Math 42041/62041 Noncommutative Algebra—Exercises 2

I will discuss these questions on Tuesday 23rd October.

1. Let $\{N_i : i \in I\}$ be submodules of a left R-module M. Prove that $\sum_{i\in I} N_i$ and $\bigcap_{i\in I} N_i$ are submodules of M.

2. Let M be a left module over a ring R and for a subset $S \subseteq M$ define $ann_R S = \{r \in R : rs = 0 \text{ for all } s \in S\}$.
(i) Prove that $ann_R S$ is always a left ideal of R.
(ii) Prove that $ann_R M$ is an ideal of R.
(iii) Show that, if $\mathbb{C}^2$ is regarded as a left $M_2(\mathbb{C})$-module in the usual way and $0 \neq m \in \mathbb{C}^2$ then $ann_R(m)$ is not an ideal of $R = M_2(\mathbb{C})$.
(iv) [Harder] Now assume that $M = R/I$ is a cyclic left R-module. Prove that $ann_R M$ is the (unique) largest ideal contained inside the left ideal I.

3. Suppose that R is a k-algebra for some field k. Since $k \subseteq Z(R)$, any R-module M is also a k-vector space. This module M is called finite dimensional if it is finite dimensional when considered as a k-vector space.

(1) Suppose that I and J are left ideals of R such that R/I and R/J are finite dimensional. Prove that $R/I \cap J$ is also finite dimensional.

(2) Suppose that there exists a non-zero, finite dimensional left R-module M; thus M is finite dimensional as a k-vector space with basis, say $\{x_1, ..., x_n\}$. Show that $ann_R(M) = \bigcap_{i=1}^n ann(x_i)$.

Hence prove that R has a two-sided ideal $I \neq R$ such that R/I is finite dimensional as a k-vector space.

Comment: Note that this shows in particular that there are no (non-zero) finite dimensional modules over the Weyl algebra $A_1(\mathbb{C})$.

4. (a) Let M be a left R-module. Recall that the set of all R-module endomorphisms of M is written $\mathrm{End}_R(M)$. How do you define the sum and product of such endomorphisms? Prove that $\mathrm{End}_R(M)$ is a ring under composition of functions.
(b) Prove that M is naturally a right module over $\mathrm{End}_R(M)$.
[This question is best answered by writing endomorphisms on the right ie write m^θ rather than $\theta(m)$. Also, the word "obvious" should be liberally applied!]

5. (a) The aim of this question is to prove that (as rings) $\mathrm{End}_R(R) \cong R$. Here we regard R as a *left* R-module.
First, prove that for any $r \in R$ there is a left R-module homomorphism $\theta_r : R \to R$ defined by $s \mapsto sr$. Now show that these are the only endomorphisms and then that the ring of endomorphisms is isomorphic to R.

Here it is important that you write endomorphisms on the *right*, so use the notation from the last question.

(b) What goes wrong if I write endomorphisms on the left? (See question 6!)

(c) Now regard R as a right R-module and write endomorphisms on the left, then again prove that $R \cong \mathrm{End}_R(R_R)$ as rings.

6. The problem in Question 5(b) is that you will find that the multiplication in $End_R(R)$ and R are not quite the same—indeed one is the "opposite ring" of the other.

This question explores this concept. Let $R = (R, +, \cdot)$ be a ring. We will define a new ring $R^{op} = (R, +, \circ)$. This notation mean that the new ring is the same set as R and has the same addition as R (thus they are the same as abelian groups) but it has a new multiplication which will be defined as $a \circ b = ba$ for $a, b \in R$.

(a) Prove that $R^{op} = (R, +, \circ)$ is indeed a ring.

(b) Let I be a left ideal of R. Prove that the set I is a *right* ideal of R^{op}.

(c) If M is a left R-module prove that M is a right R^{op}-module under the action $m * r = rm$.

(d) Prove that $M_n(R)$ (for R a field, or indeed any commutative ring) satisfies $M_n(R)^{op} \cong M_n(R)$ under $A \mapsto A^T$; the transpose of A.

Comment: (i) This question gives a formal way of saying "Whenever we prove a result for left modules it also works for right modules." This is useful because sometimes one side is more natural than the other. In particular I will tend to work with left modules, but in generalizations of Question 5 it is really is more natural to write endomorphism on the left and hence to work with right modules.

(ii) In fact "most" rings are isomorphic to their opposites; for example it is true for the Weyl algebra under the map $x \mapsto -x$ and $\partial \mapsto \partial$. Since this not so natural to prove with the way we define the Weyl algebra, I will skip the proof. Rings that cannot be isomorphic to their opposites will appear incidentally in the next chapter.

7. Complete the proof of Example 2.9 from the notes. [This is really only here to force me to write out the solution, as promised in the notes; it is not a result you need to remember!]