is obviously an ideal of the bigger ring $M_2(\mathbb{Q}[x,y])$, so is also an ideal of R.)

3) (i) Prove Schur's Lemma: *If M is a simple left module over a ring R then $End_R(M)$ is a division ring.*
(ii) Find an example of a ring R and a simple left R-module M where $\mathrm{End}_R(M)$ is not a field (i.e. is not commutative).

4) Suppose that R is a left artinian domain. Prove that R is a field. [We saw in class that if R is left *and* right artinian then every non-zero element has a multiplicative inverse; so the question is to show that left artinian is enough.]

5) Let N_1 and N_2 be submodules of a left R-module M. If M/N_1 and M/N_2 are noetherian, prove that $M/(N_1 \cap N_2)$ is also noetherian.

6) Prove Theorem 3.2, concerning equivalent conditions for a module to be artinian. [You could also, for practice/to test your understanding, do this with, for example, 3.6, which was done just for the noetherian, but not the artinian, condition.]

7) Let I be an ideal of a ring R and M a left R-module.
(i) Prove that M/IM is a left R/I-module under the natural action $[r+I] \cdot [m+IM] = [rm+IM]$, for $r \in R$ and $m \in M$.
(ii) Show that each R-submodule of M/IM is an R/I-submodule and conversely that each R/I-submodule of M/IM is an R-submodule.