The Functor Category*

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1 Lecture 1: Projective Functors

1.1 Module categories and functors

Let $\mathcal{A}$ be a **skeletally small preadditive category**.

preadditive means that, for each pair, A, B, of objects of $\mathcal{A}$, the set, $\mathcal{A}(A, B)$, we write just (A, B), of morphisms from A to B is equipped with an abelian group structure such that composition is bilinear: $f(g+h) = fg+fh$ and $(g+h)f = gf+hf$ when these are defined.

*These notes have been extracted and modified from [28].

A (skeletally) small category is one where the collection of (isomorphism classes of) objects is a set.

For example, if R is a ring then consider the category which has just one object, $*$, and which has $(*, *) = R$ with the addition on R giving the abelian group structure on $(*, *)$ and the multiplication on R giving the composition (we'll use the convention for composition of morphisms that rs means do s then r). This allows us to regard the ring R as a small preadditive category with just one object.

More generally if $\mathcal{A}$ has only finitely many objects, $A_1, \ldots, A_n$, set $R = \bigoplus_{i,j}(A_i, A_j)$. Define addition and multiplication (i.e. composition) on R pointwise, with the convention that if the domain of f is not equal to the codomain of g, where $f, g \in \bigcup_{i,j}(A_i, A_j)$, then the product fg is zero. It is easy to check that R is a ring and that $1 = e_1 + \cdots + e_n$, where e_i is the identity map of A_i, is a decomposition of $1 \in R$ into a sum of orthogonal idempotents. This ring R codes up almost all the information contained in the category $\mathcal{A}$ (the objects A_i might not be recoverable from R, but that's not a big deal).

Example 1.1. Let A_∞ be the quiver

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \ldots$$

Let k be any field and let $\mathcal{A}$ be the **k-path category** of this quiver. That is, the objects of $\mathcal{A}$ are the vertices of the quiver and the arrows of $\mathcal{A}$ are freely generated as a k-category, that is under composition and forming well-defined k-linear combinations, by the arrows of the quiver. Then $\dim_k(i, j) = 1$ if $i \leq j$ and $(i, j) = 0$ if $i > j$. This is a preadditive category.

We consider only **additive** functors between preadditive categories; that is functors F which satisfy $F(f+g) = F(f)+F(g)$ whenever $f+g$ is defined. **Our convention throughout is that "functor" means additive functor.** A **left $\mathcal{A}$-module** is a functor from $\mathcal{A}$ to the category, **Ab**, of abelian groups. For instance in the one-object $(*)$ case, obtained from a ring $R = \mathrm{End}(*)$, a functor from $\mathcal{A}$ to **Ab** is determined by the image of $*$, an abelian group - let us denote it by M - together with, for each $r \in R = (*, *)$, an endomorphism of this abelian group M. If we denote the action of this endomorphism as $m \mapsto rm$ then, by functoriality, $(sr)m = s(rm)$. Thus a functor from this category, which we may as well write as R, to **Ab** is a left R-module. It is easy to see that, conversely, every left R-module gives rise to a functor from this one-point category to **Ab** and that, furthermore, the natural transformations between functors are exactly the R-linear maps between modules. Therefore, $(R, \mathbf{Ab}) \simeq R\text{-Mod}$ where the latter is our notation for the category of left R-modules (we will use R-mod for the category of finitely presented modules).

A natural transformation τ from the functor F to the functor G (where $FG : \mathcal{C} \to \mathcal{D}$) is given by, for each object $C \in \mathcal{C}$, a morphism $\tau_C : GC \to FC$ such that these all cohere in the sense that if $C \xrightarrow{f} C'$ is a morphism of $\mathcal{C}$ then the diagram below commutes.

$$\begin{tikzcd} FC \arrow[r, "\tau_C"] \arrow[d, "Ff"'] & GC \arrow[d, "Gf"] \\ FC' \arrow[r, "\tau_{C'}"'] & GC' \end{tikzcd}$$

In the general case, where $\mathcal{A}$ is a small preadditive category, we may, therefore, write $\mathcal{A}$-Mod for $(\mathcal{A}, \mathbf{Ab})$, especially since this reminds us that functors are just generalised modules. For example, if R is a ring we could write (mod-R)-Mod instead of (mod-R, $\mathbf{Ab}$) (but we won't).

Theorem 1.2. *If $\mathcal{A}$ is a skeletally small preadditive category then the functor category $\mathcal{A}$-Mod $= (\mathcal{A}, \mathbf{Ab})$ is abelian, indeed Grothendieck. There is, moreover, a generating set of finitely generated projective objects, which we describe at 1.9.*

An abelian category is a preadditive category which has finite direct sums and a zero object, such that every morphism has a kernel and every monomorphism is a kernel and, dually every morphism has a cokernel and every epimorphism is a cokernel.

A Grothendieck category is an abelian category which has coproducts, in which direct limits are exact and which has a generator.

A directed system is a collection of objects, C_i with $i \in I$ where I is an ordered set, and morphisms f_{ij}, where $i, j \in I$ with $i < j$, such that $f_{jk} f_{ij} = f_{ik}$ for all $i < j < k$ and such that for all $i, j \in I$ there is $k \in I$ with $i, j < k$.

The direct limit (also called directed colimit) of a directed system $\big((C_i)_i, (f_{ij} : C_i \to C_j)_{i<j}\big)$ is an object C and morphisms $f_{i\infty} : C_i \to C$ with $i \in I$, such that $f_{i\infty} = f_{j\infty} f_{ij}$ for all $i < j$ and which is universal in the sense that if we also have an object D and system $(g_{i\infty} : C_i \to D)_{i \in I}$ of morphisms such that $g_{i\infty} = g_{j\infty} f_{ij}$ for all $i < j$, then there is a unique morphism $h : C \to D$ such that $g_{i\infty} = h f_{i\infty}$ for all i.

Direct limits being exact means that if we have a directed system of exact sequences $0 \to A_\lambda \xrightarrow{f_\lambda} B_\lambda \xrightarrow{g_\lambda} C_\lambda \to 0$ then the direct limit sequence $0 \to \varinjlim A_\lambda \xrightarrow{\varinjlim f_\lambda} \varinjlim B_\lambda \xrightarrow{\varinjlim g_\lambda} \varinjlim C_\lambda \to 0$ is exact.

A generating set for a preadditive category $\mathcal{C}$ is a set $\mathcal{G}$ of objects such for every nonzero morphism $f : A \to B$ in $\mathcal{C}$ there is $G \in \mathcal{G}$ and a morphism $g : G \to A$ such that $fg \neq 0$. If $\mathcal{G} = \{G\}$ then G is a **generator**.

Every Grothendieck category is a localisation of a module category (this is the Gabriel-Popescu Theorem).

One may check that the category, Mod-R, of *right* R-modules is equivalent to the category of *contravariant* functors to $\mathbf{Ab}$ from the one-point category (corresponding to) R. More generally, and bearing in mind that contravariant functors from a category are covariant functors from its opposite, we may write (in the general case, replacing R by $\mathcal{A}$) Mod-$\mathcal{A}$ for $(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$.

In a case such as Example 1.1, where the morphism sets are k-vectorspaces and composition is k-bilinear, it is natural to consider, instead of the category of additive functors to $\mathbf{Ab}$, the naturally equivalent category of k-linear functors to the category k-Mod $\simeq$ Mod-k of k-vectorspaces. It is easily checked that the forgetful functor from Mod-k to $\mathbf{Ab}$ induces a natural equivalence of functor categories $(\mathcal{A}, \mathbf{Ab}) \simeq (\mathcal{A}, \text{Mod-}k)$.

Suppose that $\mathcal{C}$ is a skeletally small preadditive category and that $\mathcal{D}$ is an abelian category (we have in mind $\mathcal{D} = \mathbf{Ab}$ or $\mathcal{D} = $ Mod-k).

The **functor category** $(\mathcal{C}, \mathcal{D})$ has, for its objects, the additive functors from $\mathcal{C}$ to $\mathcal{D}$ and, for its morphisms, from a functor G to a functor F the natural transformations from G to F. When one is trying to make sense of functor categories, there is no harm in imagining that "functor" means "module", that "subfunctor" means "submodule" and that "natural transformation between functors" means "linear map between modules". For example the notion of submodule generalises that of a **subfunctor** of F, which is simply a subobject of F in the functor category: that is, a functor G and a natural transformation $\tau : G \to F$ which is **monic** in $(\mathcal{C}, \mathcal{D})$, meaning that if $\alpha, \beta : H \rightrightarrows G$ are natural transformations such that $\tau\alpha = \tau\beta$ then $\alpha = \beta$, equivalently if $H \xrightarrow{\gamma} G$ is such that $\tau\gamma = 0$ then $\gamma = 0$.

The next lemma says that if $\mathcal{D}$ is abelian then subfunctors and quotient functors are described "pointwise".

Lemma 1.3. *Suppose that $\mathcal{C}$ is a preadditive category, that $\mathcal{D}$ is an abelian category and let $\tau : G \to F$ be a morphism in $(\mathcal{C}, \mathcal{D})$.*
(i) τ is monic iff for every object $C \in \mathcal{C}$ the component $\tau_C : GC \to FC$ at C is a monomorphism in $\mathcal{D}$;
(ii) τ is epi iff for every object $C \in \mathcal{C}$ the component $\tau_C : GC \to FC$ at C is an epimorphism in $\mathcal{D}$.

The image and kernel of a morphism between functors in $(\mathcal{C}, \mathcal{D})$, where $\mathcal{D}$ is abelian, are also given locally: if $\tau : G \to F$ is such a morphism then $\ker(\tau)$ is the subfunctor of G given at $C \in \mathcal{C}$ by $\ker(\tau) \cdot C = \ker(\tau_C) \leq GC$; also $\mathrm{im}(\tau) \leq F$ is given on objects by $\mathrm{im}(\tau) \cdot C = \mathrm{im}(\tau_C) \leq FC$. In each case the action of the functor on morphisms is the "obvious" one.

Lemma 1.4. *Suppose that $\mathcal{C}$ is preadditive and $\mathcal{D}$ is abelian. The sequence $0 \to F' \xrightarrow{\iota} F \xrightarrow{\pi} F'' \to 0$ of functors in $(\mathcal{C}, \mathcal{D})$ is exact iff for every $C \in \mathcal{C}$ the sequence $0 \to F'C \xrightarrow{\iota_C} FC \xrightarrow{\pi_C} F''C \to 0$ is an exact sequence in $\mathcal{D}$.*

In particular, in the situation of 1.4, if $F' \leq F$ is an inclusion of functors then the quotient F/F' is given on objects by $(F/F')C = FC/F'C$. Also the direct sum, $F \oplus G$, of two functors is given pointwise.

Direct limits also are computed pointwise. Let $\big((F_\lambda)_\lambda, (\gamma_{\lambda\mu} : F_\lambda \to F_\mu)_{\lambda \leq \mu}\big)$ be a directed system of functors in $(\mathcal{C}, \mathcal{D})$ where now we assume that $\mathcal{D}$ is Grothendieck abelian (for example, a module category or a functor category), so has direct limits which are exact. Then, generalising 1.2, the functor category also is Grothendieck abelian. Let $\big(F, (\gamma_{\lambda\infty} : F_\lambda \to F)_\lambda\big)$ be the direct limit of this directed system in $(\mathcal{C}, \mathcal{D})$. For any object C of $\mathcal{C}$ there is a directed system $\big((F_\lambda C)_\lambda, ((\gamma_{\lambda\mu})_C : F_\lambda C \to F_\mu C)_{\lambda \leq \mu}\big)$ in $\mathcal{D}$. Then $\big(FC, ((\gamma_{\lambda\infty})_C : F_\lambda C \to FC)_\lambda\big)$ is, one may check, the direct limit of this system. As for the action of F on morphisms, note that a morphism $f : C \to C'$ gives rise to a morphism (the obvious definition) between the corresponding directed systems in $\mathcal{D}$ and hence, using the definition of direct limit, to a morphism which is $Ff : FC \to FC'$.

1.2 The Yoneda embedding and projective functors

If C is an object of the preadditive category $\mathcal{C}$ then there is the corresponding **representable functor** $(C, -) : \mathcal{C} \longrightarrow \mathbf{Ab}$ defined on objects by $D \mapsto (C, D)$ and on morphisms by $f : D \to E$ maps to (C, f), where $(C, f) : (C, D) \to (C, E)$

is defined by $(C, f)g = fg$ for $g \in (C, D)$. There is also the corresponding contravariant representable functor $(-, C)$ defined on objects by $D \mapsto (D, C)$ and on morphisms in the obvious way.

Lemma 1.5. *(Yoneda Lemma) Let $\mathcal{C}$ be any preadditive category. Take $C \in \mathcal{C}$ and $F \in (\mathcal{C}, \mathbf{Ab})$. Then there is a natural identification*

$$((C, -), F) \simeq (C, -)$$

between the group of natural transformations from the representable fuctor $(C, -)$ to F and the value group of F at C.

Naturality includes that if $\eta : F \to G$ is a natural transformation then $((C, -), \eta) : ((C, -), F) \to ((C, -), G)$ is identified with the map $\eta_C : FC \to GC$.

Similarly, if $G \in (\mathcal{C}^{\mathrm{op}}, \mathbf{Ab})$ then $((-, C), G) \simeq GC$.

Proof. The isomorphism $\theta = \theta_{C,F} : ((C, -), F) \simeq FC$ is defined as follows. Let $\tau \in ((C, -), F)$ and consider the component of τ at C, that is, $\tau_C : (C, C) \to FC$. Define $\theta(\tau) = \tau_C(1_C)$. Note that this element $\tau_C 1_C$ determines τ as follows: the component of τ at D, $\tau_D : (C, D) \to FD$, is defined by $\tau_D f = Ff \cdot (\tau_C 1_C)$.

$$\begin{tikzcd} C \arrow[d, "f"'] \\ D \end{tikzcd} \qquad \begin{tikzcd} (C,C) \arrow[r, "\tau_C"] \arrow[d, "{(C,f)}"'] & FC \arrow[d, "Ff"] \\ (C,D) \arrow[r, "\tau_D"'] & FD \end{tikzcd}$$

It is straightforward to check that this works.

The first use of the term "natural" in the statement refers to the dependence of θ on C and F. $\square$

The Yoneda Lemma (theorem and proof) is completely general (the basic version is for arbitrary categories and functors to **Set**). In particular it also applies to the category, $(\mathcal{C}, \text{Mod-}k)$, of k-linear functors if $\mathcal{C}$ is a k-linear category.

It follows that, if $(C, -)$ and $(D, -)$ are representable functors then

$$((C, -), (D, -)) \simeq (D, C).$$

In fact, it is immediate from 1.5 that we have a full contravariant embedding of $\mathcal{C}$ to $(\mathcal{C}, \mathbf{Ab})$, called the **Yoneda embedding**.

A faithful functor $F : \mathcal{A} \to \mathcal{B}$ between preadditive categories is one for which $Ff = 0$ implies $f = 0$; in particular $FA = 0$ implies $A = 0$.

A full functor F is one such that every morphism $g : FA \to FB$ between objects in the image of F has the form Ff for some $f : A \to B$. So a full and faithful functor allows allows its domain to be identified with a full subcategory of its codomain.

Corollary 1.6. *(see e.g. [8, 5.32]) Given any preadditive category $\mathcal{C}$ the functor from $\mathcal{C}^{\mathrm{op}}$ to $(\mathcal{C}, \mathbf{Ab})$ given on objects by $C \mapsto (C, -)$ and on morphisms by sending $f : C \to D$ to $(f, -) : (D, -) \to (C, -)$ where $(f, -)$ is given by $(f, -)g = gf$ whenever $g \in (D, E)$, is a full and faithful embedding.*

Dually, the (covariant) Yoneda embedding of $\mathcal{C}$ into $(\mathcal{C}^{\mathrm{op}}, \mathbf{Ab})$ given on objects by $C \mapsto (-, C)$, and on morphisms in the obvious way, is a full and faithful embedding.

Example 1.7. Let R be a ring, regarded as a preadditive category with one object. The Yoneda embedding from R^{op} to $(R, \mathbf{Ab}) = R$-Mod is given by taking the object $*$ of R to the functor $(*, -)$. This functor takes $*$ to $(*, *) = R$ and takes $r \in R = (*, *)$ to the map $s \mapsto rs$ $(s \in R)$: $(*, *) \to (*, *)$ - this is just the left module ${}_R R$.

$$\begin{tikzcd}
& * \arrow[dl, "s"'] \arrow[dr, dashed, "rs"] & \\
* \arrow[rr, "r"'] & & *
\end{tikzcd}$$

$$\begin{tikzcd}
* \arrow[rr, "t"] \arrow[dr, dashed, "st"'] & & * \arrow[dl, "s"] \\
& * &
\end{tikzcd}$$

The action of this Yoneda embedding on morphisms is to take $t \in R = (*, *)$ to the endomorphism of ${}_R R$ which takes $s \in {}_R R$ to st. The fact that the Yoneda embedding is full and faithful is just the assertion that $\mathrm{End}({}_R R) = R^{\mathrm{op}}$ (i.e. the right action of R).

Of course the dual Yoneda embedding from R (as a category) to $(R^{\mathrm{op}}, \mathbf{Ab}) =$ Mod-R takes $*$ to the right module R_R and $R = (*, *)$ isomorphically to $R = \mathrm{End}(R_R)$.

Lemma 1.8. *If A is an object of the small preadditive category $\mathcal{C}$ then the representable functor $(A, -)$ is a finitely generated projective object of $(\mathcal{C}, \mathbf{Ab})$. In particular, it is finitely presented.*

Proof. If $(A, -) = \underrightarrow{\sum}_\lambda F_\lambda$ is a directed sum of subfunctors then $1_A \in (A, A) = \underrightarrow{\sum}_\lambda F_\lambda A$ so $1_A \in F_\lambda A$ for some λ. By the Yoneda lemma (1.5) this gives a morphism $(A, -) \to F_\lambda$ which, composed with the inclusion $F_\lambda \to (A, -)$, is $1_{(A,-)}$; for that is the endomorphism of $(A, -)$ which Yoneda-corresponds to 1_A. Therefore the inclusion $F_\lambda \to (A, -)$ is an epimorphism, as required.

For projectivity, take an epimorphism $\pi : F \to G$ of functors and a morphism $\alpha : (A, -) \to G$ which, by Yoneda, corresponds to an element $a \in GA$. Since $\pi_A : FA \to GA$ is epi there is $b \in FA$ mapping to a.

$$\begin{tikzcd}
& (A,-) \arrow[dl, dotted, "\beta"'] \arrow[d, "\alpha"] \\
F \arrow[r, "\pi"'] & G
\end{tikzcd}$$

$$\begin{tikzcd}
FA \arrow[r, "\pi_A"'] & GA
\end{tikzcd}$$

$$\begin{tikzcd}
b \arrow[r, dotted] & a
\end{tikzcd}$$

Suppose that $\beta \in ((A,-),F)$ Yoneda-corresponds to b. Then $\pi\beta = \alpha$ by the Yoneda lemma. $\square$

finitely generated An object C of a preadditive category $\mathcal{C}$ is finitely generated if it is not the direct limit of any directed system of proper subobjects. Equivalently, if $F = \sum_{i \in I} F_i$ then $F = F_{i_1} + \dots F_{i_n}$ for some $i_1, \dots, i_n \in I$.

finitely presented An object C of a preadditive category $\mathcal{C}$ is finitely presented if the representable functor $(C,-) : \mathcal{C} \to \mathbf{Ab}$ commutes with direct limits (equivalently with filtered colimits).

Filtered colimits are more general than directed colimits = direct limits in that they allow that in the underlying diagram there can be more than one morphism $f, g : C_i \to C_j$ but these also are directed in the sense that, for any such pair, there is $k > j$ and $h : C_j \to C_k$ such that $hf = hg$ (this replaces the condition $f_{ik} = f_{jk} f_{ij}$ in the definition of direct limit.

If $\mathcal{C}$ is a module or, more generally functor, category then these definitions are equivalent to the usual ones in terms of generators and relations, and to those in terms of projective resolutions.

Proposition 1.9. *(e.g. [8, p. 119]) If $\mathcal{C}$ is a skeletally small preadditive category then the representable functors generate the functor category $(\mathcal{C}, \mathbf{Ab})$, indeed, for every functor $F : \mathcal{C} \longrightarrow \mathbf{Ab}$ there is an epimorphism $\bigoplus_i (A_i,-) \to F$ for some $A_i \in \mathcal{C}$. A functor F is finitely generated iff this direct sum may be taken to be finite.*

Proof. For each isomorphism class of objects of $\mathcal{C}$ take a representative A. Define the morphism $\bigoplus_A (A,-)^{(FA)} \to F$ to have component at $a \in FA$ the morphism $f_a : (A,-) \to F$ which Yoneda-corresponds to a. This map is surjective, essentially by definition.

For the second statement, we have $F = \sum_i \mathrm{im}(f_i)$ where f_i is the i-th component map of $\bigoplus_i (A_i,-) \to F$, so F finitely generated implies that F is a sum of finitely many of these, therefore finitely many of the direct summands will do. The converse follows since, by 1.8, each $(A_i,-)$ is finitely generated (and every image of a finitely generated object is finitely generated). $\square$

Say that **idempotents split** in the preadditive category $\mathcal{C}$ if for every $A \in \mathcal{C}$ each idempotent $e = e^2 \in \mathrm{End}(A)$ has a kernel and the canonical map $\ker(e) \oplus \ker(1-e) \to A$ is an isomorphism.

Corollary 1.10. *If $\mathcal{C}$ is a small preadditive category then the finitely generated projective objects of $(\mathcal{C}, \mathbf{Ab})$ are the direct summands of finite direct sums of representable functors. If $\mathcal{C}$ has split idempotents and has finite direct sums then these are precisely the representable functors.*

Proof. The first statement follows from 1.8 and 1.9.

For the second, if $\mathcal{C}$ has finite direct sums then $(A_i,-) \oplus (A_j,-) \simeq (A_i \oplus A_j,-)$ so, if F is finitely generated and projective, F is, without loss of generality, a direct summand of a functor of the form $(A,-)$. Let $\pi : (A,-) \to F$ split the inclusion and let $f \in \mathrm{End}(A,-)$ be π followed by the inclusion: so $f^2 = f$. The Yoneda embedding is full and faithful so there is $e \in \mathrm{End}(A)$ with

$(e,-) = f$ hence with $e^2 = e$. By assumption $A = \mathrm{im}(e) \oplus \ker(e)$ and then it follows quickly that $F \simeq (\mathrm{im}(e), -)$ so F is representable. $\square$

In particular the finitely generated projectives of $(R\text{-mod}, \mathbf{Ab})$ are the functors $(A,-)$ with $A \in R\text{-mod}$ and, together, these generate the functor category.

Of course all the above applies if we replace $\mathcal{C}$ by $\mathcal{C}^{\mathrm{op}}$. In particular the representable functors $(-,A)$, $A \in \mathcal{C}$, yield (closing under direct summands of finite direct sums) the finitely generated projective objects of $(\mathcal{C}^{\mathrm{op}}, \mathbf{Ab})$. The injective objects of the functor category $(R\text{-mod}, \mathbf{Ab})$ are described in 2.15.

Example 1.11. We continue Example 1.1. We write $\alpha_{i,i+1}$ for the arrow of the quiver going from vertex i to vertex $i+1$, more generally, α_{ij} $(i \leq j)$ for compositions of these in the path category $\mathcal{A}$, with α_{ii} the identity at i. Clearly an additive functor from $\mathcal{A}$ to Mod-k is equivalent to a k-representation of the quiver.

The simple functors/representations/modules are the S_i, $i \geq 1$, where S_i has dimension 1 at vertex i and is zero elsewhere. For each vertex i there is the indecomposable projective P_i (the projective cover of the simple object S_i) which is 1-dimensional at each $j \geq i$ and 0 elsewhere. This is the representable functor $(i,-)$ and one can see that the embeddings $P_1 \leftarrow P_2 \leftarrow \dots$, of each indecomposable projective as the radical of the next, shows $\mathcal{A}^{\mathrm{op}}$ Yoneda-embedded into the functor category $(\mathcal{A}, \text{Mod-}k)$. Dually, the indecomposable injective representations are the $E_i = E(S_i)$, which have dimension 1 at each vertex $j \leq i$ and zero elsewhere, plus one more, $E_\infty = P_1$, which has dimension 1 at each vertex.

The right modules - the contravariant functors - are the covariant functors for the opposite quiver - the quiver with the same vertices but all arrows reversed.

Lemma 1.12. *(e.g. [8, §3.1, 3.21]) If $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence in* mod-R *then the induced sequence $0 \to (C,-) \xrightarrow{(g,-)} (B,-) \xrightarrow{(f,-)} (A,-)$ of representable functors is exact in* $(\text{mod-}R, \mathbf{Ab})$.

If $A \xrightarrow{f} B \xrightarrow{g} C$ is a sequence in mod-R *such that $(C,-) \xrightarrow{(g,-)} (B,-) \xrightarrow{(f,-)} (A,-)$ is exact in* $(\text{mod-}R, \mathbf{Ab})$ *then the original sequence is exact. In particular a morphism h of $\mathcal{C}$ is an epimorphism iff $(h,-)$ is a monomorphism and, if $(h,-)$ is an epimorphism then h is a monomorphism.*

1.3 Finitely presented functors in $(\text{mod-}R, \mathbf{Ab})$

We will use the following facts about finitely presented objects (they hold in any abelian category with a generating set of finitely presented objects).

Lemma 1.13. *Let $\mathcal{C}$ be a module or functor category and let*

$$0 \to A \to B \xrightarrow{\pi} C \to 0$$

be an exact sequence.
(1) Suppose that B is finitely presented. Then C is finitely presented iff A is finitely generated.
(2) If C is finitely presented and B is finitely generated then A is finitely generated.

Let F be a finitely generated functor in $(\text{mod-}R, \mathbf{Ab})$. There is, by 1.9, some $A \in \text{mod-}R$ and an epimorphism $(A,-) \to F$. If F is finitely presented then the kernel of this epimorphism will be finitely generated so there will be $B \in \text{mod-}R$ and an exact sequence $(B, -) \to (A, -) \to F \to 0$. By the Yoneda lemma every morphism from $(B, -)$ to $(A, -)$ is induced by a morphism $f : A \to B$. This gives the following description.

Lemma 1.14. *Let $F \in (\text{mod-}R, \mathbf{Ab})$ be finitely presented. Then there is a morphism $f : A \to B$ in mod-R such that $F \simeq \text{coker}\big((f, -) : (B, -) \to (A, -)\big)$. Conversely, any functor of this form, $\text{coker}(f, -)$ for some $f \in \text{mod-}R$, is finitely presented.*

Of course, given F, there are many choices for f above.

Given $f : A \to B$ in mod-R write $F_f = \text{coker}(f, -)$. Then $F_f M$ ($M \in \text{Mod-}R$) has the following description: $F_f M = (A, M)/\text{im}(f, M)$ is the abelian group, (A, M), of morphisms from A to M factored by the subgroup consisting of those which factor initially through f. The next result shows that each finitely generated subfunctor of $(A, -)$ is determined by the property of morphisms factoring through some specified morphism with domain A.

Lemma 1.15. *Suppose that $A \in \text{mod-}R$ and that G is a subfunctor of $(A, -)$. Then G is finitely generated iff G is finitely presented iff G has the form $\text{im}(f, -)$ for some $f : A \to B$ in mod-R.*

Proof. Any functor $(B, -)$ with $B \in \text{mod-}R$ is finitely generated by 1.8, so any functor of the form $\text{im}(f, -)$ is finitely generated. Conversely, if G is a finitely generated subfunctor of $(A, -)$ then it is the image of a representable functor $(B, -) \to G$ (by 1.9). Compose this with the inclusion $G \to (A, -)$ to obtain a morphism $(B, -) \to (A, -)$ which, by Yoneda, 1.5, has the form $(f, -)$ for some $A \xrightarrow{f} B$. It follows that $G = \text{im}(f, -)$.

It remains to show that any such functor is finitely presented. Retaining the notation, let $g : B \to C$ be the cokernel of f, so $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact (and C is finitely presented). By 1.12 we obtain an exact sequence of functors $0 \to (C, -) \to (B, -) \xrightarrow{(f,-)} (A, -)$. Therefore $\ker\big((B, -) \to G\big)$ is $(C, -) \to (B, -)$ and hence G is finitely presented, by 1.13. □

The following strengthening of 1.15, the fact that every finitely presented functor in $(\text{mod-}R, \mathbf{Ab})$ is coherent, is a key property, **local coherence**, of the functor category. An object is **coherent** if it is finitely presented and every finitely generated subobject is finitely presented. A ring is **right coherent** if every finitely presented right module is coherent (equivalently if every intersection of two finitely generated right ideals is finitely generated and the right annihilator of every element is finitely generated).

Corollary 1.16. *Every finitely generated subfunctor of a finitely presented functor in $(\text{mod-}R, \mathbf{Ab})$ is finitely presented. That is, $(\text{mod-}R, \mathbf{Ab})$ is locally coherent.*

Proof. Suppose $H \leq G$ with G finitely presented and H finitely generated. By 1.14 there is $B \in \text{mod-}R$ and an epimorphism $\pi : (B, -) \to G$. Since G is finitely presented $\ker(\pi)$ is finitely generated (1.13) so, if $H' = \pi^{-1}H$, then H' is finitely generated, being an extension of a finitely generated by a finitely generated object, hence is finitely presented by 1.15. By 1.13, $H \simeq H'/\ker(\pi)$ is finitely presented. □

Denote by $(\text{mod-}R, \mathbf{Ab})^{\text{fp}}$ the full subcategory of finitely presented functors from mod-R to $\mathbf{Ab}$. The above result implies that this subcategory is abelian.

Proposition 1.17. $(\text{mod-}R, \mathbf{Ab})^{\text{fp}}$ *is an abelian subcategory of* $(\text{mod-}R, \mathbf{Ab})$*: it is an abelian category and a sequence* $0 \to F \to G \to H \to 0$ *which is exact in the smaller category is also exact in the larger category.*

Proof. By [8, 3.41] (where the term "exact subcategory" is used) it is enough to check that the kernel and cokernel of every morphism $\alpha : G \to F$ in $(\text{mod-}R, \mathbf{Ab})^{\text{fp}}$ is also in this subcategory.

The cokernel of α is a finitely presented object factored by a finitely generated one, hence is finitely presented (1.13).

The image of α is a finitely generated hence, by 1.16, finitely presented, subfunctor of F. Therefore (1.13) the kernel of α is a finitely generated, therefore finitely presented, subfunctor of G, as required. $\square$

Corollary 1.18. *[2, §2] Every finitely generated subfunctor of a representable functor in* $(\text{mod-}R, \mathbf{Ab})$ *has projective dimension less than or equal to 1. Every finitely presented functor has projective dimension less than or equal to 2.*

Proof. The first statement follows from the end of the proof of 1.15, along with 1.8. Also, since the representable functors form a generating set of finitely generated projective functors (1.9) the second statement follows; for every finitely presented functor F therefore has a projective presentation $0 \to (C, -) \to (B, -) \to (A, -) \to F \to 0$. $\square$

Corollary 1.19. *A finitely presented functor in* $(\text{mod-}R, \mathbf{Ab})$ *has projective dimension* ≤ 1 *iff it embeds in a representable functor.*

Proof. For the direction not covered by 1.18 suppose that the sequence $0 \to (B, -) \xrightarrow{(f,-)} (A, -) \to F \to 0$ induced by $f : A \to B$ is exact. So (1.12) f is an epimorphism. Then the sequence $0 \to \ker(f) \to A \to B \to 0$ is exact. Now, $\ker(f)$ is a finitely generated (but not necessarily finitely presented) R-module, so there is an epimorphism $C \to \ker(f)$ with $C \in \text{mod-}R$. From the exact sequence $C \to A \to B \to 0$ one obtains (1.12) the exact sequence $0 \to (B, -) \to (A, -) \to (C, -)$ and so F, being $\text{coker}((B, -) \to (A, -))$, embeds in $(C, -)$, as required. $\square$

The next result comes from [2, p. 205]; we write $\text{gldim}(\mathcal{C}) = n$ if n is the maximum of projective dimensions of objects in the category $\mathcal{C}$.

Proposition 1.20. *A ring R is von Neumann regular iff* $\text{gldim}(\text{mod-}R, \mathbf{Ab})^{\text{fp}} = 0$. *Otherwise* $\text{gldim}(\text{mod-}R, \mathbf{Ab})^{\text{fp}} = 2$.

Proof. It has to be shown that if $\text{gldim}(\text{mod-}R, \mathbf{Ab})^{\text{fp}} \leq 1$ then $\text{gldim}(\text{mod-}R, \mathbf{Ab})^{\text{fp}} = 0$.

If $f : A \to B$ is a morphism in mod-R then, by assumption, F_f has projective dimension ≤ 1 so, by 1.19, F_f embeds in a representable functor, hence there is an exact sequence $(B, -) \xrightarrow{(f,-)} (A, -) \xrightarrow{(g,-)} (D, -)$ for some $g : D \to A$ in mod-R. It follows from 1.12 that $D \xrightarrow{g} A \xrightarrow{f} B$ is exact. Therefore mod-R has kernels, which implies that R is right coherent; for this shows that $\ker(f)$

is finitely generated and, since $\mathrm{im}(f)$ is a typical finitely generated subobject of B, it follows that every finitely generated submodule of a finitely presented module is finitely presented. Furthermore, if $f : A \to B$ is monic, then $g = 0$, hence $(f, -)$ is epi and so $f : A \to B$ is split. Thus every embedding between finitely presented modules is split and this is one of the standard equivalents to a ring being von Neumann regular.

If R is von Neumann regular let $F = F_f$, where $f : A \to B$ is in mod-R, be a typical finitely presented functor (1.14). Let $A'' = \mathrm{im}(f)$. Since R, being von Neumann regular, is coherent A'' is finitely presented so $A' = \ker(f)$ is finitely generated. The embedding of A' in A is, by von Neumann regularity, split. So $F_f \simeq (A', -)$ has projective dimension 0 (1.8), as required. $\square$

Since regularity is a right/left symmetric condition one has the immediate corollary.

Corollary 1.21. $\mathrm{gldim}(\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}} = \mathrm{gldim}(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ *for every ring* R.

2 Injective Functors

2.1 The tensor embedding

We can move the whole module category Mod-R into the functor category - though now we mean functors on finitely presented *left* modules. Here is how.

To any right R-module, M, we associate the corresponding tensor functor, $(M \otimes -) = (M \otimes_R -) : R\text{-mod} \longrightarrow \mathbf{Ab}$, given by $(M \otimes -)(L) = M \otimes_R L$ on objects $L \in R$-mod and with the obvious effect on morphisms: if $g : L \to K \in R$-mod then $(M \otimes -)g = M \otimes_R g : M \otimes L \to M \otimes K$.

Thus from $M \in$ Mod-R we obtain $(M \otimes -) \in (R\text{-mod}, \mathbf{Ab})$.

Define the functor $\epsilon : \text{Mod-}R \longrightarrow (R\text{-mod}, \mathbf{Ab})$ by $\epsilon M = (M \otimes -)$ on objects and, if $M \xrightarrow{f} N$ is a morphism of right R-modules, then $\epsilon f : (M \otimes -) \to (N \otimes -)$ is the natural transformation whose component at $L \in R$-mod is defined to be $f \otimes 1_L : M \otimes L \to N \otimes L$, that is, $(- \otimes_R L)f$.

Lemma 2.1. *Suppose that* $F, F' \in (R\text{-mod}, \mathbf{Ab})$ *with* F *right exact and let* $\tau, \tau' : F \to F'$ *be natural transformations. If* $\tau_R = \tau'_R$ *then* $\tau = \tau'$.

Proof. Let L be a finitely presented left module, say $(R^m \to) R^n \xrightarrow{\pi} L \to 0$ is exact. There is a commutative diagram with the top row exact

$$\begin{tikzcd} FR^n \arrow[r, "F\pi"] \arrow[d, "\tau^{(\prime)}_{R^n}"'] & FL \arrow[r] \arrow[d, shift right, "\tau_L"'] \arrow[d, shift left, "\tau'_L"] & 0 \\ F'R^n \arrow[r, "F'\pi"'] & F'L \arrow[r] & 0 \end{tikzcd}$$

and with vertical maps being $\tau_{R^n} = (\tau_R)^n = (\tau'_R)^n = \tau'_{R^n}$ and either τ_L or τ'_L for the other. It follows that $\tau_L = \tau'_L$ since $\tau_L.F\pi = F'\pi.\tau_{R^n} = F'\pi.\tau'_{R^n} = \tau'_L.F\pi$ and $F\pi$ is an epimorphism. $\square$

A right exact functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is one which takes right exact sequences in $\mathcal{A}$ to right exact sequences in $\mathcal{B}$. If F is, instead, contravariant then we mean that it takes left exact sequences in $\mathcal{A}$ to right exact sequences in $\mathcal{B}$. An **exact functor** is one which takes exact sequences to exact sequences.

The basic results about $M \mapsto M \otimes -$ are stated in [13] and some more detail may be found in [14, §1] (also see the exposition in [17, B16]).

Theorem 2.2. *Let R be a small preadditive category. The functor $\epsilon : \text{Mod-}R \longrightarrow (R\text{-mod}, \mathbf{Ab})$ given on objects by $M \mapsto M \otimes -$ is a full embedding and is left adjoint to the functor "evaluation at R" from $(R\text{-mod}, \mathbf{Ab})$ to Mod-R: $(M \otimes -, F) \simeq (M, F(R))$.*

adjoint functors Suppose that $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ are functors. Then F is **left adjoint** to G and G is **right adjoint** to F (and that (F, G) is an **adjoint pair**) if there is a natural bijection $\mathcal{D}(F(-), *) \simeq \mathcal{C}(-, G(*))$, meaning that for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$ there is a natural bijection $\beta_{C,D} : \mathcal{D}(FC, D) \to \mathcal{C}(C, GD)$. By "natural" we mean that for all $f : C \to C'$ in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$ the obvious diagrams commute.

Proof. If $(M \otimes -) \simeq (N \otimes -)$ then, evaluating at ${}_R R$, we obtain $M \simeq M \otimes_R R \simeq N \otimes_R R \simeq N$. If $\tau : (M \otimes -) \to (N \otimes -)$ is a morphism then its component, τ_R, at ${}_R R$ is a morphism from $M \simeq M \otimes_R R$ to $N \simeq N \otimes_R R$. The natural transformations τ and $\tau_R \otimes -$ agree at R so, since $M \otimes -$ and $N \otimes -$ are right exact, 2.1 yields $\tau = \tau_R \otimes -$. Therefore ϵ is a full embedding.

For the adjointness, first note that if $F \in (R\text{-mod}, \mathbf{Ab})$ then, since $\text{End}({}_R R) = R$ (acting on the right), $F({}_R R)$ has the structure of a right R-module: if $a \in F({}_R R)$ and $s \in R$ then set $as = F(- \times s) \cdot a$ and note that $a(st) = F(- \times st) \cdot a = F((- \times t)(- \times s)) \cdot a = (as)t$. The natural isomorphism $(M \otimes -, F) \simeq (M, FR)$ takes $\tau \in (M \otimes -, F)$ to τ_R. By 2.1 this map is monic. To define the inverse map: given $g : M \to FR$, let τ_g be the natural transformation from $M \otimes -$ to F the component of which at $L \in R\text{-mod}$ is defined by taking $m \otimes l \in M \otimes L$ to $Fl \cdot g(m)$ where Fl denotes the value of F at the morphism from ${}_R R$ to L which takes 1 to l. One may check that τ_g is a natural transformation, that $(\tau_g)_R = g$ and that these processes do define an adjunction. $\square$

Example 2.3. The functor ϵ is not left exact: we show this for the embedding, j, of the $\mathbb{Z}$-module $\mathbb{Z}$ into the $\mathbb{Z}$-module $\mathbb{Q}$.

A morphism $\tau : F \to G$ in $(\mathbf{Ab}, \mathbf{Ab})$ is monic iff for every $L \in \mathbf{Ab}$ the component $\tau_L : FL \to GL$ is monic (1.3). Set $L = \mathbb{Z}_2$ (i.e. $\mathbb{Z}/2\mathbb{Z}$). Then $\mathbb{Z} \otimes \mathbb{Z}_2 \simeq \mathbb{Z}_2$ but $\mathbb{Q} \otimes \mathbb{Z}_2 = 0$ because every element of $\mathbb{Q} \otimes \mathbb{Z}_2$ is (a linear combination of elements) of the form $a \otimes 1_2$, and $a \otimes 1_2 = \frac{1}{2}a.2 \otimes 1_2 = \frac{1}{2}a \otimes 2.1_2 = 0$. Thus we see that $j \otimes L : \mathbb{Z} \otimes \mathbb{Z}_2 \to \mathbb{Q} \otimes \mathbb{Z}_2$ is not monic.

A short exact sequence $0 \to A \xrightarrow{j} B \xrightarrow{p} C \to 0$ is **pure-exact** if it satisfies the equivalent conditions of the next result, in which case we say that j is a **pure monomorphism** (and that p is a **pure epimorphism**).

Theorem 2.4. *The following conditions on a short exact sequence $0 \to A \xrightarrow{j} B \xrightarrow{p} C \to 0$ of right R-modules are equivalent.*

(i) For every (finitely presented) left R-module L the morphism $j \otimes 1_L : A \otimes L \to B \otimes L$ is monic.
(ii) The sequence is a direct limit of split-exact sequences.

Corollary 2.5. *[20, 2.4] If $0 \to A \to B \xrightarrow{\pi} C \to 0$ is pure-exact and C is finitely presented then this sequence is split.*
In particular any pure embedding in mod-R *splits.*

Theorem 2.6. *A sequence $0 \to M \to N \to N' \to 0$ of right R-modules is pure-exact iff the image $0 \to \epsilon M \to \epsilon N \to \epsilon N' \to 0$ is an exact sequence of functors. Furthermore ϵ commutes with direct limits and products.*

Proof. The first assertion is from the definition/2.4.
For every $L \in R$-mod the functor $- \otimes L$, being a left adjoint, commutes with direct limits and, from this, the first part of the second statement follows easily.
The assertion about products follows directly from 2.7 below. $\square$

Proposition 2.7. *[21, Sätze 1,2] A module M is finitely presented iff the canonical map $M \otimes (\prod_i N_i) \to \prod_i (M \otimes N_i)$ is an isomorphism for all collections of modules $(N_i)_i$ (and M is finitely generated iff each such map is a surjection).*

The next result is immediate from the fact that ϵ is full and faithful, hence preserves and reflects idempotents (that is, projections to direct summands).

Corollary 2.8. *A module M is indecomposable iff the corresponding functor $(M \otimes -) \in (R\text{-mod}, \mathbf{Ab})$ is indecomposable.*

2.2 Injective functors and pure-injective modules

A module M is **absolutely pure** if every embedding $M \to N$ in Mod-R with domain M is pure. A module M is **fp-injective** if for every embedding $i : A \to B$ with finitely presented cokernel and every morphism $f : A \to M$ there is $g : B \to M$ such that $gi = f$.

Proposition 2.9. *For any module M the following are equivalent:*
(i) M is absolutely pure;
(ii) M is fp-injective;
(iii) $\mathrm{Ext}^1(C, M) = 0$ for every finitely presented module C.

Clearly every injective module is absolutely pure. Over a right noetherian ring absolutely pure = injective.

Proposition 2.10. *A module is absolutely pure iff it is a pure submodule of an injective module.*

Proof. One direction is by the definition (and existence of injective hulls). For the other suppose that $i : M \to E$ is a pure embedding and that E is injective. Let $j : M \to N$ be any embedding. Since E is injective there is $g : N \to E$ such that $gj = i$. Since i is pure it follows easily that so is j, as required. $\square$

An R-module N is **pure-injective** if, given any pure embedding $f : A \to B$ in Mod-R, every morphism $g : A \to N$ lifts through f: there exists $h : B \to N$

such that $hf = g$.

$$\begin{tikzcd} A \arrow[r, hook, "\forall f", "\text{pure}"'] \arrow[d, "\forall g"'] & B \arrow[dl, dotted, "\exists h"] \\ N & \end{tikzcd}$$

Equivalently, N is pure-injective iff every pure embedding $N \to M$ with domain N is split.

Any direct product of pure-injective modules is pure-injective, as is any direct summand of a pure-injective module. Any injective module is pure-injective because such a module has the lifting property over all embeddings, pure or not.

A module is of **finite endolength** if, when regarded as a module over its endomorphism ring, it has finite length.

Corollary 2.11. *Every module of finite endolength is pure-injective.*

In fact every module M of finite endolength has the stronger property of being **Σ-pure-injective**, meaning that every infinite direct sum of copies of M is pure-injective.

Duals of modules are pure-injective.

Proposition 2.12. *If ${}_SM_R$ is an (S,R)-bimodule and ${}_SE$ is an injective left S-module then $M^* = \mathrm{Hom}_S({}_SM_R, {}_SE)$ is a pure-injective left R-module.*

In particular if M is a right R-module then $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a pure-injective left R-module.

Example 2.13. [39, Thm. 1] The group ring RG is pure-injective as a right module over itself iff the same is true for R and G is finite.

The following criterion for pure-injectivity is due to Jensen and Lenzing.

Theorem 2.14. *[17, 7.1] A module M is pure-injective iff for every index set, I, the summation map $\Sigma : M^{(I)} \to M$, given by $(x_i)_i \mapsto \sum_i x_i$, factors through the natural (pure) embedding of $M^{(I)}$ into the corresponding direct product M^I.*

Theorem 2.15. *([13], [14, §1]) An exact sequence $0 \to M \to N \to N' \to 0$ in* Mod-R *is pure exact iff the sequence $0 \to \epsilon M \to \epsilon N \to \epsilon N' \to 0$ is a pure exact sequence in* (R-mod, $\mathbf{Ab}$).

If M is a right R-module then $\epsilon M = (M \otimes -)$ is an absolutely pure object of (R-mod, $\mathbf{Ab}$), *indeed every absolutely pure functor is isomorphic to one of this form.*

Furthermore, $(M \otimes -)$ is injective iff M is pure-injective.

Proof. The given pure exact sequence is, 2.4, a direct limit of split exact sequences. Since ϵ commutes with direct limits (2.2) the image sequence is a direct limit of split exact sequences, hence is pure exact. By 2.2 we also have the converse.

Next we show that $Q \in$ (R-mod, $\mathbf{Ab}$) is absolutely pure [the definition given for modules applies equally to functors] iff Q is a right exact functor.

Let $F \in (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ have projective presentation $0 \to (C,-) \to (B,-) \to (A,-) \to F \to 0$ where $A \to B \to C \to 0$ is an exact sequence in R-mod (see the proof of 1.18). Then the homology groups of the chain complex $0 \to ((A,-),Q) \to ((B,-),Q) \to ((C,-),Q) \to 0$, that is (by Yoneda) $0 \to QA \to$

$QB \to QC \to 0$, are, by definition, precisely (F,Q), $\mathrm{Ext}^1(F,Q)$ and $\mathrm{Ext}^2(F,Q)$. Therefore Q is right exact iff for all finitely presented functors F we have $\mathrm{Ext}^1(F,Q) = 0 = \mathrm{Ext}^2(F,Q)$. We have $\mathrm{Ext}^2(F,-) \simeq \mathrm{Ext}^1(F',-)$ where, with the notation above, F' is the kernel of $(A,-) \to F$. Since (1.16) $(R\text{-mod}, \mathbf{Ab})$ is locally coherent F' also is finitely presented. So the condition on Q reduces to $\mathrm{Ext}^1(F,Q) = 0$ for all finitely presented F, that is (2.9), Q is an absolutely pure functor.

It follows that every functor of the form $M \otimes -$ is absolutely pure.

For the converse, suppose that Q is absolutely pure hence, as shown above, right exact. Note that $Q({}_RR)$ is a right $(\mathrm{End}({}_RR) \simeq)R$-module. Define the natural transformation $(Q(R) \otimes -) \to Q$ to have component at $L \in R$-mod the map $Q(R) \otimes L \to QL$ defined by taking $m \otimes l$ ($m \in Q(R)$, $l \in L$) to $Q(l : {}_RR \to L) \cdot m$ (a special case of that in the proof of adjointness in 2.2). It is straightforward to check that this is, indeed, a natural transformation and that at $L = R$ it is an isomorphism. So, by 2.1, this is an isomorphism from Q to $(Q(R) \otimes -)$, as required.

Given $M \in$ Mod-R there is the pure-exact sequence $0 \to M \to H(M) \to H(M)/M \to 0$, where $H(M)$ is the pure-injective hull (see below) of M, hence, by 2.6, the sequence of functors $0 \to (M \otimes -) \to (H(M) \otimes -) \to (H(M)/M \otimes -) \to 0$ is pure-exact. If $(M \otimes -)$ is injective then this sequence is split and so, therefore (by 2.2, ϵ is full), is the first, whence M is a direct summand of $H(M)$, hence is pure-injective (and equal to $H(M)$). For the converse, take an injective hull, $E(M \otimes -)$, of $M \otimes -$. Since any injective is absolutely pure, $E(M \otimes -) \simeq (N \otimes -)$ for some $N \in$ Mod-R. So there is an exact sequence $0 \to (M \otimes -) \to (N \otimes -) \to (N \otimes -)/(M \otimes -) \simeq (N/M \otimes -) \to 0$ (the isomorphism by right exactness of $\otimes$) and, therefore, by the first part, there is the pure-exact sequence $0 \to M \to N \to N/M \to 0$ in Mod-R. If M is pure-injective this sequence is split, so the sequence of functors is split and, therefore, $M \otimes -$ is already injective. $\square$

We record the following point which was established in the course of the proof above.

Proposition 2.16. *An object of* $(R\text{-mod}, \mathbf{Ab})$ *is absolutely pure iff it is a right exact functor.*

Corollary 2.17. *There is a bijection between isomorphism classes of indecomposable pure-injective right R-modules, N, and isomorphism classes of indecomposable injective objects, Q, in the functor category* $(R\text{-mod}, \mathbf{Ab})$*, given by* $N \mapsto (N \otimes -)$ *and* $Q \mapsto Q({}_RR)$*.*

A **pure-injective hull** for a module M is a pure embedding $M \to N$ with N pure-injective and N minimal such, in the sense that there is no factorisation of this map through any direct summand of N. Existence and uniqueness of pure-injective hulls is most easily obtained by using ϵ to pull back the corresponding results for injective functors to pure-injective modules. We denote the pure-injective hull of M by $H(M)$.

Corollary 2.18. *The embedding $M \to N$ is a pure-injective hull in* Mod-R *iff $(M \otimes -) \to (N \otimes -)$ is an injective hull in* $(R\text{-mod}, \mathbf{Ab})$*:* $E(M \otimes -) \simeq (H(M) \otimes -)$*.*

Corollary 2.19. *Every module M has a pure-injective hull which is unique to isomorphism over M: if $j : M \to N$ and $j' : M \to N'$ are pure-injective hulls of M then there is an isomorphism $f : N \to N'$ such that $fj = j'$.*

Corollary 2.20. *Every indecomposable pure-injective has local endomorphism ring.*

Proof. This follows by using ϵ to pull back the corresponding fact for indecomposable injective objects in Grothendieck categories. $\square$

2.3 Duality of finitely presented functors

The two functor categories, $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ and $(\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$, are dual. The explicit definition of the duality comes from [3, §7] and [14, 5.6] and is as follows.

Let $F \in (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$. Define the functor $dF \in (\text{mod-}R, \mathbf{Ab})$ by $dF \cdot A = (F, A \otimes -)$ for $A \in \text{mod-}R$; it is the case, 2.21 below, that $A \otimes -$ is a finitely presented functor if A is a finitely presented module. If $g : A \to B$ is in mod-R, then $dF \cdot g : (F, A \otimes -) \to (F, B \otimes -)$ is defined to be $(F, g \otimes -)$, so is given by $dF \cdot g \cdot \tau = (g \otimes -)\tau$. That is, dF, the **dual** of F, is given by the representable functor $(F, -)$ restricted to the image of the embedding, (2.2) $A \mapsto A \otimes -$, of mod-R into $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$. The fact that dF is a finitely presented functor is 2.24.

Theorem 2.21. *([2, 6.1]) Let $M \in \text{Mod-}R$. The functor $(M \otimes_R -) : R\text{-mod} \longrightarrow \mathbf{Ab}$ is finitely presented iff M is finitely presented.*

The definition of d on morphisms is as follows: given $f : F \to G$ in $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ define $df : dG \to dF$ to be $(f, -)$, that is, if $M \in \text{mod-}R$ then the component of df at M is given by taking $\tau \in (G, M \otimes -)$ to $\tau f \in (F, M \otimes -)$.

Dually, if $F \in (\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$, then we use the same notation for the functor $dF \in (R\text{-mod}, \mathbf{Ab})$ given on objects by $dF \cdot L = (F, - \otimes L)$.

Example 2.22. If $A \in \text{mod-}R$ and $L \in R\text{-mod}$ then, $d(L, -) \simeq (- \otimes L)$ and $d(A \otimes -) \simeq (A, -)$. For the first, $d(L, -) \cdot B = \big((L, -), B \otimes -\big)$ which (1.5) is naturally isomorphic to $B \otimes L = (- \otimes L)B$ and, for the second, $d(A \otimes -) \cdot B = (A \otimes -, B \otimes -)$ which is naturally isomorphic to $(A, B) = (A, -)B$.

Lemma 2.23. *d is a contravariant exact functor from $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ to $(\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$.*

Proof. Suppose that $0 \to H \to F \to G \to 0$ is an exact sequence in $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$. For any $A \in \text{mod-}R$ the sequence $0 \to (G, A \otimes -) \to (F, A \otimes -) \to (H, A \otimes -) \to \mathrm{Ext}^1(G, A \otimes -)$ is exact. By 2.15, $(A \otimes -)$ is an absolutely pure functor hence (see 2.9), since G is finitely presented, $\mathrm{Ext}^1(G, A \otimes -) = 0$, so d is indeed exact. That dF is finitely presented is shown next. $\square$

Proposition 2.24. *If $F \in (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ then $dF \in (\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$.*

Proof. Suppose that $(L, -) \xrightarrow{(f,-)} (K, -) \to F \to 0$ is a projective presentation of F with $K \xrightarrow{f} L \in R\text{-mod}$ (1.14). Apply d to obtain the, exact by 2.23, sequence $0 \to dF \to d(K, -) \to d(L, -)$ that is, by 2.22, $0 \to dF \to (- \otimes K) \to (- \otimes L)$. Since K, L are finitely presented so, by 2.21, are the functors $- \otimes K$

and $- \otimes L$ so the image of $- \otimes K$, being finitely generated, is finitely presented (1.16). Therefore (1.13) the kernel, dF, of this map is finitely generated, hence (1.16 again) finitely presented, as required. $\square$

Theorem 2.25. *The functor*

$$d : ((R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}})^{\mathrm{op}} \longrightarrow (\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$$

is an equivalence of categories. If we use d also to denote the corresponding functor from $((\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}})^{\mathrm{op}}$ *to* $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ *then* d^2 *is naturally equivalent to the identity functor on* $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$.

Proof. Applying d to a presentation $(L, -) \to (K, -) \to F \to 0$ of $F \in (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ gives the exact sequence $0 \to dF \to (- \otimes K) \to (- \otimes L)$ in $(\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$, by the proof of 2.24. Applying d again gives, by 2.23 and 2.22, the exact sequence $(L, -) \to (K, -) \to d^2 F \to 0$. From this and what has been shown already the assertions follow. $\square$

Proposition 2.26. *If M is any right R-module and if F is any functor in* $(\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$ *then there is a natural isomorphism* $(dF, M \otimes -) \simeq FM$ *(more accurately,* $\overrightarrow{F}M$ *where* $\overrightarrow{F}$ *denotes the unique extension of F to a functor on all of* Mod-R *which commutes with direct limits - this uses that every module is a direct limit of finitely presented modules).*

Proof. For M finitely presented we have this, by 2.25 and definition of d, since $FM \simeq d.dF.M \simeq (dF, M \otimes -)$. For arbitrary modules we can argue as follows. Suppose that $(B, -) \to (A, -) \to F \to 0$ is a projective presentation of F in $(\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$. By 2.23 and 2.22 this gives the exact sequence $0 \to dF \to (A \otimes -) \to (B \otimes -)$ in $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$. In turn this gives the exact sequence $(B \otimes -, M \otimes -) \to (A \otimes -, M \otimes -) \to (dF, M \otimes -) \to 0$ (exactness at the last place because, 2.15, $M \otimes -$ is absolutely pure), that is, $(B, M) \to (A, M) \to (dF, M \otimes -) \to 0$. Since $(B, M) \to (A, M) \to FM \to 0$ also is exact we deduce $(dF, M \otimes -) \simeq FM$. $\square$

2.4 Injectives and projectives in the category of finitely presented functors

We can see from the first lecture that the category of finitely presented functors has enough projectives (since the whole category has a generating set of projectives which are themselves finitely presented). So, by the duality 2.25, we deduce that there are **enough injectives**, meaning that every functor embeds into a functor which, *as an object of* $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$, is injective. We identify these functors.

Proposition 2.27. *[14, 5.5] The category* $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ *of finitely presented functors has enough injectives; the injective objects are the functors (isomorphic to one) of the form* $A \otimes -$ *with* $A \in \text{mod-}R$.

Proof. Any functor of the form $A \otimes_R -$ with $A \in \text{mod-}R$ is finitely presented (2.21) and is, by 2.15, absolutely pure in $(R\text{-mod}, \mathbf{Ab})$, hence is injective in

$(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$; for every pure embedding in the latter category is split since it has finitely presented cokernel - see 2.5.

Conversely, if $G \in (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ is injective in this category then $\mathrm{Ext}^1(F, G) = 0$ for all $F \in (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$, that is, by 2.9, G is absolutely pure in $(R\text{-mod}, \mathbf{Ab})$ so, by 2.15, G is isomorphic to a functor of the form $A \otimes -$ with $A \in \text{Mod-}R$. By 2.21, $A \in \text{mod-}R$.

That proves the second statement: to see the first, let $F \in (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$. There is, by 1.9, an epimorphism $(A, -) \to dF$ for some $A \in \text{mod-}R$. This dualises (2.25) to an embedding $F \to d(A, -) \simeq (A \otimes_R -)$ (by 2.22), as required. $\square$

In general $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ does not have injective hulls (i.e. minimal injective extensions) equivalently, by 2.25, $(\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$ does not have projective covers. A ring R is **Krull-Schmidt** if every finitely presented right R-module is a direct sum of indecomposable modules with local endomorphism rings. For instance artin algebras are Krull-Schmidt. As defined, this is "right Krull-Schmidt" but, in fact, the notion is right/left symmetric.

Proposition 2.28. *Let R be a ring. Then*
(i) the category $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ has injective hulls iff
(ii) the category $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ has projective covers iff
(iii) R is Krull-Schmidt.

3 Simple Functors

3.1 Simple functors and indecomposable modules

Every simple functor in the category $(R\text{-mod}, \mathbf{Ab})$ has, since it is finitely generated and by 1.9, the form $S = (A, -)/G$ where $A \in \text{mod-}R$ and G is a maximal subfunctor of $(A, -)$. There are two cases: if G is a finitely generated functor then the quotient S is a finitely presented functor and has a projective presentation of the form $(B, -) \xrightarrow{(f,-)} (A, -) \to S \to 0$ where $A \xrightarrow{f} B$ is a morphism in mod-R; otherwise S, though finitely generated, does not lie in $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$. In the case that S is finitely presented the dual, $dS \in (\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$, is, by 2.25, a finitely presented simple functor.

In both cases, by 2.17, the injective hull of S in the functor category $(R\text{-mod}, \mathbf{Ab})$ has the form $(N \otimes -)$ for a unique indecomposable pure-injective module N_R.

Assume that R is Krull-Schmidt. Let $S \in (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ be a finitely presented simple functor and let N_R be the indecomposable pure-injective such that $(N \otimes -)$ is its injective hull, that is such that $(S, N \otimes -) \neq 0$. Consider the dual $dS \in (\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$ of S and let $(A, -)$, where A is an indecomposable finitely presented right module, be its projective cover; so we have an exact sequence $0 \to F \to (A, -) \to dS \to 0$ of finitely presented functors (F is finitely presented by coherence of the functor category and 1.13). Dualise this sequence to obtain, using 2.22, the exact sequence $0 \to ddS \simeq S \to (A \otimes -) \to dF \to 0$. If we assume now that A is a pure-injective module (which, by 2.11, will be the case if R is an artin algebra), then we have $(N \otimes -) \simeq (A \otimes -)$, hence $A \simeq N$ (since there is just one indecomposable injective to which S has a nonzero map). This gives the following.

Theorem 3.1. *Suppose that R is an artin algebra. Then there are natural bijections between indecomposable right R-modules A of finite length and simple finitely presented functors S on* mod-R *(respectively, on* R-mod*). This is given by $S \mapsto A$ where $(A,-)$ is the projective cover of S (resp. $A \otimes -$ is the injective hull of S) and, in the other direction, $A \mapsto (A,-)/J(A,-)$ where $J(A,-)$ denotes the unique maximal proper subfunctor of $(A,-)$ (resp. by $A \mapsto \mathrm{soc}(A \otimes -)$, the unique minimal nonzero subfunctor of $A \otimes -$).*

Regarding the detailed part of the statement, one can check that the functor F which is defined by taking $B \in$ mod-R to the set of all non-isomorphisms from A to B is the unique maximal proper subfunctor of $(A,-)$.

In fact, over an artin algebra every simple functor is finitely presented and this is a starting point for Auslander-Reiten theory.

3.2 An example

Very occasionally we can get a complete picture of the functor category.

Let K be a field and let $R = K[\epsilon] = K[X]/\langle X^2 \rangle$. Then R_R is indecomposable, projective and injective, with top and socle both isomorphic to the unique simple module U. It follows that every module has the form $R^{(I)} \oplus U^{(J)}$ for some I, J. Thus R is of **finite representation type**, which means that every R-module is a direct sum of indecomposable modules and there are, up to isomorphism, only finitely many indecomposable modules. The Auslander-Reiten quiver of R is easily computed (for Auslander-Reiten theory see, e.g., [4]). The vertices are R and U and it has arrows the embedding $i : U \to R$ of $\mathrm{rad}(R)$ into R and the epimorphism $\pi : R \to U$ of R to $R/\mathrm{rad}(R)$. Note that $\pi i = 0$ and $i\pi = \epsilon$ (i.e. multiplication by ϵ).

The **Auslander algebra** of R is the endomorphism ring $S = \mathrm{End}(M)$ where M is a direct sum of one copy of each of the indecomposable (finitely presented) R-modules, which in this case is $S = \mathrm{End}(R \oplus U)$. We may represent S as the matrix ring $\begin{pmatrix} (R,R) & (U,R) \\ (R,U) & (U,U) \end{pmatrix} \simeq \begin{pmatrix} R = K1_R \oplus K\epsilon & Ki \\ K\pi & K1_U \end{pmatrix}$ and this decomposes as a left module as, say, $Q_1 \oplus Q_2$ where Q_i is the i-th column. Set $T_1 = Q_1/\mathrm{rad}(Q_1)$. One can check that $Q_2 \simeq \mathrm{rad}(Q_1)$. Right multiplication by $\begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix}$ gives an epimorphism $Q_1 \to \mathrm{rad}(Q_2)$ so, noting the lengths of these modules, we conclude that $\mathrm{rad}(Q_2) \simeq T_1$. Let T_2 denote the other simple module (the top of Q_2).

Thus far we have four indecomposable modules - the two indecomposable projectives and the two simples. One can also see I_2 - the injective hull of T_2, with socle T_2 and $I_2/T_2 \simeq T_1$. It is not difficult to check that there are no more indecomposable modules so, since S is an artin algebra, S is of finite representation type (this is not always the case for the Auslander algebra of a finite representation type algebra). We can compute the Auslander-Reiten quiver for left S-modules to be as shown, where the 1st (respectively 2nd) and 5th (respectively 6th) columns should be identified (and dotted lines indicate Auslander-Reiten translates).

$$
\begin{tikzcd}[column sep=small]
 & Q_1 = I_1 \arrow[dr] & & & & Q_1 \\
Q_2 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & I_2 \arrow[dr] & & Q_2 \arrow[ur] \arrow[dr] \arrow[r, dotted, no head] & {} \\
{} \arrow[r, dotted, no head] & T_2 \arrow[ur] \arrow[rr, dotted, no head] & & T_1 \arrow[ur] \arrow[rr, dotted, no head] & & T_2
\end{tikzcd}
$$

Now we use the fact, see e.g. [43, 4.9.4], that the functor category (mod-R, **Ab**) is equivalent to the category of left modules over the Auslander algebra[1]: S-Mod $\simeq$ (mod-R, **Ab**). We deduce that, in our example, there are just five indecomposable functors. To identify these we can use the explicit description, see [43, p. 121], of this equivalence, which takes a functor F to FM regarded as a left S-module and, in the other direction, a finitely presented left S-module ${}_SA$ is sent to the functor $\mathrm{Hom}_S({}_S\mathrm{Hom}_R(-, {}_SM_R), {}_SA)$. In our example the functors are easily identified directly:
Q_1 is visibly the representable functor $(R_R, -)$, which is also $(- \otimes_R R)$, it is also the injective hull of T_1 and the projective cover of T_1;
Q_2 is visibly the representable functor $(U, -)$ and is the projective cover of T_2;
I_2 is the injective hull of T_2 and is also $(- \otimes_R U)$;

To illustrate the localisation process described in the next section, consider the Serre subcategory, $\langle T_2 \rangle$, generated by T_2. In the quotient category S-mod$/\langle T_2 \rangle$, since T_2 becomes isomorphic to 0, all of Q_2, T_1 and I_2 become isomorphic and we see that there are just two indecomposables, a simple and a non-split self-extension of that simple - which looks like, and can checked to be, (equivalent to) mod-R (in general, however, a localisation of a module or functor category need not be equivalent to a module or functor category).

3.3 Serre subcategories and localisation

Suppose that $\mathcal{B}$ is an abelian category and that $\mathcal{S}$ is a **Serre subcategory**, that is, a subcategory of $\mathcal{B}$ such that whenever $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathcal{B}$ then $B \in \mathcal{S}$ iff $A, C \in \mathcal{S}$. Then there is a "localisation functor" from $\mathcal{B}$ to a category of fractions which is obtained by inverting those morphisms of $\mathcal{B}$ whose kernels and cokernels are in $\mathcal{S}$ (see, e.g., [23, §4.3] for details). This category is denoted $\mathcal{B}/\mathcal{S}$ and referred to as the **quotient** of $\mathcal{B}$ by $\mathcal{S}$. As stated in the next result, this is again an abelian category, so this is not the same as factoring by an ideal of $\mathcal{S}$ (the process used to obtain the stable module category).

Theorem 3.2. *[23, pp. 172-174] Suppose that $\mathcal{B}$ is an abelian category and that $\mathcal{S}$ is a Serre subcategory of $\mathcal{B}$. The quotient ategory $\mathcal{B}/\mathcal{S}$ is abelian and the localisation functor $F : \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{S}$ is exact. The kernel of F in the sense of* $\ker(F) = \{B \in \mathcal{B} : FB = 0\}$ *is $\mathcal{S}$. Every exact functor $G : \mathcal{B} \longrightarrow \mathcal{B}'$ from $\mathcal{B}$ to an abelian category $\mathcal{B}'$ with* $\ker(G) \supseteq \mathcal{S}$ *factors uniquely through $F : \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{S}$ via an exact and faithful functor.*

[1] All that is going on here is that mod-R is the additive closure of, hence has the same category of representations as, the preadditive category of indecomposables which, having only finitely many objects, is equivalent to a ring. We could make the same definition in the general case but we'd get a ring with many objects rather than an actual ring.

If $\mathcal{C}$ is a locally coherent Grothendieck abelian category, in particular if $\mathcal{C}$ is a functor category, then the closure under direct limits, $\overrightarrow{\mathcal{S}}$, of any Serre subcategory $\mathcal{S}$ of $\mathcal{C}^{\mathrm{fp}}$ is a **hereditary torsion class** in $\mathcal{C}$, that is, a class closed under subobjects, factor objects, extensions and arbitrary direct sums. An object is defined to be **torsionfree** if it contains no non-zero torsion subobject. The associated (hereditary) **torsion functor**, $\tau : \mathcal{C} \to \mathcal{C}$ assigns to every object its largest torsion subobject.

We may form the localisation (in the sense of [10], see [38]) of $\mathcal{C}$ at $\overrightarrow{\mathcal{S}}$. As above, this is done by forcing every object of $\overrightarrow{\mathcal{S}}$ to become 0 in the new, quotient, abelian category, but we also require the localisation functor to commute with direct limits.

There are a couple of ways of constructing this functor. The first method we describe is most naturally presented as a functor from $\mathcal{C}$ to a full subcategory. Take an object $C \in \mathcal{C}$. First we make its torsion subobject zero, by forming $C/\tau(C)$. Then we embed this into its injective hull $E(C/\tau(C))$ and take the maximal extension of $C_1 = C/\tau(C)$ within $E(C_1)$ by a torsion object, namely $C_\tau = \pi^{-1}\big(\tau\big(E(C_1)/C_1\big)\big)$ where $\pi : E(C_1) \to E(C_1)/C_1$ is the natural projection. Then C_τ is **τ-injective**, meaning that it has no non-split extension by a torsion object. This is the action of the localisation functor, which we denote Q_τ, on objects; there is a naturally induced action on morphisms. We denote by $\mathcal{C}_\tau$ the image of this functor; it is abelian and a full, but not exact=abelian, subcategory of $\mathcal{C}$. This category is referred to as the **localisation** (or **quotient category**) of $\mathcal{C}$ at τ.

Theorem 3.3. *(see [38, §§IX.1, X.1], [23, 4.3.8, 4.3.11, §4.4, 4.6.2]) Let τ be a hereditary torsion functor on a Grothendieck abelian category $\mathcal{C}$. Then the localised category $\mathcal{C}_\tau$ also is Grothendieck abelian, the localisation functor $Q_\tau : \mathcal{C} \longrightarrow \mathcal{C}_\tau$ is exact and, if $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is any exact functor to a Grothendieck category $\mathcal{C}'$ such that F commutes with direct limits and $\ker(F) \supseteq \mathcal{T}_\tau$ - the torsion class corresponding to τ - then F factors uniquely through Q_τ.*

As said already, the localisation functor $Q_\tau : \mathcal{C} \longrightarrow \mathcal{C}_\tau$ has a right adjoint, namely the inclusion, i, of $\mathcal{C}_\tau$ in $\mathcal{C}$: $\mathcal{C}(C, iD) \simeq \mathcal{C}_\tau(C_\tau, D)$ for every $C \in \mathcal{C}$ and $D \in \mathcal{C}_\tau$. The image of i is, up to natural equivalence, the full subcategory of τ-torsionfree, τ-injective objects of $\mathcal{C}$.

For any τ-torsionfree, τ-injective object C, one has $Q_\tau C \simeq C$ and the injective objects of $(i)\mathcal{C}_\tau$ are exactly the τ-torsionfree injective objects of $\mathcal{C}$.

If $\mathcal{G}$ is a generating set for $\mathcal{C}$ then $Q_\tau\mathcal{G}$ is a generating set for $\mathcal{C}_\tau$.

The inclusion functor $i : \mathcal{C}_\tau \to \mathcal{C}$ is not in general right exact: if $0 \to D' \to D \to D'' \to 0$ is an exact sequence in $\mathcal{C}_\tau$ then, regarded as a sequence in $\mathcal{C}$, the quotient D/D', though certainly τ-torsionfree, need not be τ-injective, so the corresponding exact sequence in $\mathcal{C}$ would replace D'' by the, perhaps proper, submodule D/D' (of which D'' would be the τ-injective hull). Under the alternative construction, outlined below, the functor Q_τ can be seen as sheafification with respect to a certain Grothendieck-type topology and then this corresponds to the fact that the inclusion of the category of sheaves in the category of presheaves is not right exact. Also the inclusion functor does not, for instance, commute with infinite direct sums: in general, the direct sum of objects in $\mathcal{C}_\tau$, regarded as embedded in $\mathcal{C}$, is obtained by taking their direct sum in $\mathcal{C}$ and then taking the τ-injective hull of that.

The alternative construction is essentially a sheafification process, and can be found in [1], or see [38, §IX.1]. In this construction no change is made in the objects but the morphisms of the new category are defined by $\mathcal{C}_\tau(C, D) = \varinjlim_{C'} \varinjlim_{D_0}(C', D/D_0)$ where C' ranges over subobjects of C such that C/C' is torsion (these are directed by intersection) and D_0 ranges over τ-torsion subobjects of D (directed by sum). In this category, an enlargement of $\mathcal{C}$ by morphisms, each object C is isomorphic to the corresponding object C_τ obtained by the first construction.

Theorem 3.4. *(see [23, 4.4.9]) Suppose that $\mathcal{C}$ and $\mathcal{C}'$ are abelian categories, that $\mathcal{C}$ is Grothendieck and that $Q : \mathcal{C} \longrightarrow \mathcal{C}'$ is an exact functor. Suppose also that Q has a full and faithful right adjoint. Then* $\ker(Q) = \{C \in \mathcal{C} : QC = 0\}$ *is a hereditary torsion class in $\mathcal{C}$ and the right adjoint of Q induces an equivalence between $\mathcal{C}'$ and the corresponding localisation of $\mathcal{C}$.*

Example 3.5. Take $\mathcal{C} = \mathbf{Ab}$ and take "torsion" to mean 2-torsion by declaring an abelian group M to be torsion if each element of M is annihilated by a power of 2.

Then the objects of the quotient category are the abelian groups D such that D has no 2-torsion and such that D has no non-trivial extension by a 2-torsion module.

To get from an arbitrary abelian group, C, to the corresponding object of the quotient category, first factor out the 2-torsion subgroup, τC, the subgroup consisting of all elements of C which are annihilated by some power of 2; set $C_1 = C/\tau C$ - a 2-torsionfree group. Let $E(C_1)$ denote the injective hull of C_1 (this will be a direct sum of $(p \neq 2)$-Prüfer groups and copies of $\mathbb{Q}$) and consider the factor group $E(C_1)/C_1$; consider the 2-torsion subgroup $\tau(E(C_1)/C_1)$ (clearly there will be no contribution from the Prüfer components) and let C_2 be its full inverse image in $E(C_1)$ - so $C_1 \leq C_2 \leq E(C_1)$ and $C_2/C_1 = \tau(E(C_1)/C_1)$ (thus C_2 is obtained by "making C_1 fully divisible by powers of 2"). The localisation functor takes C to C_2. For instance, if $C = \mathbb{Z}$ then $C_2 = \mathbb{Z}[\frac{1}{2}]$. Indeed this localisation functor is easily seen to be equivalent to tensoring with $\mathbb{Z}[\frac{1}{2}]$.

Say that τ (or the corresponding torsion class) is of **finite type** if the torsion class is generated as such by the finitely presented torsion objects. For example, if $\mathcal{C}$ is locally coherent and $\mathcal{S}$ is a Serre subcategory of $\mathcal{C}^{\mathrm{fp}}$ then $\overrightarrow{\mathcal{S}}$ is a torsion class of finite type.

Corollary 3.6. *If $\mathcal{C}$ is a locally coherent abelian category and τ is of finite type then $\mathcal{C}_\tau$ is locally coherent and has, for its class (up to isomorphism) of finitely presented objects, the localisations of finitely presented objects of $\mathcal{C}$: $(\mathcal{C}_\tau)^{\mathrm{fp}} = (\mathcal{C}^{\mathrm{fp}})_\tau = \{C_\tau : C \in \mathcal{C}^{\mathrm{fp}}\}$.*

3.4 Krull-Gabriel dimension

Let $\mathcal{C}$ be a locally coherent abelian (hence, [7, 2.4], Grothendieck) category; so the subcategory $\mathcal{C}^{\mathrm{fp}}$ of finitely presented=coherent objects is abelian.

Let $\mathcal{S}$ be the Serre subcategory of $\mathcal{C}^{\mathrm{fp}}$ generated by all the *finitely presented* simple functors[2]. This consists of all finitely presented objects of $\mathcal{C}$ of finite

[2]If we were to use *all* the simple functors then we would obtain Gabriel dimension [12].

length. Then $\overrightarrow{\mathcal{S}}$ is a finite type torsion class in $\mathcal{C}$ and we may form the localisation $\mathcal{C}_1 = \mathcal{C}/\overrightarrow{\mathcal{S}}$. By 3.6, $\mathcal{C}_1$ is locally coherent, so we can repeat the process, with $\mathcal{C}_1$ in place of $\mathcal{C}$, and obtain $\mathcal{C}_2$. Etc. Transfinitely. We obtain a sequence of locally coherent categories $\mathcal{C}_\alpha$ indexed by ordinals. By the last statement in 3.6 this sequence eventually must stabilise, either with the trivial category, or with a nontrivial category with no finitely presented simple object. In the first case, if α is the smallest ordinal such that $\mathcal{C}_\alpha = 0$ then we say that $\alpha - 1$ (or α if this is a limit ordinal) is the **Krull-Gabriel dimension** of $\mathcal{C}$, $\mathrm{KGdim}(\mathcal{C}) = \alpha$. In the second case the Krull-Gabriel dimension of $\mathcal{C}$ is undefined and we write $\mathrm{KGdim}(\mathcal{C}) = \infty$. If $\mathcal{C}$ is the functor category $(\text{mod-}R, \mathbf{Ab})$ where R is a ring or small preadditive category then we write $\mathrm{KG}(R)$ for $\mathrm{KGdim}(\text{mod-}R, \mathbf{Ab})$ (α above cannot be a limit ordinal in this case).

Since the duality of Section 2.3 takes finitely presented simple functors to finitely presented simple functors, $\mathrm{KG}(R)$ is right/left symmetric.

Corollary 3.7. $\mathrm{KGdim}\big((\text{mod-}R, \mathbf{Ab})\big) = \mathrm{KGdim}\big((R\text{-mod}, \mathbf{Ab})\big)$.

That there is some connection between this dimension and representation type of artin algebras can be seen in the following results.

Dimension 0 $\mathrm{KG}(R) = 0$ iff R is of finite representation type.

Dimension 1 This value is not attained by any artin algebra ([16, 3.6], [19, 11.4]).

Dimension 2 Algebras with Krull-Gabriel dimension 2 include the tame hereditary algebras ([11], see also [25] and [33]).

Dimension defined and > 2 The domestic string algebras Λ_n (see [6, 2.3, §4], [35, Thm. 1]) have $\mathrm{KG}(\Lambda_n) = n + 1$. It is conjectured that any domestic string algebra has finite Krull-Gabriel dimension.

Dimension ∞ Wild finite-dimensional algebras have Krull-Gabriel dimension ∞ (see [24, pp. 281-2] or [18, 8.15]).

Hereditary artin algebras In particular for hereditary artin algebras the possible representation types - finite, tame/domestic, wild - correspond to the values 0, 2, ∞ for Krull-Gabriel dimension.

String algebras/canonical algebras There are examples of domestic string algebras with any finite Krull-Gabriel dimension ≥ 2. Exactly what values of Krull-Gabriel dimension can occur for domestic (string) algebras is a subject of conjecture (see [29], also [27], [37]). On the other hand, non-domestic string algebras have Krull-Gabriel dimension ∞ (see [26], [36], and also [30]). Tubular algebras have Krull-Gabriel dimension ∞ (see [26], [15]).

Some group rings Puninski, Puninskaya and Toffalori [31] showed that the integral group ring of a nontrivial finite group has Krull-Gabriel dimension ∞. For group rings KG over a field they determined [32, 4.11] the value of Krull-Gabriel dimension (which turns out to be 0, 2 or ∞), except in the case that K has characteristic 2, does not contain a primitive cube root of 1 and G has a generalised quaternion group for a Sylow 2-subgroup.

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