6 An example: dense linear orders

Take $\mathcal{L}$ to be $\mathcal{L}_0$ augmented by a binary relation symbol, which we will write as "$\leq$". Let T_{dlo} be the theory of densely linearly ordered sets without endpoints:
(poset) $\forall x\,(x \leq x)$; $\quad \forall x, y, z\,(x \leq y \wedge y \leq z \rightarrow x \leq z)$;
$\forall x, y\,(x \leq y \wedge y \leq x \rightarrow x = y)$
(linear) $\forall x, y\,(x \leq y \vee y \leq x)$
(densely ordered) $\forall x, y\,(x < y \rightarrow \exists z(x < z \wedge z < y))$
(w/o endpoints) $\forall x \exists y \exists z\,(x < y \wedge z < x)$
where we use the symbol "$<$" as an abbreviation (so "$x < y$ stands for "$x \leq y \wedge \neg(x = y)$); similarly for $>$ and $\geq$.
For instance, $(\mathbb{Q}; \leq)$ is a model of T_{dlo}, as is $(\mathbb{R}; \leq)$.

Theorem 6.1. *Let $\mathcal{M}, \mathcal{N} \models T_{\mathrm{dlo}}$ be countable and let $a_1 < \cdots < a_k$ be elements of M and $b_1 < \cdots < b_k$ be elements of N. Then there is an isomorphism $\alpha : \mathcal{M} \longrightarrow \mathcal{N}$ taking a_i to b_i $(i = 1, \ldots, k)$.*

Proof. The proof is a back-and-forth argument - see Section [5.2]. It is a slight elaboration of that argument in that we start with part of the isomorphism, let's call it α, already determined (namely a_i has to be sent to b_i).

We enumerate M as $a_1, \ldots, a_k, m_1, \ldots, m_t, \ldots$ and we also enumerate N as $b_1, \ldots, b_k, n_1, \ldots, n_t, \ldots$ We build up the isomorphism, starting by sending each a_i to b_i. Then we have to decide where to send m_1: so we look for an element - which we will set to be $\alpha(m_1)$ - which bears the same relation to $b_1, \ldots, b_k$ that m_1 does to $a_1, \ldots, a_k$ (we can find such an element because the ordering is dense and because there are no end-points). We move on to m_2 - we find an element which bears the same relation to $b_1, \ldots, b_k, \alpha(m_1)$ that m_2 does to $a_1, \ldots, a_k, m_1$ and set this to be $\alpha(m_2)$, etc. If we continue in this way we build up an order-preserving injection from $\mathcal{M}$ to $\mathcal{N}$: but we want an isomorphism and there's no reason why n_1, say, should be in the image of α - that is just a "forth" construction. For a back-and-forth we modify the construction as follows: at odd-numbered stages we follow the above; at even-numbered stages we reverse the roles of $\mathcal{M}$ and $\mathcal{N}$ (so at stage 2 we will find an element which bears the same relation to $a_1, \ldots, a_k, m_1$ that n_1 does to $b_1, \ldots, b_k, \alpha(m_1)$ and set this to be $\alpha^{-1}(n_1), etc.$).

We end up with an isomorphism as described. $\square$

Corollary 6.2. *If $\mathcal{M}$, $\mathcal{N}$ are countable models of T_{dlo} then $\mathcal{M} \simeq \mathcal{N}$.*

Proof. Take $\bar{a} = \emptyset = \bar{b}$ in 6.1. $\square$

Corollary 6.3. *If $\mathcal{M}$ is a countable densely linearly ordered set without endpoints then $\mathcal{M} \simeq (\mathbb{Q}, \leq)$.*

Definition 6.4. *A theory T is κ-***categorical** *(κ a cardinal) if it has, up to isomorphism, just one model of cardinality κ.*

Corollary 6.5. *The theory of densely linearly ordered sets without endpoints is $\aleph_0$-categorical and hence complete.*

Proof. Suppose that $\mathcal{M}, \mathcal{N}$ are models of $T_{\rm dlo}$. Since $T_{\rm dlo}$ has no finite model, these are infinite and so, by downwards Löwenheim-Skolem ([5.1]), there are countable $\mathcal{M}'$, $\mathcal{N}'$ with $\mathcal{M}' \prec \mathcal{M}$ and $\mathcal{N}' \prec \mathcal{N}$. By 6.2, $\mathcal{M}' \simeq \mathcal{N}'$ and hence ([3.14]) $\mathcal{M}' \equiv \mathcal{N}'$. So $\mathcal{M} \equiv \mathcal{N}$. $\square$

Exercise 6.6. Show that if T is a theory in a countable language $\mathcal{L}$, has no finite model, and is $\aleph_0$-categorical, then T is complete.

Exercise 6.7. Show that $T_{\rm dlo}$ is not κ-categorical where κ is the cardinal of the continuum.

Proposition 6.8. *Let $\mathcal{M} \models T_{\rm dlo}$ and let $\bar{a} = (a_1, \dots, a_n)$ and $\bar{b} = (b_1, \dots, b_n)$ be tuples from M with the same order-type in $\mathcal{M}$ - that is, $a_i < a_j$ iff $b_i < b_j$ for all i, j.*

Then for all $\phi(\bar{x}) \in \mathcal{L}$ we have $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \models \phi(\bar{b})$.

Proof. By Downwards Löwenheim-Skolem we can replace $\mathcal{M}$ by a countable elementary substructure still containing $\bar{a}$ and $\bar{b}$, so wlog $\mathcal{M}$ is countable. Renumbering if necessary we may assume that we are in the situation of the hypothesis of Proposition 6.1 and so we may conclude that there is an automorphism of $\mathcal{M}$ taking $\bar{a}$ to $\bar{b}$. An appeal to [3.13] finishes the proof. $\square$

Definition 6.9. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $\bar{a}$ a tuple from M. The* **type** *of $\bar{a}$ in $\mathcal{M}$ is the set of formulas (in some fixed tuple, $\overline{x}$, of free variables) satisfied by $\bar{a}$ in $\mathcal{M}$:*

$$\mathrm{tp}^{\mathcal{M}}(\bar{a}) = \{\phi(\bar{x}) \in \mathcal{L} : \mathcal{M} \models \phi(\bar{a})\}.$$

Corollary 6.10. *Let $\mathcal{M} \models T_{\rm dlo}$ and let $\bar{a}$ be in M. Then the type of $\bar{a}$ in $\mathcal{M}$ is completely determined by the order-type of $\bar{a}$.*

Definition 6.11. *A complete $\mathcal{L}$-theory T has* **elimination of quantifiers** *if, for every formula $\phi(\overline{x})$ in $\mathcal{L}$ there is a quantifier-free formula $\theta(\overline{x})$ in $\mathcal{L}$ such that ϕ is equivalent to θ in every model of T; that is,*

$\mathcal{M} \models T$ *implies* $\phi(\mathcal{M}) = \theta(\mathcal{M})$,

equivalently,

$\mathcal{M} \models T$ *implies* $M \models \forall\overline{x}\, (\phi(\overline{x}) \leftrightarrow \theta(\overline{x}))$.

Corollary 6.12. *$T_{\rm dlo}$ has elimination of quantifiers.*

Proof. Given $\phi(\bar{x}) \in L$ we must show that there is a quantifier-free formula $\theta(\bar{x})$ equivalent to ϕ modulo $T_{\rm dlo}$. By 6.10, whether or not a tuple $\bar{a}$ satisfies ϕ is determined by the order-type of $\bar{a}$. The order-type τ of a tuple can be described by a quantifier-free formula, ρ_τ say, so we let θ be the disjunction of the ρ_τ such that some (hence by 6.8 every) n-tuple (where $n = l(\bar{x})$) of order-type τ satisfies ϕ. $\square$