DEFINABLE SUBCATEGORIES OF MAXIMAL COHEN-MACAULAY MODULES

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We investigate extensions to the class of maximal Cohen-Macaulay modules over Cohen-Macaulay rings from the viewpoint of definable subcategories. We pay particular attention to two definable classes that arise naturally from the definition of the Cohen-Macaulay property.

A comparison between these two definable subcategories is made, as well as a consideration of how well these classes reflect the properties of the maximal Cohen-Macaulay modules and relate to Hochster's balanced big Cohen-Macaulay modules. In doing so we explore homological and categorical properties of both classes.

A more detailed consideration is given to the role cotilting plays in this discussion. We show both these definable classes are cotilting and completely describe their cotilting structure, including properties of their corresponding cotilting modules. This approach yields further information about the classes and their closure properties.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction and background

1.1 Introduction

Over a commutative Noetherian local ring R, regular sequences are a tool that bridge the gap between homological and commutative algebra. One of the traditional invariants formed by regular sequences is that of *grade*: given an R-module M and an ideal $\mathfrak{a}$ such that $M \neq \mathfrak{a}M$, the grade of $\mathfrak{a}$ on M is the supremum of the lengths of all regular M-sequences contained in $\mathfrak{a}$, denoted $\operatorname{grade}(\mathfrak{a}, M)$. A classic result of Rees states that when M is a finitely generated R-module, the grade of a module can be measured by the non-vanishing of Ext-modules. More precisely there is an equality

$$\operatorname{grade}(\mathfrak{a}, M) = \inf\{i \geq 0 : \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \neq 0\}. \tag{1.1}$$

Over a local ring, the grade of the maximal ideal $\mathfrak{m}$ on a finitely generated module M has a particular name, the *depth* of M. A particularly notable class of modules, that will provide the starting point to the contents of the thesis, are the *maximal Cohen-Macaulay modules*. These are the finitely generated modules whose depth is equal to the Krull dimension of R. Over a Cohen-Macaulay ring the representation theory of the category $\operatorname{CM}(R)$ of maximal Cohen-Macaulay modules has been, and continues to be, extensively studied.

However, if one wishes to extend the concept of being maximal Cohen-Macaulay beyond finitely generated modules, some immediate obstacles start to arise. In general, the equality given in 1.1 fails to hold, while regular sequences start to become more unruly. For example, if one finds a maximal M-sequence for a finitely generated R-module M, then all other maximal M-sequences have the same length, hence it is enough to determine depth on one maximal M-sequence. This does not happen with non finitely generated modules. A consequence of these obstacles is that there is no definitive way to extend the property of being maximal Cohen-Macaulay to arbitrarily generated R-modules.

The aim of this thesis is to provide some extensions of the class of maximal Cohen-Macaulay modules from the viewpoint of *definable subcategories*, which are classes that are closed under direct limits, direct products and pure submodules. More specifically, we consider two classes that naturally arise from the class $\mathrm{CM}(R)$. These are

$$\varinjlim \mathrm{CM}(R),$$

the closure of $\mathrm{CM}(R)$ under direct limits, and what we write as

$$\mathrm{CohCM}(R) = \{M \in \mathrm{Mod}(R) : \mathrm{Ext}^i_R(R/\mathfrak{m}, M) = 0 \text{ for all } i < \dim R\},$$

which is the class formed by considering the invariant on the right hand side of 1.1. The finitely generated modules in both these classes are precisely the maximal Cohen-Macaulay modules, and we will prove that these classes are both definable, at least with minimal assumptions on the ring. Definable categories appear in several areas of representation theory, as well as in model theoretic algebra and we will encounter both these settings. The model theoretic viewpoint provides an alternate motivation for taking definable extensions of the class $\mathrm{CM}(R)$: one can form a topological space, the Ziegler spectrum, from the category of R-modules by taking as its underlying set the set of isomorphism classes of indecomposable pure injective R-modules, where a basis of closed sets is given by definable subcategories. In particular, any class of modules containing an indecomposable pure injective module will be visible in the Ziegler spectrum. Therefore in order to understand the part of the Ziegler spectrum that corresponds to a class one has to consider the definable subcategories containing the class, and consequently understanding definable subcategories that extend the class of maximal Cohen-Macaulay modules is necessary in determining how $\mathrm{CM}(R)$ appears in the Ziegler spectrum.

While the Ziegler spectrum provides a motivation for our study, it will appear infrequently and we will usually speak about definable subcategories in terms of the closure properties stated above. The structure of the thesis, and how these classes will be studied is as follows. The first chapter provides background material specific to the thesis. This includes the necessary background in commutative and homological algebra over commutative Noetherian local rings, as well as providing an introduction to definable categories and the Ziegler spectrum. As Cohen-Macaulay rings and modules are the central point of the thesis, an introduction to these classes is given, as well as the homological properties of $\mathrm{CM}(R)$ that are needed in subsequent chapters. Lastly, the classes of Gorenstein projective, flat and injective modules are introduced. A basic background in homological algebra and commutative algebra is assumed.

In the second chapter the classes $\varinjlim \mathrm{CM}(R)$ and $\mathrm{CohCM}(R)$ are introduced. We show that $\mathrm{CohCM}(R)$ is always definable, and that when R admits a canonical module so is $\varinjlim \mathrm{CM}(R)$, moreover this is the smallest definable subcategory containing $\mathrm{CM}(R)$. Necessary and sufficient conditions are given for when these two classes coincide. We then consider some of the representation theoretic properties of $\mathrm{CM}(R)$ and whether they extend to $\varinjlim \mathrm{CM}(R)$ and $\mathrm{CohCM}(R)$. We pay particular attention to the case when R admits a canonical module Ω, in which case the functor $\mathrm{Hom}_R(-, \Omega)$ is an endofunctor on $\mathrm{CM}(R)$ and provides a duality on this category. We show that if R is a complete Cohen-Macaulay ring then $\mathrm{Hom}_R(-, \Omega)$ is an endofunctor on both $\varinjlim \mathrm{CM}(R)$ and $\mathrm{CohCM}(R)$, at least partially extending the property from $\mathrm{CM}(R)$. We then consider these classes in relation to Hochster's *balanced big Cohen-Macaulay modules*, a class of modules that generalise the maximal Cohen-Macaulay modules through properties in commutative algebra. We then consider an advantage of considering $\varinjlim \mathrm{CM}(R)$ over $\mathrm{CohCM}(R)$ by looking at a generalisation of Knörrer periodicity for Cohen-Macaulay modules over hypersurfaces.

The third chapter looks more at the categorical properties of $\mathrm{CohCM}(R)$ and $\varinjlim \mathrm{CM}(R)$ in their own right rather than in relation to the class of maximal Cohen-Macaulay modules. We begin this chapter by introducing some background information on Kaplansky classes, covers, envelopes and cotorsion pairs. Using this background we are able to deduce the existence of cotorsion pairs containing $\varinjlim \mathrm{CM}(R)$ and $\mathrm{CohCM}(R)$ over any Cohen-Macaulay ring. This enables us to deduce a stronger result than the existence of covers, which holds for any definable subcategory. We then consider some dimensions with respect to $\mathrm{CohCM}(R)$ before turning our attention to further closure properties of the classes. In particular, we show that $\mathrm{CohCM}(R)$ is closed under injective hulls, so has enough injectives. We then turn our attention to inverse limits to show that over a complete local ring there is an inclusion $\varprojlim \mathrm{CM}(R) \subseteq \varinjlim \mathrm{CM}(R)$, which enables an alternative proof of the canonical dual being an endofunctor on $\varinjlim \mathrm{CM}(R)$. Moreover, we show that $\mathrm{CohCM}(R)$ is usually not closed under arbitrary inverse limits before considering certain inverse systems for which $\mathrm{CohCM}(R)$ contains the inverse limit. This leads to a discussion of Mittag-Leffler systems and stationary modules, which we consider for $\varinjlim \mathrm{CM}(R)$ over Gorenstein local rings.

In the fourth chapter we consider definable subcategories from a different perspective, namely that of cotilting (and to a lesser extent tilting). We show that both $\varinjlim \mathrm{CM}(R)$ and $\mathrm{CohCM}(R)$ are d-cotilting, where d is the Krull dimension of R. Moreover, using the classification of cotilting classes over commutative Noetherian rings, we provide a complete description of the cotilting structure for both these classes, including giving properties of the corresponding cotilting modules, such as $\varinjlim \mathrm{CM}(R)$ being the cotilting class for a balanced big Cohen-Macaulay module. Describing this structure enables us to give the minimal injective resolutions of modules in both $\varinjlim \mathrm{CM}(R)$ and $\mathrm{CohCM}(R)$, and as a corollary we are able to deduce minimal injective resolutions of Gorenstein flat modules over commutative Gorenstein rings. Since cotilting modules have particularly nice closure properties, we are able to use existing results to show $\varinjlim \mathrm{CM}(R)$ is closed under injective hulls. Moreover, implicit within this cotilting discussion is a classification of all d-cotilting classes containing $\mathrm{CM}(R)$; since cotilting classes are definable we are also able to compare $\mathrm{CohCM}(R)$ and $\varinjlim \mathrm{CM}(R)$ as definable subcategories and see how different they are, reinforcing the results obtained in Chapter 2.

Chapter five is focussed on the particular case when R is a one-dimensional Cohen-Macaulay ring. In this setting there is only one definable subcategory containing $\mathrm{CM}(R)$, which is $\varinjlim \mathrm{CM}(R) = \mathrm{CohCM}(R)$. There are several phenomena in this setting that do not appear in higher dimensions, such as this class being closed under inverse limits. Moreover, we pay particular attention to the class

$$\{M \in \mathrm{Mod}(R) : \mathrm{Hom}_R(k, M) = 0 = k \otimes_R M\},$$

which lies in $\varinjlim \mathrm{CM}(R)$. We show that this is a definable Grothendieck abelian category equipped with an internal hom and tensor product. Moreover, since the class $\varinjlim \mathrm{CM}(R)$ is closed under submodules, we are able to localise the class $\varinjlim \mathrm{CM}(R)$ with respect to the above class in a manner similar to that of a Serre subcategory of an abelian category.

1.2 Notation and preliminaries

Throughout we will assume some general facts from category theory, homological algebra and commutative algebra. More specific concepts will be introduced in the background section or when they become relevant to the thesis. To this end, if R is a ring we will let R-Mod (resp. Mod-R) denote the category of left (resp. right) R-modules. When R is commutative, we may use $\mathrm{Mod}(R)$ to denote either of these classes, although in general we will assume the ring acts on the left. The category of abelian groups will be denoted by **Ab**.

The term *exact category* we will mean an extension closed subcategory of an abelian category (this is compatible with the usual categorical definition by [15, A.1]). Such a category $\mathcal{E}$ inherits an exact structure from the abelian category $\mathcal{A}$, by saying that a short exact sequence is exact in $\mathcal{E}$ if it is also exact in $\mathcal{A}$. The exact sequences in an exact category $\mathcal{E}$ are sometimes referred to as *conflations*. Assuming $\mathcal{E}$ is an exact category, we say a morphism $L \longrightarrow M$ is an *admissible monomorphism* if there is a conflation $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ in $\mathcal{E}$. Conversely, say a morphism $M \longrightarrow N$ is an *admissible epimorphism* if it is the cokernel of an admissible monomorphism.

Definition 1.2.1. If $\mathcal{E}$ is an exact category, we will call an object I of $\mathcal{E}$ *injective* if the functor $\mathrm{Hom}_{\mathcal{E}}(-, I) : \mathcal{E} \longrightarrow \mathbf{Ab}$ is exact on conflations. Dually, we say an object P of $\mathcal{E}$ is *projective* if $\mathrm{Hom}_{\mathcal{E}}(P, -) : \mathcal{E} \longrightarrow \mathbf{Ab}$ is exact on conflations.

If R is any ring we will let R-Inj denote the full subcategory of R-Mod consisting of all injective left R-modules. We similarly alter the notation for the right modules and commutative cases and extend such notation to other classes of modules.

Recall that an exact category $\mathcal{E}$ has *enough injectives* if for every object X in $\mathcal{E}$ there is an admissible monomorphism $X \longrightarrow I$ with I an injective object in $\mathcal{E}$, while it has *enough projectives* if there is an admissible epimorphism $P \longrightarrow X$ with P a projective object in $\mathcal{E}$. In an exact category with enough injectives (resp. projectives) one can form *injective resolutions* (resp. *projective resolutions*) of an object X, that is an exact sequence

$$0 \longrightarrow X \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

where each E^i is an injective object in $\mathcal{E}$, and similarly on the left for projective resolutions. We will encounter more general resolutions in due course. If R is a ring, we will denote the class of left R-modules with injective dimension at most n by $\mathcal{I}_n$, and similarly $\mathcal{P}_n$ for projective dimension. Important classes of modules can be defined in terms of properties of their projective resolutions.

Definition 1.2.2. Let R be a ring and M a left R-module with projective resolution

$$\cdots \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0.$$

The image of f_i (which is the same as the kernel of f_{i-1}) is called the *i-th syzygy* of M, and will be denoted $\Omega^i(M)$. We say that M is FP_n if P_i is finitely generated for all $i \leq n$.

There is a dual notion using injective resolutions, called cosyzygies, that will be introduced later. Returning to the case as stated in the definition, it is clear that module is finitely generated if and only if it is FP_0, while a module is finitely presented if and only if M is FP_1. We will let R-mod denote the class of left R-modules that are FP_n for all $n < \omega$. In general the rings we will be considering will be Noetherian, over which the class R-mod coincides with the class of finitely generated modules which coincides with the class of finitely generated modules.

If $\mathcal{C}$ is a class of modules, we will let $\mathrm{Add}(\mathcal{C})$ denote the class of all R-modules that can be realised as direct summands of arbitrary direct sums of modules in $\mathcal{C}$. Similarly we will let $\mathrm{Prod}(\mathcal{C})$ denote the class of R-modules that can be realised as direct summands of direct products of elements of $\mathcal{C}$.

Throughout the thesis, we will regularly use the notions of direct and inverse systems and limits. Recall that a set I is *directed* if it is equipped with a reflexive transitive binary relation $\leq$ such that if $a, b \in I$ then there is a $c \in I$ such that $a \leq c$ and $b \leq c$.

Let $\mathcal{A}$ be an abelian category, and I a directed set. A *directed system* in $\mathcal{A}$ is a collection

$$(A_i, f_{ij} : A_i \longrightarrow A_j)_{i \leq j \in I}$$

of objects and morphisms in $\mathcal{A}$ such that whenever $i \leq j \leq k$ we have $f_{ik} = f_{jk} \circ f_{ij}$. The *direct limit* of this directed system is an object $\varinjlim_I A_i$ and collection of morphisms $\varphi_i : A_i \longrightarrow \varinjlim_I A_i$ with the following properties:

1. For all $i \leq j$ in I, the following diagram commutes:

$$\begin{tikzcd} A_i \arrow[r, "\varphi_i"] \arrow[d, "f_{ij}"] & \varinjlim A_i \\ A_j \arrow[ur, "\varphi_j"'] & \end{tikzcd}$$

2. If $(G, \{\psi_i : A_i \longrightarrow G\}_{i \in I})$ is another object and collection of morphisms such that $\psi_i = \psi_j \circ f_{ij}$ for all $i \leq j$ in I, then there is a unique morphism $\alpha : \varinjlim_I A_i \longrightarrow G$ in $\mathcal{A}$ such that the following diagram commutes:

$$\begin{tikzcd} \varinjlim_I A_I \arrow[rr, dashed, "\alpha"] & & G \\ & A_i \arrow[ul, "\varphi_i"'] \arrow[ur, "\psi_i"] \arrow[d, "f_{ij}"'] & \\ & A_j \arrow[uul, bend left, "\varphi_j"] \arrow[uur, bend right, "\psi_j"'] & \end{tikzcd}$$

Assume that $\mathcal{A}$ has all direct sums. Consider the inclusions $\varepsilon_i : A_i \longrightarrow \bigoplus_i A_i$ and the submodule L generated by all elements of the form $(\varepsilon_j \circ f_{ij} - \varepsilon_i)(a_i)$ for all $i \leq j$ and $a_i \in A_i$. Then the quotient $\bigoplus_i A_i/L$ is isomorphic to $\varinjlim_I A_i$.

The following is a classic result relating elements of R-Mod to FP_1, that is finitely presented, modules.

Lemma 1.2.3. [29, 2.5] Let R be a ring and M an R-module. Then M is a direct limit of a directed system of FP_1 modules. $\square$

The dual notion is that of an *inverse system* and an *inverse limit*. An inverse system is a collection $(A_i, f_{ij} : A_j \longrightarrow A_i)_{i \leq j \in I}$ such that if $i \leq j \leq k$ we have $f_{ik} = f_{ij} \circ f_{jk}$. The inverse limit of this system is an object $\varprojlim_I A_i$ and collection of morphisms $\{\gamma_i : \varprojlim_I A_i \longrightarrow A_i\}_{i \in I}$ such that

1. if $i \leq j$ then $\gamma_i = f_{ij} \circ \gamma_j$;
2. If $(G, \{\nu_i : G \longrightarrow A_i\}_I)$ is another collection such that $\nu_i = f_{ij}\nu_j$ for all $i \leq j$, then there is a unique map $\beta : G \longrightarrow \varprojlim_I A_i$ such that for all $i \leq j$ the following

diagram commutes

$$\begin{tikzcd}
G \arrow[rr, dashed, "\beta"] \arrow[dr, "\nu_i"] \arrow[ddr, bend right, "\nu_j"'] & & \varprojlim_I A_i \arrow[dl, "\gamma_i"'] \arrow[ddl, bend left, "\gamma_j"] \\
& A_i & \\
& A_j \arrow[u, "f_{ij}"] &
\end{tikzcd}$$

If $\mathcal{A}$ has all direct products, then the submodule of $\prod_I A_i$ consisting of all sequences (a_i) such that $a_i = f_{ij}(a_j)$ for all $i \leq j$ is isomorphic to $\varprojlim_I A_i$. The direct corresponding result to 1.2.3 is not true, that is there are rings with modules that are not realisable as an inverse limit of finitely generated modules. There is a similar result, due to Bergman, that is also of interest.

Lemma 1.2.4. [9, Theorem 2] Let R be a ring, then every left R module can be realised as an inverse limit of an inverse system of injective modules. $\square$

If $\mathcal{A}$ is an abelian category with direct limits and $\mathcal{C}$ is a category with direct limits, we say a functor $F : \mathcal{A} \longrightarrow \mathcal{C}$ *commutes* with (or *preserves*) direct limits if $F(\varinjlim_I A_i) = \varinjlim_I F(A_i)$ for any directed system $(A_i, f_{ij})_I$. Similarly we have functors that commute with inverse limits.

Example 1.2.5. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor, then F preserves (co)homology. Indeed, let $\mathcal{C}$ be a chain complex and $d^n : C^n \longrightarrow C^{n+1}$ be the differential. Applying F to the exact sequence $0 \longrightarrow \mathrm{Ker}(d^n) \longrightarrow C^n \longrightarrow \mathrm{Im}(d^n) \longrightarrow 0$ shows that $F(\mathrm{Ker}(d^n)) \simeq \mathrm{Ker}(F(d^n))$. Similarly, $F(\mathrm{Im}(d^n)) \simeq \mathrm{Im}(F(d^n))$. From the short exact sequence $0 \longrightarrow \mathrm{Im}(d^{n-1}) \longrightarrow \mathrm{Ker}(d^n) \longrightarrow H_n(\boldsymbol{C}) \longrightarrow 0$ we see that

$$F(H_n(\boldsymbol{C})) = F(\mathrm{Ker}(d^n))/F(\mathrm{Im}(d^{n-1})) \simeq \mathrm{Ker}(F(d^n))/\mathrm{Im}(F(d^{n-1})) = H_n(F\boldsymbol{C}).$$

If $\mathcal{A}$ satisfies the AB5 axiom, that is $\mathcal{A}$ has all direct sums and direct limits of exact sequences are exact, it follows that direct limits preserve homology.

Let us now recall some concepts from commutative algebra. If R is a commutative ring and M is an R-module, a prime $\mathfrak{p} \in \mathrm{Spec(R)}$ is an *associated prime* of M if there is an element $m \in M$ such that $\mathfrak{p} = \mathrm{Ann}_R(m)$. This is equivalent to M having a cyclic submodule isomorphic to $R/\mathfrak{p}$. We let $\mathrm{Ass}\, M$ denote the set of associated primes of

M. If R is Noetherian, then $\operatorname{Ass} M = \varnothing$ if and only if $M = 0$. We will also consider the *support* of M, which is

$$\operatorname{Supp} M = \{\mathfrak{p} \in \mathrm{Spec(R)} : M_{\mathfrak{p}} \neq 0\}.$$

Again, if $M \neq 0$ then $\operatorname{Supp} M \neq 0$. Proofs of these claims can be found in, for instance, [6].

1.3 The depth of a module

For this section, R will denote a commutative Noetherian ring. By the *dimension* of R, sometimes written $\dim R$, we will mean the Krull dimension; if M is an R-module we set

$$\dim M := \dim R/\mathrm{Ann}_R(M).$$

1.3.1 Local cohomology and Ext

For any R-module M and ideal $\mathfrak{a}$ of R, there is a natural identification $\mathrm{Hom}_R(R/\mathfrak{a}, M) = \{m \in M : \mathfrak{a}m = 0\}$. If $\mathfrak{a}^i m = 0$ then $\mathfrak{a}^j m = 0$ for all $j \geq i$ so one obtains a directed system $(\mathrm{Hom}_R(R/\mathfrak{a}^i, M), f_{i,j})_{i,j<\omega}$ where $f_{i,j}$ is just the inclusion $\mathrm{Hom}(R/\mathfrak{a}^i, M) \longrightarrow \mathrm{Hom}(R/\mathfrak{a}^j, M)$. Since this collection is directed, the direct limit of this system is the union of these submodules, which gives the submodule of M containing elements annihilated by some power of $\mathfrak{a}$.

Definition 1.3.1. With R, $\mathfrak{a}$ and M as above, we define the $\mathfrak{a}$-torsion functor, $\Gamma_{\mathfrak{a}} : \mathrm{Mod}(R) \longrightarrow \mathrm{Mod}(R)$ to be

$$\Gamma_a(M) = \varinjlim \mathrm{Hom}_R(R/\mathfrak{a}^t, M) \simeq \{m \in M : \exists t \geq 0 \text{ such that } m\mathfrak{a}^n = 0\}$$

on objects, and restriction on morphisms.

Example 1.3.2.

- Let $\mathfrak{a} \subset R$ be any ideal, then since $\mathfrak{a} \subset \mathrm{rad}(\mathfrak{a})$, it is clear that for any R-module M one has an inclusion $\Gamma_{\mathrm{rad}(\mathfrak{a})}(M) \subset \Gamma_{\mathfrak{a}}(M)$. However, there is a non-negative integer N such that $\mathrm{rad}(\mathfrak{a})^N \subset \mathfrak{a}$. Consequently if $m \in \Gamma_{\mathfrak{a}}(M)$, we know $m\mathfrak{a}^t = 0$ for some t, and thus $m\mathrm{rad}(\mathfrak{a})^{tN} = 0$ showing the reverse inclusion. Consequently the torsion function is invariant under radical, that is $\Gamma_{\mathfrak{a}}(-) = \Gamma_{\mathrm{rad}(\mathfrak{a})}(-)$.
- Suppose we are given two ideals $\mathfrak{a}$ and $\mathfrak{b}$. If $m \in \Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(M))$ for some R-module M, then there are non-negative integers s and t such that $m\mathfrak{a}^s\mathfrak{b}^t = 0$. It is then apparent that $m(\mathfrak{a}+\mathfrak{b})^{s+t} = 0$, so $m \in \Gamma_{\mathfrak{a}+\mathfrak{b}}(M)$. Conversely, if $m \in \Gamma_{\mathfrak{a}+\mathfrak{b}}(M)$, there is an N such that $m(\mathfrak{a}+\mathfrak{b})^N = 0$; in particular $m\mathfrak{b}^N = 0$ so $m \in \Gamma_{\mathfrak{b}}(M)$, yet it is also clear that $m\mathfrak{a}^N = 0$, so $m \in \Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(M))$. Therefore $\Gamma_{\mathfrak{a}} \circ \Gamma_{\mathfrak{b}} = \Gamma_{\mathfrak{a}+\mathfrak{b}}$.

It is shown in [11, Lemma 1.1.6] that the functor $\Gamma_{\mathfrak{a}}$ is left exact, and its right derived functors will be of great use.

Definition 1.3.3. Let R be as above, $\mathfrak{a}$ an ideal of R and M an R-module. We define the i^{th} *local cohomology of M with support in $\mathfrak{a}$*, denoted $H^i_{\mathfrak{a}}(M)$, to be

$$H^i_{\mathfrak{a}}(M) := R^i\Gamma_{\mathfrak{a}}(M).$$

As R-Mod satisfies the AB5 condition, taking homology preserves direct limits. Therefore the following result is immediate.

Lemma 1.3.4. With $R, \mathfrak{a}$ and M as in the above definition, for each $i \geq 0$ there is an isomorphism

$$H^i_{\mathfrak{a}}(M) \simeq \varinjlim_t \operatorname{Ext}^i_R(R/\mathfrak{a}^t, M).$$

Proof. If $0 \longrightarrow M \longrightarrow E^\bullet$ is an injective resolution of M, there are isomorphisms

$$\begin{aligned} H^i_{\mathfrak{a}}(M) = H_i(\Gamma_{\mathfrak{a}}(E^\bullet)) &\simeq H_i(\varinjlim_t \operatorname{Hom}_R(R/\mathfrak{a}^t, E^\bullet)) \\ &\simeq \varinjlim_t H_i(\operatorname{Hom}_R(R/\mathfrak{a}^t, E^\bullet)) = \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^t, M). \end{aligned}$$

□

We will not be particularly interested in the local cohomology modules themselves, but instead when they are, or are not, zero. Fortunately, in the case we will most often be considering, there are only a finite number of modules that need to be considered.

Definition 1.3.5. Let R be as above and $\mathfrak{a}$ an ideal of R. We define the *cohomological dimension of* $\mathfrak{a}$, denoted $\operatorname{cohd}(\mathfrak{a})$ to be the largest integer n such that there is an R-module M with $H^n_{\mathfrak{a}}(M) \neq 0$. If no such integer exists, we set $\operatorname{cohd}(\mathfrak{a}) = -\infty$.

There is a test module to determine cohomological dimension.

Proposition 1.3.6. [32, Theorem 9.6] With R and $\mathfrak{a}$ as above, $\operatorname{cohd}(\mathfrak{a}) = \sup\{n \geq 0 : H^n_{\mathfrak{a}}(R) \neq 0\}$. □

Usually, we will be interested in the case when R is a local ring with maximal ideal $\mathfrak{m}$. In this situation, there is a result due to Grothendieck that enables one to determine the cohomological dimension of $\mathfrak{m}$ immediately.

Theorem 1.3.7. [32, Theorem 9.3] Let $(R, \mathfrak{m}, k)$ be a local ring and M a finitely generated R-module. Then

$$\sup\{n \geq 0 : H^n_{\mathfrak{m}}(M) \neq 0\} = \dim M.$$

In particular, $\operatorname{cohd}(\mathfrak{m}) = \dim R$. □

There is not a global lower bound for the non-vanishing of local cohomology modules. Yet for each individual R-module, this lower bound is of note, and can be measured using Ext functors, as the following result shows.

Proposition 1.3.8. [32, Theorem 9.1] Let R be a Noetherian ring, $\mathfrak{a}$ an ideal of R and M an R-module. Then the following numbers coincide.

$$\inf\{n \geq 0 : H^n_{\mathfrak{a}}(M) \neq 0\}$$

$$\inf\{n \geq 0 : \operatorname{Ext}^n_R(R/\mathfrak{a}, M) \neq 0\}.$$

□

These lower bounds will be discussed in more depth in subsequent sections. For now, we will consider further properties of local cohomology that will be of use to us. First, we consider how the local cohomology functors relate to direct limits.

Proposition 1.3.9. [11, Theorem 3.4.10] Suppose R is as before and $\mathfrak{a}$ is an ideal of R. If $(M_i, f_{i,j})_{i.j\in I}$ is a directed system of R-modules, then for each $n \geq 0$ there are natural isomorphisms

$$\varinjlim_I H^n_{\mathfrak{a}}(M_i) \simeq H^n_{\mathfrak{a}}(\varinjlim_I M_i).$$

□

In the case that $(R, \mathfrak{m}, k)$ is a local ring of dimension n, then the top local cohomology functor $H^n_{\mathfrak{m}}(-)$ has a particularly nice form.

Lemma 1.3.10. [32, 9.7] If $(R, \mathfrak{m}, k)$ is a local ring with $\dim R = n$, then there is a natural isomorphism of R-modules

$$H^n_{\mathfrak{m}}(-) \simeq - \otimes_R H^n_{\mathfrak{m}}(R).$$

□

The above result holds in more generality, and this statement can be found at the citation. One particularly useful property of local cohomology is how it deals with a change of rings. Suppose $f : R \longrightarrow S$ is a homomorphism between Noetherian rings and let $\mathfrak{a} \subset R$ be an ideal. Let $\mathfrak{a}S$ denote the ideal of S generated by the image of the generators of $\mathfrak{a}$. If M is an S-module, we may view it as an R-module through f, which we denote by $M|_R$.

Theorem 1.3.11. [11, Theorem 4.2.1] With the set up given, for every $i \geq 0$ there is an isomorphism of R-modules

$$H^i_{\mathfrak{a}}(M|_R) \simeq H^i_{\mathfrak{a}S}(M)|_R.$$

$\square$

This theorem is sometimes known as the *independence theorem*, and we will refer to it as such.

1.3.2 Regular sequences and Rees's theorem

Throughout this section all rings will be assumed to be commutative and Noetherian. If R is such a ring and M is an R-module, we say that $x \in R$ is a *M-regular element* if $xm = 0$ for $m \in M$ implies $m = 0$. That is, x is a non-zerodivisor on M.

Definition 1.3.12. Let M be an R-module. Let $\boldsymbol{x} = x_1, \cdots, x_n$ be a sequence of elements in R, and consider the following two conditions:

1. x_i is a regular element of $M/(x_1, \cdots, x_{i-1})M$, for $1 \leq i \leq n$;
2. $M/\boldsymbol{x}M \neq 0$.

If both the above conditions hold, we say that $\boldsymbol{x}$ is an *M-regular sequence*, or simply an *M-sequence*, while if only the first condition holds we say $\boldsymbol{x}$ is a *weak M-regular sequence*, or simply a *weak M-sequence.*

In general, one may wish to consider regular sequences contained within a specific ideal $\mathfrak{a} \subset R$. For example, when R is a local ring, one often considers regular sequences contained within the maximal ideal $\mathfrak{m}$. In this situation, if M is a nonzero module then a weak M-sequence is just an M-sequence.

Example 1.3.13.

- A standard example of a regular sequence is the indeterminates $\boldsymbol{x} = x_1, \cdots, x_n$ in the polynomial ring $\mathbb{C}[\boldsymbol{x}]$. For each i, $\mathbb{C}[\boldsymbol{x}]/(x_1, \cdots, x_i)$ is an integral domain, so x_{i+1} is a regular element on it.
- Consider the ring $\mathbb{C}[x, y, z]$ and the sequence (x^2y, yz). This is not a regular sequence: indeed, the kernel of the map

$$\cdot yz : \mathbb{C}[x, y, z]/(x^2y) \longrightarrow \mathbb{C}[x, y, z]/(x^2y)$$

contains the non-zero element x^2.

- Over a non-local ring, a permutation of a regular sequence need not be regular. The following example can be found at [48, 5.1.2(2)]. If $R = k[x, y, z]$ with k a field, then $x, y(x-1), z(x-1)$ is an R-sequence, but $y(x-1), z(x-1), x$ is not.

Given any R-module M, we can construct M-sequences just by picking elements that satisfy the properties of the definition. In this way, one obtains an increasing chain of ideals $(x_1) \subset (x_1, x_2) \subset \cdots$. Since R is Noetherian this process will eventually terminate, meaning that there is an M-sequence $x_1, \cdots, x_n$ that cannot be extended.

Definition 1.3.14. Let M be an R-module. We say that an M-sequence $\boldsymbol{x}$ is *maximal* if it cannot be extended in length.

The above reasoning shows that every maximal M-sequence is finite. However, since the length of an M-sequence depends on the choice of elements, there is no reason to assume that all maximal M-sequences are of the same length. Indeed, in general one can find maximal M-sequences of different length. However, when M is finitely generated, a classical result of Rees shows that this phenomenon cannot happen.

Theorem 1.3.15 (Rees)**.** [12, Theorem 1.2.5] Let R be a Noetherian ring, M a finitely generated R-module and $\mathfrak{a}$ an ideal such that $\mathfrak{a}M \neq M$. Then all maximal M-sequences contained in $\mathfrak{a}$ have the same length, n, given by

$$n = \min\{i \geq 0 : \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \neq 0\}.$$

$\square$

We give a name to this invariant.

Definition 1.3.16. Let R be a Noetherian ring, M a finitely generated R-module and $\mathfrak{a}$ an ideal such that $\mathfrak{a}M \neq M$. We call the common maximal length of all M-sequences contained in $\mathfrak{a}$ the *grade of* $\mathfrak{a}$ *on* M, and denote it

$$\operatorname{grade}(\mathfrak{a}, M).$$

We have seen the formula for grade before, in 1.3.8, so we can rephrase the formula for grade given in 1.3.15 in terms of local cohomology.

Corollary 1.3.17. Let R, $\mathfrak{a}$ and M be as in 1.3.15, then

$$\operatorname{grade}(\mathfrak{a}, M) = \min\{i \geq 0 : \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \neq 0\} = \min\{i \geq 0 : H^i_{\mathfrak{a}}(M) \neq 0\}.$$

□

When $(R, \mathfrak{m}, k)$ is a local ring, the grade of $\mathfrak{m}$ on a finitely generated R-module is of sufficient importance to warrant its own name.

Definition 1.3.18. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a finitely generated R-module such that $\mathfrak{m}M \neq M$. Then the grade of $\mathfrak{m}$ on M is called the *depth* of M.

We restate 1.3.17 for this special case.

Corollary 1.3.19. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M an R-module such that $\mathfrak{m}M \neq M$. Then

$$\operatorname{depth} M = \min\{i \geq 0 : \operatorname{Ext}^i_R(k, M) \neq 0\} = \min\{i \geq 0 : H^i_{\mathfrak{m}}(M) \neq 0\}.$$

□

If M is a finitely generated R-module and $\mathfrak{a}$ is an ideal, we can combine 1.3.17 and the concept of cohomological dimension to see that $H^i_{\mathfrak{a}}(M) = 0$ whenever $i < \operatorname{grade}(\mathfrak{a}, M)$ and $i > \operatorname{cohd}(\mathfrak{a})$. This upper bound is not strict, since it is possible that $\sup\{n \geq 0 : H^n_{\mathfrak{a}}(M) \neq 0\} < \operatorname{cohd}(\mathfrak{a})$. In the case that R is a local ring, it is possible to refine this upper bound. The following result is due to Grothendieck.

Theorem 1.3.20. [12, Theorem 3.5.7] Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a finitely generated R-module of depth t and dimension n. Then

$$\sup\{i \geq 0 : H^i_{\mathfrak{m}}(M) \neq 0\} = n.$$

Consequently $H^i_{\mathfrak{m}}(M) = 0$ whenever $i < t$ and $i > n$. □

Therefore, if $(R, \mathfrak{m}, k)$ is a local ring, there is a chain of inequalities

$$\operatorname{depth} M \leq \dim M \leq \dim R.$$

We are now in a position to define the key notion of our study.

Definition 1.3.21. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. We say that a finitely generated R-module M is a *maximal Cohen-Macaulay module*, or simply a *Cohen-Macaulay module*, if the above inequalities are equalities, that is

$$\operatorname{depth} M = \dim R.$$

We say that R is a *Cohen-Macaulay ring* if it is a Cohen-Macaulay module over itself.

1.3.3 Types of Cohen-Macaulay ring

The class of Cohen-Macaulay local rings is quite broad and contains other classes of local rings, some of which will play a larger role than others. Before considering these different classes, we need a concept from commutative algebra.

Definition 1.3.22. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of Krull dimension d. A collection of elements $\boldsymbol{y} = y_1, \cdots, y_d$ is a *system of parameters* if $\sqrt{(\boldsymbol{y})} = \mathfrak{m}$.

It is known that every Noetherian local ring admits a system of parameters, and that every R-sequence is part of a system of parameters. A more precise statement can be found at [12, Prop. 1.2.12].

Let us now assume that $(R, \mathfrak{m}, k)$ is a Cohen-Macaulay ring of Krull dimension d. Since R is Cohen-Macaulay, every maximal R-sequence has length d, and is therefore just a system of parameters.

Definition 1.3.23. A Cohen-Macaulay ring is *regular* if its maximal ideal is generated by a system of parameters. Such a system of parameters is called a *regular system of parameters*.

Equivalently, a local ring is regular if and only if its Krull dimension is equal to the minimal number of generators of the maximal ideal.

A field is an example of a regular local ring. Moreover, if A is a regular local ring, so is $A[[x_1, \cdots, x_n]]$ for any $n \geq 0$. If p is a prime number, then $\mathbb{Z}_p$, the p-adic integers, is a regular local ring. Indeed, a discrete valuation ring is just a one-dimensional regular local ring. If K is a (sufficiently nice) field, then the localisation of $K[x_1, \cdots, x_n]$ at any prime ideal is a regular local ring.

A more homological classification of regular local rings is due to Auslander, Buchsbaum and Serre.

Theorem 1.3.24. [12, Theorem 2.2.7] Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. The following are equivalent.

1. R is regular;
2. proj dim $M < \infty$ for every R-module M;
3. proj dim $k < \infty$.

$\square$

In fact, the proof of the above statement shows that the global dimension of a regular local ring is equal to its Krull dimension. Thus, if R is regular with $\dim R = d$, it is clear that $\mathrm{Ext}_R^{d+i}(k, M) = 0$ for all R-modules M and $i > 0$. In particular, since every regular local ring is Cohen-Macaulay, we see that $\sup\{i \geq 0 : \mathrm{Ext}_R^i(k, R) \neq 0\} = d$.

Lemma 1.3.25. [12, Prop. 3.1.14] Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a finitely generated R-module. Then

$$\text{inj dim } M = \sup\{i \geq 0 : \mathrm{Ext}_R^i(k, M) \neq 0\}.$$

$\square$

Consequently, it is clear that every regular local ring has finite injective dimension over itself, and inj dim ${}_R R = \dim R$. However, these are not the only rings that satisfy this property.

Definition 1.3.26. We say that a Noetherian local ring $(R, \mathfrak{m}, k)$ is *Gorenstein* if it has finite injective dimension over itself.

It is not immediately obvious from the definition that a Gorenstein ring is Cohen-Macaulay. It follows, however, from the following result.

Proposition 1.3.27. [12, Theorem 3.1.17] Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a finitely generated R-module of finite injective dimension. Then

$$\dim M \leq \text{inj dim } M = \operatorname{depth} R.$$

$\square$

Indeed, since the depth of a finitely generated module is always less than its dimension, we see that $\dim R = \operatorname{depth} R$ whenever R is Gorenstein. Let us consider some examples of Gorenstein rings.

Example 1.3.28.

1. Let $(A, \mathfrak{n})$ be a regular local ring and $x \in \mathfrak{n}$, we call the quotient $R = A/(x)$ a *hypersurface ring*. Such a ring is always Gorenstein.

   Proof. Since A is a regular local ring it is an integral domain by [12, Prop. 2.2.3] and thus any element $x \in \mathfrak{n}$ is A-regular. Consequently, by [12, Cor. 3.1.15], there is an equality
   $$\text{inj dim}_R(R) = \text{inj dim}_A(A) - 1.$$
   Since we chose A to be regular, it is of finite injective dimension and therefore so is R. □

2. If $\boldsymbol{x}$ is an R-sequence and R is a Noetherian local ring, then $R/(\boldsymbol{x})$ is Gorenstein if and only if R is Gorenstein.
3. Not all Gorenstein rings are regular rings: for example, if k is a field and $R = k[[x,y]]/(x^2)$, then R is Gorenstein as it is a hypersurface. It is clear that $\dim R = 1$ but the maximal ideal is generated by two elements. It is also not a domain so cannot be regular.
4. Not all Cohen-Macaulay rings are Gorenstein. Let $G \subset \mathrm{GL}_2(\mathbb{C})$ be a finite subgroup and let it act on the $\mathbb{C}$-vector space with basis x and y. One can extend this to an action on $\mathbb{C}[[x,y]]$ through multiplication. The invariant ring $\mathbb{C}[[x,y]]^G$ is Cohen-Macaulay by the Hochster-Roberts theorem [12, 6.5.1], but is Gorenstein if and only if $G \subset \mathrm{SL}_2(\mathbb{C})$ due to a result of Watanabe [12, 6.4.10]. The classical Kleinian, or Du Val, singularities can be realised as such invariant rings. More details can be found in [51, Chapter 10] and [35, Chapters 5,6].

We therefore have an hierarchy of Noetherian local rings

$$\text{regular rings} \implies \text{Gorenstein rings} \implies \text{Cohen-Macaulay rings.}$$

One class of local rings is missing from the above chain, namely *complete intersection rings*, which are Gorenstein but not necessarily regular, but we will not need these.

Recall that if R is a local ring, the $\mathfrak{m}$-adic completion of an R-module M is the inverse limit

$$\hat{M} = \varprojlim_{t} M/\mathfrak{m}^t M$$

of the inverse system formed by the inverse system $(M/\mathfrak{m}^i M, f_{ij} :)_{i \leq j < \omega}$, where

$$f_{ij} : M/\mathfrak{m}^j M \longrightarrow M/\mathfrak{m}^i M \qquad x + \mathfrak{m}^j M \mapsto x + \mathfrak{m}^i M.$$

We say that the ring R is *complete* if $\hat{R} = R$ when viewed as an R-module. One result that will be of some use is the following, which is known as *Cohen's Structure Theorem*.

Proposition 1.3.29. [12, Theorem A.21] Let $(R, \mathfrak{m})$ be a complete Noetherian local ring. Then there is a ring R_0, that is either a field or a discrete valuation ring, such that $R \simeq R_0[[x_1, \cdots, x_n]]/I$ for some ideal I.

1.4 Injective modules over commutative Noetherian rings

1.4.1 Canonical modules and the trivial extension

Throughout this section, $(R, \mathfrak{m}, k)$ will denote a Cohen-Macaulay local ring. We will let $\mathrm{CM}(R)$ denote the full subcategory of $\mathrm{mod}(R)$ consisting of all Cohen-Macaulay modules. This category is very well understood, and much more information than we will need can be found in the monographs [35] and [51].

If the Krull dimension of R is d, it is clear that a finitely generated R-module M is in $\mathrm{CM}(R)$ if and only if $\mathrm{Ext}^i_R(k, M) = 0$ for all $i < d$ and $\mathrm{Ext}^d_R(k, M) \neq 0$. Since $\mathrm{Ext}^d_R(k, M)$ is a non-zero finite dimensional k-vector space, the minimal possible k-dimension of $\mathrm{Ext}^d_R(k, M)$, as M runs over $\mathrm{CM}(R)$, is one, although it is possible that this value is never attained. When it is, the modules that satisfy this minimal dimension have a particular name.

Definition 1.4.1. Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay ring. A module $\Omega \in \mathrm{CM}(R)$ is said to be a *canonical module* if

$$\dim_k \mathrm{Ext}^d_R(k, M) = 1.$$

Before considering existence conditions for a canonical module, the following shows that should a canonical module exist, it is unique up to isomorphism.

Theorem 1.4.2. [12, Theorem 3.3.4.(b)] Let R be a Cohen-Macaulay ring. If Ω_1 and Ω_2 are canonical modules, then $\Omega_1 \simeq \Omega_2$. □

Let us now turn our attention to when a canonical module exists. Necessary and sufficient conditions were independently determined by Foxby and Reiten in 1972.

Theorem 1.4.3. [27, Theorem 4.1][41, Theorem (3)] Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay ring. The following are equivalent.

1. R admits a canonical module,
2. R is the homomorphic image of a Gorenstein local ring.

□

Corollary 1.4.4. Let R be a complete local ring. Then R admits a canonical module.

Proof. If R is complete then Cohen's structure theorem states that $R \simeq R_0[[x_1, \cdots, x_n]]/I$, where I is some ideal and R_0 is a regular local ring or field. Consequently R is isomorphic to the quotient of a regular, and thus Gorenstein, local ring. □

From the preceding theorem, it is automatic that every Gorenstein ring admits a canonical module. Moreover, the nature of the canonical module is uniquely determined in this case.

Theorem 1.4.5. [12, Theorem 3.3.7(a)] Let R be a Cohen-Macaulay local ring admitting a canonical module Ω. Then $\Omega \simeq R$ if and only if R is Gorenstein. □

If we assume that R admits a canonical module, the functor $\mathrm{Hom}_R(-, \Omega)$ plays a significant role on the category $\mathrm{CM}(R)$. The following result highlights some of the properties of this functor.

Theorem 1.4.6. [12, Theorem 3.3.10] Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring with canonical module Ω.

1. $\mathrm{Hom}_R(-, \Omega)$ is an endofunctor on $\mathrm{CM}(R)$;
2. $\mathrm{Ext}^i_R(M, \Omega) = 0$ for all $i > 0$ and $M \in \mathrm{CM}(R)$;
3. $\mathrm{End}_R(\Omega) \simeq R$;
4. For every $M \in \mathrm{CM}(R)$, the canonical morphism $M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, \Omega), \Omega)$ is an isomorphism.

□

Beyond these properties, $\mathrm{Hom}_R(-, \Omega)$ plays a key role in understanding the underlying structure of $\mathrm{CM}(R)$ due to its relationship to the Auslander-Reiten translate. We will not cover any AR theory, but [51] provides a good background to it in the context of Cohen-Macaulay rings.

One implication of the proof of 1.4.3 is immediate once one introduces the construction known as a trivial extension. This construction can be done in great generality in an abelian category, but we will only consider the case when R is Cohen-Macaulay.

Definition 1.4.7. Let R be as above and M an R-module. The *trivial extension* of R by M is the ring $R \ltimes M$ whose underlying abelian group is $R \oplus M$ and whose ring action is given by

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1).$$

It is clear that since R is commutative, so is $R \ltimes M$. Since M becomes an ideal of $R \ltimes \Omega$, the above process is sometimes called *idealisation*. Since R and $R \ltimes M$ are intimately related, we can see how their ring theoretic properties compare.

Proposition 1.4.8. [1, Theorems 3.1, 3.2] Let R be as above and M a finitely generated R-module. Let $\mathfrak{a}$ be an ideal of R and $N \subset M$ a submodule. Then $\mathfrak{a} \ltimes N$ is an ideal of $R \ltimes M$ if and only if $\mathfrak{a}M \subset N$. If $\mathfrak{a} \ltimes N$ is such an ideal, then $(R \ltimes M)/(\mathfrak{a} \ltimes N) \simeq (R/\mathfrak{a}) \ltimes (M/N)$. In particular the following hold:

1. The prime ideals of $R \ltimes M$ are of the form $\mathfrak{p} \ltimes M$ for $\mathfrak{p} \in \mathrm{Spec(R)}$. In particular $\mathrm{hp}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p} \ltimes M)$ and the dimensions of R and $R \ltimes M$ are equal.
2. The maximal ideals of $R \ltimes M$ are of the form $\mathfrak{m} \ltimes M$, for maximal ideals $\mathfrak{m}$ of R.

Thus, if R is local, so is $R \ltimes M$ and they have the same residue fields. □

The above theorem is much more dependent on the properties of R than of M, but it is precisely these properties that can endow $R \ltimes M$ with a richer structure.

Proposition 1.4.9. [1, Theorem 4.13] With R as above and M an R-module, there is an equality

$$\mathrm{depth}\,_{R \ltimes \Omega} R \ltimes \Omega = \mathrm{depth}\,_R (R \oplus \Omega) = \min\{\mathrm{depth}\,_R R, \mathrm{depth}\,_R M\}.$$

Consequently, when $M \in \mathrm{CM}(R)$, $R \ltimes \Omega$ is a Cohen-Macaulay ring. □

However, the most special (and important) case is when one considers the trivial extension of R by its canonical module Ω. The previous proposition tells us that $R \ltimes \Omega$ is at least a Cohen-Macaulay ring, but, as we have seen, there are stronger classes of ring that we may obtain.

Theorem 1.4.10. [41, Theorem 7] Let R be a Cohen-Macaulay ring with canonical module Ω. If M is an R-module, then $R \ltimes M$ is Gorenstein if and only if $M \simeq \Omega$. □

There are two clear ring homomorphisms $i : R \longrightarrow R \ltimes M$ and $\pi : R \ltimes M \longrightarrow R$ given by inclusion and projection. The corresponding restriction of scalars functors are $U : \mathrm{Mod}(R \ltimes M) \longrightarrow \mathrm{Mod}(R)$ and $Z : \mathrm{Mod}(R) \longrightarrow \mathrm{Mod}(R \ltimes M)$. This naming convention is found in [26], where the U stands for *underlying* because the functor U simply forgets the action of M, while Z stands for *zero* since M acts as zero on the image of Z. There are certain properties of both U and Z that are of particular use to us.

Proposition 1.4.11. Let R, U and Z be as above.

1. Both U and Z are exact functors;
2. Both U and Z commute with direct and inverse limits;
3. $UZ = \mathrm{Id}$ on $\mathrm{Mod}(R)$.
4. If M is a finitely generated R-module, then $\operatorname{depth}_{R\ltimes\Omega} Z(M) = \operatorname{depth}_R M$.

Proof. The proofs of the first two claims can be found in [26], and all references are from there. The exactness of U and Z is Corollary 1.6. In Proposition 1.3 it is shown that U is both a left and right adjoint, so preserves both direct and inverse limits. The same is shown for Z. The third claim is clear by the definitions of U and Z as restrictions of scalars. The final claim is Lemma 5.1.(v). □

Consequently, if we consider $R \ltimes \Omega$ where Ω is the canonical module of R, then $R \ltimes \Omega$ is a Gorenstein ring and we have a functor $Z : \mathrm{CM}(R) \longrightarrow \mathrm{CM}(R \ltimes \Omega)$. Since $R \ltimes \Omega$ is a Gorenstein ring, it is its own canonical module. Consequently we can consider the functor $\mathrm{Hom}_{R\ltimes\Omega}(-, R \ltimes \Omega)$. It is shown in the proof of [26, Theorem 5.4] that there are isomorphisms

$$\mathrm{Hom}_{R\ltimes\Omega}(R, R \ltimes \Omega) \simeq \Omega \oplus \mathrm{ann}_R(\Omega),$$

but since Ω is a faithful module, we have $\mathrm{Hom}_{R\ltimes\Omega}(R, R \ltimes \Omega) \simeq \Omega$, viewed as a submodule of $R \ltimes \Omega$.

Remark. The trivial extension can be made in a much more general setting and [26] provides a good resource for this. Consequently many of the properties of the functors U and Z given in the previous result extend to a more general setup. However, we will only be concerned with the case as stated above.

1.4.2 Injective modules over commutative Noetherian rings

If $(R, \mathfrak{m}, k)$ is a Cohen-Macaulay ring with a canonical module there is a direct relationship between the canonical dual $\mathrm{Hom}_R(-, \Omega)$ and the local cohomology modules

$H^i_{\mathfrak{m}}(-)$, which was first noted by Grothendieck. Before going into this in detail, we will need to look at injective modules over commutative Noetherian rings. To begin with, we will let R be any ring and increase the assumptions on R incrementally until we get to Cohen-Macaulay.

Definition 1.4.12. Let M be an R-module. We say that an injective module E is an *injective envelope* of M, and write $E(M)$, if for every submodule $I \subset E$ with $I \cap M = 0$ we have $I = 0$.

Theorem 1.4.13. [24, Theorem 3.1.14] Let R be any ring. Every R-module has an injective envelope which is unique up to isomorphism. □

Injective modules enable an extension of the Hom-Tensor adjunction to the Ext and Tor functors, to obtain the following relations.

Lemma 1.4.14. [29, Lemma 2.16] Let R and S be rings and A a right R-module.

1. Let ${}_RB_S$ be an R-S-bimodule and C_S a right S-module. Then there is a natural isomorphism
$$\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \simeq \mathrm{Hom}_S(A \otimes_R B, C).$$
2. Let $n < \omega$, ${}_RB_S$ be an R-S-bimodule and C_S an injective right S-module. Then
$$\mathrm{Ext}^n_R(A, \mathrm{Hom}_S(B, C)) \simeq \mathrm{Hom}_S(\mathrm{Tor}^R_n(A, B), C).$$
3. If A is finitely presented, ${}_SB_R$ an S-R-bimodule and C an injective left S-module. Then
$$A \otimes_R \mathrm{Hom}_S(B, C) \simeq \mathrm{Hom}_S(\mathrm{Hom}_R(A, B), C).$$
4. Let $m < \omega$ and assume A is FP_{m+1}. Moreover, let $B \in S$-Mod-R and C an injective left S-module. Then
$$\mathrm{Tor}^R_i(A, \mathrm{Hom}_S(B, C)) \simeq \mathrm{Hom}_S(\mathrm{Ext}^i_R(A, B), C)$$
for each $i \leq m$.

□

Let us now assume that R is commutative and Noetherian. The category of injective R-modules is completely understood due to the classification results of E. Matlis.

Theorem 1.4.15. [24, Theorem 3.3.7] The following properties hold:

1. For every $\mathfrak{p} \in \mathrm{Spec(R)}$, the injective module $E(R/\mathfrak{p})$ is indecomposable.
2. If E is an indecomposable injective module, then there is a prime $\mathfrak{p} \in \mathrm{Spec(R)}$ such that $E \simeq E(R/\mathfrak{p})$.

Having described the objects, we can also describe the morphisms between injective modules.

Theorem 1.4.16. [24, Theorem 3.3.8] Let R be as above and $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec(R)}$.

1. $E(R/\mathfrak{p}) \simeq E(R/\mathfrak{q})$ if and only if $\mathfrak{p} = \mathfrak{q}$.
2. Ass $E(R/\mathfrak{p}) = \{\mathfrak{p}\}$.
3. $\mathrm{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{q})) \neq 0$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$.

□

Understanding these indecomposable injective modules is sufficient to provide to complete description of all injective R-modules.

Theorem 1.4.17. [24, Theorem 3.3.10] Let R be a commutative Noetherian ring. If E is an injective R-module, then

$$E \simeq \bigoplus_{\mathfrak{p} \in \mathrm{Spec(R)}} E(R/\mathfrak{p})^{(\mu_{\mathfrak{p}})},$$

where $\mu_{\mathfrak{p}} = \dim_{k(\mathfrak{p})} \mathrm{Hom}_R(k(\mathfrak{p}), E_{\mathfrak{p}})$. □

Moreover, we can use the above result to give the precise structure of a module's injective envelope.

Corollary 1.4.18. [24, Cor. 3.3.11] Let M be an R-module, then

$$E(M) \simeq \bigoplus_{\mathfrak{p} \in \mathrm{Spec(R)}} E(R/\mathfrak{p})^{(\mu_{\mathfrak{p},M})},$$

and

$$\mu_{\mathfrak{p},M} = \dim_{k(\mathfrak{p})} \mathrm{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}}) = \dim_{k(\mathfrak{p})} \mathrm{Hom}_R(R/\mathfrak{p}, M)_{\mathfrak{p}}.$$

□

Given any R-module M and its minimal injective resolution $0 \longrightarrow M \longrightarrow E^{\bullet}$, each E^i has a direct sum decomposition as above. For each $\mathfrak{p} \in \mathrm{Spec(R)}$ we let $\mu_i(\mathfrak{p}, M)$

denote the cardinality of the summands of E^i isomorphic to $E(R/\mathfrak{p})$, and call $\mu_i(\mathfrak{p}, M)$ the *Bass invariants*. It is possible to give an explicit value to each Bass invariant for a given R-module M.

Lemma 1.4.19. [24, Theorem 9.24] If M is an R-module and $\mathfrak{p} \in \mathrm{Spec(R)}$, then

$$\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) = \dim_{k(\mathfrak{p})} \mathrm{Ext}^i_R(R/\mathfrak{p}, M)_\mathfrak{p}.$$

□

Definition 1.4.20. If R is any ring and E is an injective R-module, we say E is an *injective cogenerator* if $\mathrm{Hom}_R(M, E) = 0$ if and only if $M \simeq 0$. More generally, if $\mathcal{E}$ is an exact category we say $E \in \mathcal{E}$ is an *injective cogenerator* if $\mathrm{Hom}_\mathcal{E}(M, E) = 0$ if and only if $M \simeq 0$.

Lemma 1.4.21. Let $\mathrm{mSpec}(R)$ denote the maximal spectrum of R, then

$$\bigoplus_{\mathfrak{m} \in \mathrm{mSpec}(R)} E(R/\mathfrak{m})$$

is an injective cogenerator in $\mathrm{Mod}(R)$.

Proof. Let $E = \bigoplus_{\mathrm{mSpec}(R)} E(R/\mathfrak{m})$ and suppose M is a non-zero R-module with $0 \neq x \in M$. Then there is an isomorphism $R/\mathrm{Ann}_R(x) \simeq Rx$ through multiplication by x. Since $\mathrm{Ann}_R(x)$ is an ideal of R, there is a maximal ideal $\mathfrak{m} \in \mathrm{mSpec}(R)$ such that $\mathrm{Ann}_R(x) \subseteq \mathfrak{m}$, which yields a non-zero morphism $Rx \longrightarrow R/\mathfrak{m}$ which extends to a non-zero $g : Rx \longrightarrow E(R/\mathfrak{m}) \longrightarrow E$. Since E is injective, there is a morphism $f : M \longrightarrow E$ such that $f| = g$; in particular f is nonzero. □

Remark. We have already seen that if R is a Cohen-Macaulay ring with a canonical module Ω, then Ω is an injective cogenerator in $\mathrm{CM}(R)$.

The injective modules play an important role in the context of local cohomology.

Lemma 1.4.22. Let R be a commutative Noetherian ring and $\mathfrak{a}$ an ideal. The injective R-modules are $\Gamma_\mathfrak{a}$-acyclic, that is $H^i_\mathfrak{a}(E) = 0$ for all injective modules E and $i > 0$.

Proof. Directly from 1.3.4 there are isomorphisms

$$H^i_\mathfrak{a}(M) \simeq \varinjlim_t \mathrm{Ext}^i_R(R/\mathfrak{a}^t, M)$$

for every $i \geq 0$ and $M \in \mathrm{Mod}(R)$. It is clear that if $i > 0$ and M is injective one has $\mathrm{Ext}^i_R(-, M) = 0$, so the result is clear. □

Matlis duality and Grothendieck local duality

Let us now assume that $(R, \mathfrak{m}, k)$ is a commutative Noetherian local ring. Since $\mathfrak{m}$ is the unique maximal ideal of R, 1.4.21 shows that $E(k)$ is an injective cogenerator in $\mathrm{Mod}(R)$.

Definition 1.4.23. Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring. We call the injective module $E(k)$ the *Matlis module*. For an R-module M, we let $M^\vee := \mathrm{Hom}_R(M, E(k))$ denote the *Matlis dual* of M.

For any R-module M, there is a canonical homomorphism $M \longrightarrow M^{\vee\vee}$. When this map is an isomorphism, we say that M is *Matlis reflexive*.

Theorem 1.4.24. [24, Theorem 3.4.1, 3.4.7] Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring. Let $\mathcal{A}(R)$ denote the class of Artinian modules.
1. For every R-module M, the map $M \longrightarrow M^{\vee\vee}$ is injective.
2. $R^\vee \simeq E(k)$ and $E(k)^\vee \simeq R$.
3. If $M \in \mathrm{mod}(R)$, then $M^\vee \in \mathcal{A}(R)$ and M is Matlis reflexive.
4. If $N \in \mathcal{A}(R)$, then $N^\vee \in \mathrm{mod}(R)$ and N is Matlis reflexive.
5. $(-)^\vee : \mathrm{mod}(R) \longrightarrow \mathcal{A}(R)$ is an anti-equivalence of categories.

□

The assumption of completeness is necessary for both finitely generated and Artinian modules to be reflexive, as well as the duality between these classes. One does not need completeness for the third and fourth statements of the theorem. Proofs of the above statements, as well as more details about Matlis duality, including necessary and sufficient conditions for a module to be Matlis reflexive, can be found in [24, §3.4].

Having introduced Matlis duality, we are in position to introduce the previously mentioned connection between the canonical module and local cohomology. The following result is known as *Grothendieck local duality*, although we may refer to it simply as local duality. Due to its significance, we will include the proof which can be found at the citation.

Theorem 1.4.25. [12, Theorem 3.5.8] Let $(R, \mathfrak{m}, k)$ be a complete Cohen-Macaulay ring of Krull dimension d, whose canonical module is Ω. Then for every R-module M

and all integers i there is an isomorphism

$$\mathrm{Ext}^i_R(M,\Omega) \simeq \mathrm{Hom}_R(H^{d-i}_{\mathfrak{m}}(M), E(k)).$$

Moreover, when M is a finitely generated there is an isomorphism

$$H^i_{\mathfrak{m}}(M) \simeq \mathrm{Hom}_R(\mathrm{Ext}^{d-i}_R(M,\Omega), E(k)).$$

□

Proof. For $i \geq 0$ define the functor $T^i(-) = \mathrm{Hom}_R(H^{d-i}_{\mathfrak{m}}(-), E(k))$. For $i = 0$ we obtain a contravariant left exact functor that sends direct sums to direct products. Thus there is an R-module C such that $T^0(-) \simeq \mathrm{Hom}(-, C)$, where $C \simeq T^0(R)$. The collection $\{T^i(-) : i \geq 0\}$ form a δ-functor, so if each $T^i(F)$ vanishes on free modules F it follows that the T^i are the right derived functors of T^0. It suffices to show that $H^i_{\mathfrak{m}}(F)$=0 for all $i < d$ and every free module F, but since R is Cohen-Macaulay and $H^i_{\mathfrak{m}}(-)$ preserves direct sums, we see that this is true. Therefore for each R-module M and $i \geq 0$ there are isomorphisms

$$\mathrm{Hom}_R(H^i_{\mathfrak{m}}(M), E(k)) \simeq \mathrm{Ext}^{d-i}_R(M, C).$$

Now

$$H^i_{\mathfrak{m}}(k) \simeq \begin{cases} k & \text{for } i = 0, \\ 0 & \text{for } i > 0, \end{cases}$$

so

$$\mathrm{Ext}^i_R(k, C) \simeq \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$$

It follows from the definition and uniqueness of the Canonical module that $C \simeq \Omega$. The second isomorphism, when M is finitely generated, follows from the properties of the Matlis dual. □

1.5 Extending depth and the failure of Rees's theorem

Initially assume that R is a commutative Noetherian ring. As stated in 1.3.17, for a finitely generated R-module M we can measure the grade of an ideal $\mathfrak{a}$ on M by considering the homological invariant given in 1.3.8. Moreover, we assumed there was a non-degeneracy condition on the modules by assuming that $\mathfrak{a}M \neq M$. In defining both of these invariants, there was, in fact, no need to restrict to finitely generated R-modules. That is, one could easily extend the notion of grade, by an ideal $\mathfrak{a}$, to any R-module such that $\mathfrak{a}M \neq M$. Similarly, the invariant given in 1.3.8 can be considered for any R-module, even those with $\mathfrak{a}M = M$. Let us formalise these.

Definition 1.5.1. Let R be a Noetherian ring and $\mathfrak{a}$ an ideal. If M is an R-module such that $\mathfrak{a}M \neq M$, define the *grade* of $\mathfrak{a}$ on M to be the supremum of all maximal M-sequences contained in $\mathfrak{a}$, and denote it $\operatorname{grade}(\mathfrak{a}, M)$. When $(R, \mathfrak{m}, k)$ is local, let $\operatorname{depth} M$ denote the grade of $\mathfrak{m}$ on M.

As for the finitely generated case, for any R-module M such that $\mathfrak{a}M \neq M$ we see $0 \leq \operatorname{grade}(\mathfrak{a}, M) < \infty$, since we assumed that R was Noetherian.

Definition 1.5.2. Let R be a Noetherian ring and $\mathfrak{a}$ an ideal. For an R-module M, define the *Ext-grade* of $\mathfrak{a}$ on M, which we denote $\text{E-gr}(\mathfrak{a}, M)$, via

$$\begin{aligned} \text{E-gr}(\mathfrak{a}, M) &= \inf\{n \geq 0 : \operatorname{Ext}^n_R(R/\mathfrak{a}, M) \neq 0\} \\ &= \inf\{n \geq 0 : H^n_{\mathfrak{a}}(M) \neq 0\}. \end{aligned}$$

When $(R, \mathfrak{m}, k)$ is local, we write $\text{E-dp}(M)$ for $\text{E-gr}(\mathfrak{m}, M)$.

For a finitely generated R-module M we have seen that $\operatorname{grade}(\mathfrak{a}, M) = \text{E-gr}(\mathfrak{a}, M)$, but, in general, this no longer holds when one removes the assumption that M is finitely generated (even if grade is defined). The following example is due to Strooker.

Example 1.5.3. [48, p. 91] Set $R = k[[x, y]]$ where k is a field and define

$$M = \bigoplus_{f \in \mathfrak{m}} R/(f).$$

It is obvious that the depth of M is zero, since no element of $\mathfrak{m}$ is regular on M. Since R is a domain, for every $f \in \mathfrak{m}$ the ideal (f) is isomorphic to R via the map $R \xrightarrow{\cdot f} (f)$. By applying $\mathrm{Hom}_R(k, -)$ to the short exact sequence $0 \longrightarrow (f) \longrightarrow R \longrightarrow R/(f) \longrightarrow 0$ we see that $\mathrm{Hom}_R(k, R/(f)) = 0$. Since

$$\mathrm{Hom}_R(k, M) \simeq \bigoplus_{f \in \mathfrak{m}} \mathrm{Hom}_R(k, R/(f)),$$

it follows $\text{E-dp}(M) > 0$. Consequently Rees's theorem does not fully extend from the finitely generated situation.

There is, however, a relationship between grade and Ext-grade, and a criterion is known for when these two quantities are equal.

Proposition 1.5.4. [48, 5.3.7, 5.3.8] Let $\mathfrak{a}$ be an ideal of R and M an R-module such that $\mathfrak{a}M \neq M$.

1. $\mathrm{grade}(\mathfrak{a}, M) \leq \text{E-gr}(\mathfrak{a}, M)$.
2. The following are equivalent
   (a) $\mathrm{grade}(\mathfrak{a}, M) = \text{E-gr}(\mathfrak{a}, M) < \infty$;
   (b) There is an M-sequence $\boldsymbol{x} = x_1, \cdots, x_n \in \mathfrak{a}$ and an element $z \in M/\mathfrak{a}M$ such that $\mathfrak{a}z = 0$.

□

The clear advantage of Ext-grade over grade is that it is applicable to every R-module; as we will see, there are non-trivial modules that do not satisfy the condition $\mathfrak{a}M \neq M$.

Let us introduce a complementary notion to Ext-depth.

Definition 1.5.5. Let R be a Noetherian ring and $\mathfrak{a} \subset R$ an ideal. For an R-module M, define the *Tor-cograde* of $\mathfrak{a}$ on M, denoted $\text{T-cogr}(\mathfrak{a}, M)$, to be

$$\text{T-cogr}(\mathfrak{a}, M) = \inf\{n \geq 0 : \mathrm{Tor}_n^R(R/\mathfrak{a}, M) \neq 0\}.$$

When $(R, \mathfrak{m}, k)$ is local, we set $\text{T-codp}(M) = \text{T-cogr}(\mathfrak{m}, M)$.

If $(R, \mathfrak{m}, k)$ is a local ring, we can use Matlis duality to relate Ext-grade and Tor-cograde.

Lemma 1.5.6. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M an R-module. Then

$$\text{E-gr}(\mathfrak{a}, M) = \text{T-cogr}(\mathfrak{a}, M^\vee),$$

and

$$\text{T-cogr}(\mathfrak{a}, M) = \text{E-gr}(\mathfrak{a}, M^\vee)$$

.

Proof. Since $E(k)$ is an injective cogenerator, for each $i \geq 0$ we have

$$\operatorname{Ext}^i_R(R/\mathfrak{a}, M) = 0 \iff \operatorname{Ext}^i_R(R/\mathfrak{a}, M)^\vee = 0 \iff \operatorname{Tor}^R_i(R/\mathfrak{a}, M^\vee) = 0,$$

where the last isomorphism is from 1.4.14(4) as $R/\mathfrak{a}$ is finitely presented and R is Noetherian. The second equality follows from 1.4.14(2), using the same argument. □

In particular, we have equalities

$$\text{E-dp}(M) = \text{T-codp}(M^\vee) \qquad \text{and} \qquad \text{T-codp}(M) = \text{E-dp}(M^\vee)$$

We do not require the ring to be complete, since we are only using the property that $E(k)$ is an injective cogenerator. There is a close relation between the Ext-grade and the Tor-cograde of an R-module.

Proposition 1.5.7. [48, Cor. 6.1.8] Let R be a commutative Noetherian ring and $\mathfrak{a}$ an ideal. Suppose $r_1, \cdots, r_n$ generate $\mathfrak{a}$ and let M be an R-module. Then $\text{E-gr}(\mathfrak{a}, M) < \infty$ if and only if $\text{T-cogr}(\mathfrak{a}, M) < \infty$, in which case

$$\text{E-gr}(\mathfrak{a}, M) + \text{T-cogr}(\mathfrak{a}, M) \leq n.$$

□

Corollary 1.5.8. Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring and M an R-module. Then

$$\text{E-dp}(M) + \text{T-codp}(M) \leq \dim R.$$

Proof. Let $\boldsymbol{y}$ be a system of parameters for $\mathfrak{m}$, so $(\boldsymbol{y})$ is generated by exactly $\dim R$ elements. By 1.3.2 there is an isomorphism of functors $\Gamma_\mathfrak{m} \simeq \Gamma_{(\boldsymbol{y})}$, and therefore by 1.3.8 we have $\text{E-dp}(M) = \text{E-gr}((\boldsymbol{y}), M)$. Using 1.5.6 one can show that $\text{T-dp}(M) = \text{T-codp}((\boldsymbol{y}), M)$. Applying the proposition gives the result. □

Consequently if $(R, \mathfrak{m}, k)$ is local with $\dim R = d$, we see that if $\mathrm{Ext}^i_R(k, M) = 0$ for all $i \leq d$, then $\text{E-dp}(M) = \infty$. The zero module is an example of an infinite depth module, but, as we will see, there are many more which are nontrivial.

The following result shows how grade behaves with respect to short exact sequences. We call this result the *depth lemma*.

Lemma 1.5.9. Let R be a commutative Noetherian ring and $\mathfrak{a}$ an ideal. If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is a short exact sequence of R-modules, then

$$\text{E-gr}(\mathfrak{a}, M) \geq \min\{\text{E-gr}(\mathfrak{a}, L), \text{E-gr}(\mathfrak{a}, N)\},$$

$$\text{E-gr}(\mathfrak{a}, L) \geq \min\{\text{E-gr}(\mathfrak{a}, M), \text{E-gr}(\mathfrak{a}, N) + 1\},$$

$$\text{E-gr}(\mathfrak{a}, N) \geq \min\{\text{E-gr}(\mathfrak{a}, L) - 1, \text{E-gr}(\mathfrak{a}, M)\}.$$

Proof. We prove the first one and the other two are similar. Applying $\mathrm{Hom}_R(R/\mathfrak{a}, -)$ to the short exact sequence in the statement gives, for each $i \geq 0$,

$$\cdots \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{a}, L) \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{a}, N) \longrightarrow \cdots$$

If $\mathrm{Ext}^i_R(R/\mathfrak{a}, M) \neq 0$, then one of the sandwiching terms must be non-zero as well. $\square$

1.6 Definable subcategories

Before considering the consequences of the previous section, we introduce two notions that will provide motivation for this study, that of purity and definable subcategories. For this section we will assume that R is any ring.

Definition 1.6.1. Let M be a right R-module and L a submodule of M. We say that L is a *pure submodule* of M if for every left R-module T the sequence $0 \longrightarrow L \otimes_R T \longrightarrow M \otimes_R T$ is exact. We say that a short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is *pure exact* if L is a pure submodule of M. A morphism $M \longrightarrow N \longrightarrow 0$ is a *pure epimorphism* if it arises from a pure exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$.

Definition 1.6.2. Let P be an R-module. We say that P is *pure injective* if for every pure exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ of R-modules

$$0 \longrightarrow \mathrm{Hom}_R(N, P) \longrightarrow \mathrm{Hom}_R(M, P) \longrightarrow \mathrm{Hom}_R(L, P) \longrightarrow 0$$

is exact.

Clearly the class of injective R-modules is contained within the class of pure injective R-modules. Pure injective modules are plentiful.

Proposition 1.6.3. [39, 4.3.9] Let M be a right R-module and $E \in$ Mod-R an injective module. Then $\mathrm{Hom}_R(M, E)$ is a pure-injective left R-module. □

There are several important properties that determine, and are unique to, pure injective R-modules. If R is an S-algebra, say that a right R-module N is a *dual module* if there is an injective cogenerator E for Mod-S such that $N \simeq \mathrm{Hom}_S(M, E)$ for a left R-module M. In this case say that $N = M^d$ is the *dual module* of M.

Proposition 1.6.4. [29, Theorem 2.27] Let R be a ring and M an R-module. The following are equivalent:

1. M is pure-injective,
2. Any pure-exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ splits,
3. The canonical embedding $M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ splits,

4. M is a direct summand of a module of a dual module.

$\square$

Another application of pure-injective modules is how they interact with direct limits. Recall there are isomorphisms

$$\operatorname{Hom}_R(\varinjlim_I M_i, N) \simeq \varprojlim_I \operatorname{Hom}_R(M_i, N) \tag{1.2}$$

and

$$\operatorname{Hom}_R(M, \varprojlim_J N_j) \simeq \varprojlim_J \operatorname{Hom}_R(M, N_j) \tag{1.3}$$

for appropriate directed and inverse systems $\{M_i\}_I$ and $\{N_j\}_J$. However, these isomorphisms do not, in general, extend to Ext-functors. We will give an example where these fail for each of the above isomorphisms.

Example 1.6.5.

1. Let R be a non-perfect commutative local ring. Since R is not perfect, we can find a non-free flat module F. Write $F = \varinjlim_I F_i$ with each F_i a finitely presented free module. Since F is not projective, there is an R-module M such that $\operatorname{Ext}^1_R(F, M) \neq 0$. If the isomorphism in 1.2 extended, we would have
$$\operatorname{Ext}^1_R(F, M) \simeq \operatorname{Ext}^1_R(\varinjlim_I F_i, M) \simeq \varprojlim_I \operatorname{Ext}^1_R(F_i, M) = 0$$
as each F_i is projective. But by choice of F this does not happen, so said isomorphism does not extend in general.
2. In [9, Theorem 2] it is shown that over any ring every module can be written as an inverse limit of injective modules. Let R be a d-dimensional Cohen-Macaulay ring, expressed as $\varprojlim_I E_i$ where each E_i is injective and $d > 0$. Since R is Cohen-Macaulay, $\operatorname{Ext}^d_R(k, R) \neq 0$, but if the isomorphism in 1.3 extended, we would have
$$\operatorname{Ext}^d_R(k, R) \simeq \operatorname{Ext}^d_R(k, \varprojlim_I E_i) \simeq \varprojlim_I \operatorname{Ext}^d_R(k, E_i) = 0$$
as each E_i is injective. But this is not true, so the isomorphism does not extend in general.

There is no real way to remedy the failure of the second isomorphism, but the first one can be resolved, using pure-injective R-modules. The following is originally due to Auslander.

Lemma 1.6.6. [29, Lemma 6.28] Let R be a ring and N a pure injective R-module. Then for any directed system $(M_i, f_{i,j})_I$ and $n \geq 0$ there are isomorphisms

$$\mathrm{Ext}^n_R(\varinjlim_I M_i, N) \simeq \varprojlim_I \mathrm{Ext}^n_R(M_i, N).$$

□

For each $n \geq 0$ and directed system $(M_i, f_{i,j})_I$ there is a canonical homomorphism $\theta^n_N : \mathrm{Ext}^n_R(\varinjlim_I M_i, N) \longrightarrow \varprojlim \mathrm{Ext}^n_R(M_i, N)$. It is shown in [29, Lemma 6.29] that if θ^1_N is an isomorphism for every such directed system then N is pure-injective. On the other hand, if M is pure injective then θ^n_N is an isomorphism for any directed system for the above lemma. Consequently an R-module N is pure injective if and only if θ^1_N is an isomorphism, in which case θ^n_N is an isomorphism for all $n \geq 0$.

The module M chosen in the first of the above examples can never be chosen to be pure-injective. This is because every pure-injective module is a cotorsion module, that is $\mathrm{Ext}^1_R(F, N) = 0$ for all flat modules whenever N is cotorsion.

There is another interaction between direct limits and Ext functors that is well known.

Lemma 1.6.7. [29, Lemma 6.6] Let R be a Noetherian ring and M a finitely generated R-module. Then for every $n \geq 0$ and directed system $(N_i, f_{i,j})_I$ there is an isomorphism

$$\mathrm{Ext}^n_R(M, \varinjlim_I N_i) \simeq \varinjlim_I \mathrm{Ext}^n_R(M, N_i).$$

□

1.6.1 Definable subcategories

Definable subcategories are closely related to pure-injective modules.

Definition 1.6.8. Let $\mathcal{D}$ be a class of right R-modules. We say that $\mathcal{D}$ is a *definable subcategory* of $\mathrm{Mod}(R)$ if it is closed under direct products, direct limits and pure submodules.

For certain classes, it can be easier to determine whether they are definable than by checking each part of the above definition.

Definition 1.6.9. [39, 10.2.1] A functor $F : \text{mod-}R \longrightarrow \mathbf{Ab}$ is *finitely presented* if and only if there is a morphism $f : A \longrightarrow B$ in mod-R such that $F \simeq \text{Coker}\,(f, -) : \text{Hom}_R(B, -) \longrightarrow \text{Hom}_R(A, -)$, where $(f, -) = \text{Hom}_R(f, -)$.

Any functor $F : \text{mod-}R \longrightarrow \mathbf{Ab}$ can be extended uniquely to a functor $\text{Mod-}R \longrightarrow \mathbf{Ab}$ that commutes with direct limit: if M is an R-module, it can be expressed as the direct limit of a directed system $(M_i, f_{i,j})_{i,j\in I}$ in mod-R. Applying F to this directed system gives a directed system $(F(M_i), F(f_{i,j}))_{i,j\in I}$ of abelian groups. We let $\overrightarrow{F} : \text{Mod-}R \longrightarrow \mathbf{Ab}$ be given by $\overrightarrow{F}(M) = \varinjlim_I F(M_i)$, where M is given by the described directed system. In the case that the functor F is the restriction of a functor that preserves direct limits $\overrightarrow{F}(\varinjlim M_i) \simeq \varinjlim F(M_i)$. Moreover, the functor $\overrightarrow{F}$ always preserves direct limits and is unique, that is, it is independent on the choice of the directed system representing an R-module. A proof of this can be found at [39, Prop. 10.2.41].

Extending functors along direct limits is a good way to obtain definable categories. Moreover, it can help determine whether or not a class of modules is definable.

Proposition 1.6.10. [39, Cor. 10.2.32] A subcategory $\mathcal{D}$ of Mod-R is definable if and only if there is a set $\{F_i : \text{mod-}R \longrightarrow \mathbf{Ab}\}_{i\in I}$ of finitely presented functors such that

$$\mathcal{D} = \{M \in \text{Mod-}R : \overrightarrow{F_i}(M) = 0 \text{ for all } i \in I\}.$$

□

Let us give some examples of finitely presented functors and, in doing so, some definable subcategories.

Proposition 1.6.11. [39, Theorem 10.2.35, 10.2.36] Let R be a ring.

1. If M is a left R-module, the functor $\text{Ext}^n_R(M, -) : R\text{-mod} \longrightarrow \mathbf{Ab}$ is finitely presented if M is FP_{n+1}.
2. If N is a right R-module, the functor $\text{Tor}^R_n(N, -) : R\text{-mod} \longrightarrow \mathbf{Ab}$ is finitely presented if N is FP_{n+1}.

□

Corollary 1.6.12. If R is Noetherian, the class of injective right R-modules is definable.

Proof. If R is right Noetherian, R/I is a finitely presented right R-module for each right ideal I. Consequently each functor $\mathrm{Ext}^1_R(R/I,-)$ is finitely presented. The injective modules are precisely the modules that vanish on the set

$$\{\mathrm{Ext}^1_R(R/I,-) : I \text{ a right ideal of } R\}$$

of finitely presented functors. □

The converse to the above corollary is also true, as shown in [39, Theorem 3.4.28].

If $\mathcal{C}$ is a class of R-modules, there is a definable subcategory generated by $\mathcal{C}$, which we denote by $\langle \mathcal{C} \rangle$, which is the intersection of all definable subcategories containing $\mathcal{C}$. If one considers the class $\varinjlim \mathcal{C}$, the direct limit closure of $\mathcal{C}$, it is clear that $\varinjlim \mathcal{C} \subseteq \langle \mathcal{C} \rangle$ since definable subcategories are direct limit closed. However, in general there is no reason for $\varinjlim \mathcal{C}$ to be definable, since there is no reason for it to be closed under products. In particular, every class $\mathcal{C}$ of finitely presented modules generates a definable subcategory of $\mathrm{Mod}(R)$. In this situation it is possible to determine whether or not $\varinjlim \mathcal{C}$ is definable.

Definition 1.6.13. If $\mathcal{C}$ is an additive category, a subcategory $\mathcal{D}$ is *covariantly finite* in $\mathcal{C}$ if for every $C \in \mathcal{C}$ there is a morphism $f : C \longrightarrow D$ with $D \in \mathcal{D}$ such that for every $f' : C \longrightarrow D'$ with $D' \in \mathcal{D}$ there is a morphism $h : D \longrightarrow D'$ such that $f' = hf$.

Proposition 1.6.14. [39, Cor. 3.4.37] Let $\mathcal{C} \subset \text{mod-}R$ be a class of finitely presented R-modules that is closed under finite direct sums. Then $\langle \mathcal{C} \rangle = \varinjlim \mathcal{C}$ if and only if $\mathcal{C}$ is covariantly finite in mod-R. □

Example 1.6.15. If R is a left-coherent ring, that is every finitely generated left ideal of R is finitely presented, then the category proj-R of finitely generated projective right R-modules is covariantly finite (see [39, 3.4.41]). Consequently $\varinjlim \text{proj-}R = \text{Flat-}R$ is definable. A class of functors that defines the flat R-modules is $\{\mathrm{Tor}^R_1(-,R/I) : I \text{ a finitely presented left ideal of } R\}$.

Let $\mathcal{D} \subset \text{Mod-}R$ be a definable subcategory and consider the character dual $(-)^+ := \mathrm{Hom}_R(-,\mathbb{Q}/\mathbb{Z})$. There is a definable subcategory of R-Mod, which we denote $\mathcal{D}^d$ characterised by the properties that $M \in \mathcal{D} \iff M^+ \in \mathcal{D}^d$ and $\mathcal{D}^{dd} \simeq \mathcal{D}$ (see

[39, §3.4.2] for more details). We call this definable subcategory the *dual definable subcategory* of $\mathcal{D}$. One can replace $\mathbb{Q}/\mathbb{Z}$ with any injective cogenerator and obtain the same result. In particular, over a local ring we will often use the Matlis duality to consider dual definable subcategories.

Example 1.6.16. Let R be a right Noetherian ring, so Inj-R, the class of injective right R-modules, is definable. The dual definable subcategory of Inj-R is the category R-Flat of flat left R-modules. This can be seen by using the functorial description of both these categories and the relations in 1.4.14.

1.6.2 The Ziegler spectrum

We will now consider how pure injective modules and definable subcategories relate to each other. Given any ring R, there is only a set of indecomposable pure-injective R-modules up to isomorphism: [39, 4.3.38] tells us that there at most $2^{\operatorname{card}(R)+\aleph_0}$ such isomorphism classes. We denote this set by pinj_R. One forms a topological space out of pinj_R as follows: for every definable subcategory $\mathcal{D}$ of Mod-R, let

$$\mathcal{D} \cap \operatorname{pinj}_R = \{M \in \operatorname{pinj}_R : M \in \mathcal{D}\},$$

and set the closed sets of the topology to be of the form $\mathcal{D} \cap \operatorname{pinj}_R$.

Theorem 1.6.17. [39, Theorem 5.1.1] The above closed sets form a topology on pinj_R.

Definition 1.6.18. We call the space pinj_R with the above topology the *Ziegler spectrum* of R and denote it by Zg_R.

Consequently, given any class of modules $\mathcal{C}$ there is a corresponding closed subset of Zg_R containing its indecomposable pure injectives, namely $\operatorname{pinj}_R \cap \langle \mathcal{C} \rangle$. One may wonder whether many definable subcategories can yield the same closed subset of Zg_R. As the following result shows, this is not possible.

Lemma 1.6.19. [39, 5.1.5] If $\mathcal{D} \neq 0$ is a definable subcategory of Mod-R, then $\mathcal{D} \cap \operatorname{pinj}_R \neq \varnothing$. Moreover, if $\mathcal{D}'$ is another definable subcategory, then $\mathcal{D} = \mathcal{D}'$ if and only if $\mathcal{D} \cap \operatorname{pinj}_R = \mathcal{D}' \cap \operatorname{pinj}_R$.

The above theorem shows how one can obtain a closed subset of Zg_R from a definable subcategory of Mod-R; one can also go in the other direction. If $\mathcal{X}$ is a closed subset of Zg_R, consider the R-module

$$M_{\mathcal{X}} := \bigoplus_{X \in \mathcal{X}} X$$

and let $\langle M_{\mathcal{X}} \rangle$ be the definable subcategory of Mod-R generated by M. By the above lemma, if $\mathcal{X}_1$ and $\mathcal{X}_2$ are closed subsets of Zg_R, we have $\langle M_{\mathcal{X}_1} \rangle = \langle M_{\mathcal{X}_2} \rangle$ if and only if $\mathcal{X}_1 = \mathcal{X}_2$. In particular, the following result shows that $\langle M_{\mathcal{X}} \rangle \cap \mathrm{pinj}_R = \mathcal{X}$, hence we have a bijection between closed subsets of the Ziegler spectrum of R and definable subcategories of Mod-R.

Lemma 1.6.20. [39, Cor. 5.1.4] If $\mathcal{X}$ is a definable subcategory of Mod-R, then $\mathcal{X}$ is generated by the indecomposable pure injectives in it.

1.7 Gorenstein modules

We will now introduce the different types of Gorenstein modules, some of which will play a key role in the thesis. To begin with we do this in most generality, by letting R denote any ring.

Definition 1.7.1. Let $\mathcal{F}$ and $\mathcal{G}$ be two classes of R-modules. We say that a complex

$$\boldsymbol{F} : \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

of modules in $\mathcal{F}$ is $\mathrm{Hom}(-, \mathcal{G})$ *exact* if for every $G \in \mathcal{G}$ the complex $\mathrm{Hom}_R(\boldsymbol{F}, G)$ is exact. We similarly define $\mathrm{Hom}_R(\mathcal{G}, -)$ *exact* and corresponding notions for the tensor product functor.

Definition 1.7.2. Let M be an R-module. We say that M is *Gorenstein injective* if there is a $\mathrm{Hom}_R(R\text{-Inj}, -)$ exact exact sequence

$$\cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

of injective R-modules such that $M = \mathrm{Ker}(E^0 \longrightarrow E^1)$.

There are also corresponding notions of Gorenstein projective and Gorenstein flat, which we now introduce.

Definition 1.7.3. Let M be an R-module. We say that M is *Gorenstein projective* if there is a $\mathrm{Hom}_R(-, R\text{-Proj})$ exact exact sequence

$$\cdots P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

of projective R-modules such that $M = \mathrm{Ker}(P^0 \longrightarrow P^1)$.

Definition 1.7.4. Let M be an R-module. We say that M is *Gorenstein flat* if there is a $\text{Inj-}R \otimes_R -$ exact exact sequence

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

of flat R-modules such that $M = \mathrm{Ker}(F^0 \longrightarrow F^1)$.

In general, we will be most interested in Gorenstein flat modules, and the other classes will only really be used in relation to these. Over particular classes of rings, it is possible to give a more intuitive definition of Gorenstein flat.

Definition 1.7.5. Let R be a Noetherian ring. We say that R is an *Iwanaga-Gorenstein* ring, or simply a *Gorenstein* ring, if the injective dimension of ${}_RR$ is finite. We say a Gorenstein ring R is *n-Gorenstein* if inj dim ${}_RR = n$.

Remark. If R is a commutative Noetherian local ring, it is immediate from the definitions that R is Iwanaga-Gorenstein if and only if it is Gorenstein as defined in 1.3.26. Indeed, from the discussions on injective dimension in 1.3.27 it is clear that if R is a d-dimensional Gorenstein local ring it is d-Gorenstein.

One vital property about Gorenstein rings is that modules of finite injective and projective dimension have an upper bound on these dimensions.

Theorem 1.7.6. [24, 9.1.19] If R is n-Gorenstein, then the following are equivalent for a left R-module.

1. inj dim $M < \infty$;
2. proj dim $M < \infty$;
3. flat dim $M < \infty$;
4. inj dim $M \leq n$;
5. proj dim $M \leq n$;
6. flat dim $M \leq n$;

Recall that $\mathcal{I}_n$ denotes the class of modules of injective dimension at most n, and similarly for $\mathcal{P}_n$. The above theorem therefore shows that $\mathcal{I}_{<\infty} = \mathcal{I}_n = \mathcal{P}_n = \mathcal{P}_{<\infty}$ over any Gorenstein ring and using this theorem it is easier to provide more intuitive descriptions of the above classes of Gorenstein modules.

Theorem 1.7.7. [24, Theorem 10.3.8] Let R be an n-Gorenstein ring and M an R-module. The following are equivalent.

1. M is Gorenstein flat;
2. There is a directed system $(M_i, f_{i,j})_I$, where each M_i is a finitely presented Gorenstein projective R-module such that $M = \varinjlim_I M_i$.
3. $\mathrm{Tor}_i^R(E, M) = 0$ for all $i \geq 1$ and injective right R-modules E.
4. The dual module M^d is a Gorenstein injective right R-module.
5. $\mathrm{Tor}_1^R(L, M) = 0$ for all $L \in \mathcal{I}_{<\infty}$.
6. $\mathrm{Tor}_i^R(L, M) = 0$ for all $L \in \mathcal{I}_{<\infty}$.

□

There is a converse to 1.7.7.4.

Corollary 1.7.8. [24, 10.3.9] If R is (n-)Gorenstein and M is a Gorenstein injective right (left) R-module, then M^+ is a Gorenstein flat left (right) R-module. □

Corollary 1.7.9. [24, 10.3.10] If R is Gorenstein, then every Gorenstein projective module is Gorenstein flat.

If we now move to the situation where $(R, \mathfrak{m}, k)$ is a Cohen-Macaulay ring, we will see that we have already encountered some of these classes.

Lemma 1.7.10. [24, 10.2.7] If R is a Cohen-Macaulay ring, then every finitely generated Gorenstein projective module is Cohen-Macaulay. □

Over a commutative Gorenstein local ring more can be said.

Lemma 1.7.11. [24, 11.5.4] Let R be a commutative Gorenstein local ring, then a finitely generated R-module is Gorenstein projective if and only if it is Cohen-Macaulay.

Let us now give alternative descriptions of Gorenstein injective and projective modules.

Proposition 1.7.12. [24, 11.2.2] If R is n-Gorenstein, the following are equivalent for any R-module M.

1. M is Gorenstein injective;
2. $\mathrm{Ext}^i(L, M) = 0$ for all $L \in \mathcal{I}_n$ and $i \geq 1$;
3. $\mathrm{Ext}^1(L, M) = 0$ for all $L \in \mathcal{I}_n$;
4. $\mathrm{Ext}^i(E, M) = 0$ for all injective R-modules E and $i \geq 1$.

Proposition 1.7.13. [24, 11.5.3] If R is n-Gorenstein, the following are equivalent for any R-module M.

1. M is Gorenstein projective;
2. $\mathrm{Ext}^i(M, L) = 0$ for all $L \in \mathcal{I}_n$ and $i \geq 1$;
3. $\mathrm{Ext}^1(M, L) = 0$ for all $L \in \mathcal{I}_n$;

Moreover, if M is finitely generated this is equivalent to the following

4. $\mathrm{Ext}^i(M, P) = 0$ for all projective R-modules P and $i \geq 1$.

It is clear that over a Gorenstein ring, the class of flat modules is fully contained within the Gorenstein flat modules, and similarly for the injective and projective cases.

Chapter 2

Definable classes of Cohen-Macaulay modules

Unless stated otherwise, throughout this chapter $(R, \mathfrak{m}, k)$ will denote a Cohen-Macaulay ring of Krull dimension d. As previously discussed, if one removes the assumption of finite generation from the definition of maximal Cohen-Macaulay, the equivalence between the length of regular sequences and Ext-depth fails, and this provides an obstacle to a uniform definition of maximal Cohen-Macaulay in the whole module category. We consider extensions of the class $\mathrm{CM}(R)$ from the perspective of definable subcategories, one by extending the Ext-depth requirement, and the other by considering the definable subcategory of $\mathrm{Mod}(R)$ generated by $\mathrm{CM}(R)$.

2.1 Two definable extensions of $\mathbf{CM}(R)$

Let us first introduce the class of modules obtained by extending depth to Ext-depth.

Definition 2.1.1. Let M be an R-module. We will say M is *cohomologically Cohen-Macaulay* if

$$\mathrm{Ext}^i_R(k, M) = 0 \text{ for all } i < d.$$

That is, $\text{E-dp}(M) \geq d$. We let $\mathrm{CohCM}(R)$ denote the category of cohomologically Cohen-Macaulay modules.

It is clear that the finitely generated modules in $\mathrm{CohCM}(R)$ are precisely the Cohen-Macaulay R-modules, so $\mathrm{CohCM}(R)$ does extend $\mathrm{CM}(R)$. From Nakayama's lemma and 1.5.8, the Ext-depth of any finitely generated module is bounded above by d. This does not hold for arbitrary modules, so there will be modules of infinite Ext-depth in CohCM(R).

The following result is a straightforward corollary to 1.6.11.

Lemma 2.1.2. CohCM(R) is a definable subcategory of Mod(R). □

As previously discussed, the class $\langle \mathrm{CM}(R)\rangle$ will be equal to $\varinjlim \mathrm{CM}(R)$, the class obtained from closing CM(R) under direct limits, if and only if CM(R) is covariantly finite in mod(R). Over a large class of Cohen-Macaulay rings, that this property holds was determined by Holm.

Theorem 2.1.3. [30, Theorem C] Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaualy ring admitting a canonical module. Then CM(R) is covariantly finite in mod(R). □

Corollary 2.1.4. With R as in 2.1.3, we have $\langle \mathrm{CM}(R)\rangle = \varinjlim \mathrm{CM}(R)$.

Proof. It is clear CM(R) is closed under finite direct sums so this follows immediately from 1.6.14. □

Definition 2.1.5. Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaualy ring admitting a canonical module Ω. We say an R-module is *limit Cohen-Macaulay* if there is a directed system $(M_i, f_{ij})_{i,j\in I}$ in CM(R) such that $\varinjlim_I M_i = M$.

Holm also gives necessary and sufficient conditions for a module to be limit Cohen-Macaulay.

Theorem 2.1.6. [30, Theorem A, B] Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay ring admitting a canonical module Ω. Then the following are equivalent for an R-module M:

1. $M \in \varinjlim \mathrm{CM}(R)$;
2. Every system of parameters of R is a weak M-sequence;
3. M is Gorenstein flat when viewed as an $R\ltimes\Omega$-module, that is $Z(M) \in \mathrm{GFlat}(R\ltimes\Omega)$;
4. For every R-sequence $\boldsymbol{x}$, $\mathrm{Tor}_1^R(R/(\boldsymbol{x}), M) = 0$.

□

In fact, the equivalences of the first and fourth statements show that $\varinjlim \mathrm{CM}(R)$ is definable in this situation, since the fourth statement gives a set of finitely presented functors defining $\langle \mathrm{CM}(R)\rangle$. In the situation that R does not have a canonical module, certain implications of the above theorem still hold, namely the first implies the fourth (see Exercise 1.1.12 in [12]). However, it is far from clear whether $\mathrm{CM}(R)$ remains covariantly finite in this setting. There will, of course, be an inclusion $\varinjlim \mathrm{CM}(R) \subseteq \langle \mathrm{CM}(R)\rangle$.

If we consider the case when R is a Gorenstein local ring, we can deduce that $\varinjlim \mathrm{CM}(R) = \mathrm{GFlat}(R)$ from 1.7.7 and 1.7.11. Since R is Gorenstein, its canonical module is itself, so 2.1.6 tells us that the functor $Z : \mathrm{Mod}(R) \longrightarrow \mathrm{Mod}(R \ltimes \Omega)$ restricts to a functor $Z : \mathrm{GFlat}(R) \longrightarrow \mathrm{GFlat}(R \ltimes R)$. In fact, the classes $\mathrm{GFlat}(R)$ and $\mathrm{GFlat}(R \ltimes R)$ are more related than this.

Proposition 2.1.7. Let R be a Gorenstein local ring. If M is a Gorenstein flat $R \ltimes R$-module, then $U(M)$ is a Gorenstein flat R-module.

Before proving this, we need a definition.

Definition 2.1.8. A ring extension $R \subset A$ is a *Frobenius extension* if the functors $A \otimes_R -$ and $\mathrm{Hom}_R(A, -)$ are equivalent.

Proof of 2.1.7. Let M be a Gorenstein flat $R \ltimes R$-module. By 1.7.7 and 1.4.10 we have $M = \varinjlim M_i$ where $(M_i, f_{i,j})_I$ is a directed system of finitely generated Gorenstein projective R-modules. The ring homomorphism $R[x] \longrightarrow R \ltimes R$ given by

$$r_0 + r_1 + \sum_{i \geq 2} r_i x^i \mapsto (r_0, r_1)$$

induces an isomorphism $R \ltimes R \simeq R[x]/(x^2)$, and the ring extension $R \subset R[x]/(x^2)$ is a Frobenius extension by [42, 3.1]. By [42, Theorem 2.5], each M_i is a Gorenstein projective $R \ltimes R$-module if and only if $U(M_i)$ is Gorenstein projective. It was shown in 1.4.11 that U preserves direct limits, consequently there are isomorphisms

$$U(M) \simeq U(\varinjlim M_i) \simeq \varinjlim U(M_i).$$

But each $U(M_i)$ is Gorenstein projective, and as we assumed that R was itself Gorenstein, it follows from 1.7.7 that $U(M)$ is a Gorenstein flat R-module. $\square$

One can also use a similar argument to show that if M is an $R \ltimes R$-module such that $U(M)$ is a Gorenstein flat R-module, then there is a Gorenstein flat $R \ltimes R$-module N such that $U(M) \simeq U(N)$. It does not necessarily follow that M is itself Gorenstein flat.

Remark. Suppose R is a Gorenstein local ring. By 1.7.7 we know that $\varinjlim \mathrm{Gproj}(R) = \mathrm{GFlat}(R)$. Yet the previous result and 2.1.6 show that $\varinjlim \mathrm{CM}(R) = \mathrm{GFlat}(R)$ as well. Consequently one gets an alternative proof of 1.7.11.

The following result is really a corollary to 2.1.6.

Lemma 2.1.9. Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring. Then the class of Gorenstein injective R-modules consists of all R-modules M such that

$$\mathrm{Ext}^1_R(R/(\boldsymbol{x}), M) = 0,$$

for every R-sequence $\boldsymbol{x}$.

Proof. Since R is Gorenstein, we have $\varinjlim \mathrm{CM}(R) = \mathrm{GFlat}(R)$, and by 2.1.6 we have that the defining functors for this definable subcategory are

$$\{\mathrm{Tor}^R_1(R/(\boldsymbol{x}), -) : \boldsymbol{x} \text{ is an } R\text{-sequence}\}.$$

Applying 1.4.14 shows that the dual definable category to this will be given by the set of functors

$$X = \{\mathrm{Ext}^1_R(R/(\boldsymbol{x}), M) = 0 : \boldsymbol{x} \text{ is an } R\text{-sequence}\}.$$

If M lies in this definable subcategory, then $M^\vee$ is a Gorenstein flat R-module, yet by 1.7.7 it follows that $M^{\vee\vee}$ is Gorenstein injective. Since definable subcategories are closed under pure submodules and $M \longrightarrow M^{\vee\vee}$ is a pure embedding (by [39, 1.3.16]) it follows that M is Gorenstein injective. Moreover, any Gorenstein injective module lies in this definable category by the same reasoning. $\square$

Since k is finitely generated, the functors $\mathrm{Ext}^i_R(k, -)$ preserve direct limits for all $i < \omega$, hence $\varinjlim \mathrm{CM}(R) \subset \mathrm{CohCM}(R)$ for any Cohen-Macaulay ring. Consequently there is always a chain of classes

$$\varinjlim \mathrm{CM}(R) \subseteq \langle \mathrm{CM}(R) \rangle \subseteq \mathrm{CohCM}(R).$$

In the case that R admits a canonical module, we may use Holm's result to consider the inclusion $\varinjlim \mathrm{CM}(R) \subseteq \mathrm{CohCM}(R)$ in more detail.

Theorem 2.1.10. Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay ring. If $\dim R = 1$ then $\operatorname{CohCM}(R) = \varinjlim \operatorname{CM}(R)$. If R admits a canonical module then $\operatorname{CohCM}(R) = \varinjlim \operatorname{CM}(R)$ if and only if $\dim R = 1$.

Proof. Suppose that $\dim R = 1$, so $\operatorname{CohCM}(R)$ consists of all R-modules M such that $\operatorname{Hom}_R(k, M) = 0$. Let M be such a module, and express $M = \varinjlim M_i$ as a direct limit of finitely presented R-modules. For each i the sequence $0 \longrightarrow M_i \longrightarrow M$ is exact, and consequently so is $0 \longrightarrow \operatorname{Hom}_R(k, M_i) \longrightarrow \operatorname{Hom}_R(k, M)$. Yet since $\operatorname{Hom}_R(k, M) = 0$, it follows that each $M_i \in \operatorname{CM}(R)$, and thus $M \in \varinjlim \operatorname{CM}(R)$.

Let us now assume that $\dim R \geq 2$ and R admits a canonical module. Since R is Cohen-Macaulay, we can pick an R-regular element $x \in \mathfrak{m}$. Let $\mathfrak{p}$ be a minimal prime over the principal ideal (x). By Krull's Hauptidealsatz [6, Cor. 11.17], the height of $\mathfrak{p}$ is one so $\mathfrak{p}$ is not a maximal ideal. Consider the indecomposable injective R-module $E := E(R/\mathfrak{p})$. We claim that $E \in \operatorname{CohCM}(R)$ but E is not a limit Cohen-Macaulay R-module. As E is injective, it suffices to show $\operatorname{Hom}_R(k, E) = 0$ in order for M to be in $\operatorname{CohCM}(R)$. Any non-zero module homomorphism $k \longrightarrow E$ will factor through $E(k)$ by properties of injective envelopes. If such a module homomorphism exists, it follows that $\operatorname{Hom}_R(E(k), E) \neq 0$. But since $\mathfrak{p} \neq \mathfrak{m}$, 1.4.16 tell us we must have $\operatorname{Hom}_R(E(k), E(R/\mathfrak{p})) = 0$, so we conclude $E \in \operatorname{CohCM}(R)$.

To prove that E is not a limit Cohen-Macaulay module, we find a system of parameters that is not a weak E-sequence. As x is regular, we can extend it to a system of parameters $\boldsymbol{x}$. Since $E = E(R/\mathfrak{p})$, the unique associated prime of E is $\mathfrak{p}$, and therefore there is an element $e \in E$ such that $\mathfrak{p} = \operatorname{ann}_R(e)$. Yet by construction we have $x \in \mathfrak{p}$ so $xe = 0$ and therefore x is not a regular element on E; thus $\boldsymbol{x}$ is not a weak E-sequence. □

Example 2.1.11. If R is a regular local ring it is possible to find explicit examples of modules that are in $\operatorname{CohCM}(R)$ but are not limit Cohen-Macaulay. Since $\operatorname{CM}(R)$ is just the category of finitely generated free modules, the limit Cohen-Macaulay modules are just the flat R-modules. Consequently an R-module M that is not flat but has Ext-depth at least $\dim R$ will lie in the difference. We give one possible source of such modules: consider the ring extension $R[[\boldsymbol{t}]]$, where $\boldsymbol{t} = t_1, \cdots, t_n$ is a list of indeterminates. Given an ideal $\mathfrak{a} \subset R[[\boldsymbol{t}]]$ we can consider $R[[\boldsymbol{t}]]/\mathfrak{a}$ as an R-module.

The following result shows it is possible to determine whether or not this module is a flat R-module.

Proposition. [21, Theorem 5.12] Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and $\mathfrak{a} = (g_1, \cdots, g_r)$ be an ideal of $R[[t_1, \cdots, t_j]]$. For $1 \leq i \leq r$, denote by $\overline{g_i} \in k[[t_1, \cdots, t_j]]$ the image of g_i under the projection $R[[\boldsymbol{t}]] \longrightarrow k[[\boldsymbol{t}]]$. Then the following are equivalent.

1. $R[[\boldsymbol{t}]]/\mathfrak{a}$ is a flat R-module;
2. The syzygies on $\overline{g}_1, \cdots, \overline{g}_r$ are generated by the reductions mod $\mathfrak{m}$ of the syzygies on $g_1, \cdots, g_r$.

Suppose we have determined $R[[\boldsymbol{t}]]/\mathfrak{a}$ is not flat, all that is left is to determine its Ext-depth over R. We can use the independence theorem of local cohomology 1.3.11 to do this: considering the embedding $R \longrightarrow R[[\boldsymbol{t}]]$ we get, for any $R[[t]]$-module N and $i \geq 0$, an isomorphism of R-modules

$$H^i_{\mathfrak{m}}(N) \simeq H^i_{\mathfrak{m}R[[\boldsymbol{t}]]}(N).$$

Now, we see that

$$\begin{aligned}
\text{E-dp}_R(R[[\boldsymbol{t}]]/\mathfrak{a}) \geq d &\iff H^i_{\mathfrak{m}}(R[[\boldsymbol{t}]]/\mathfrak{a}) = 0 \text{ for all } i < d \\
&\iff H^i_{\mathfrak{m}R[[\boldsymbol{t}]]}(R[[\boldsymbol{t}]]/\mathfrak{a}) = 0 \text{ for all } i < d \\
&\iff \text{E-grade}(\mathfrak{m}R[[\boldsymbol{t}]], R[[\boldsymbol{t}]]/\mathfrak{a}) \geq d.
\end{aligned}$$

Since $R[[\boldsymbol{t}]]/\mathfrak{a}$ is finitely generated as an $R[[\boldsymbol{t}]]$-module, one can determine this last quantity without much difficulty.

Let us now give a concrete example. Let $R = \mathbb{Q}[[a, b]]$ and $S = \mathbb{Q}[[a, b, x, y, z]]$. Let

$$\mathfrak{a} = (-y^7z^3 + x^5y + a^4bz, a^4b + ab^2x^4 - x^9y^4z^7, z5 - 7a^3y + b^4x - x^8y^4z^3) \subset S$$

and consider the S-module $S/\mathfrak{a}$. We show that this is not flat over R using `Singular`

```
> ring Rxyz = 0, (x,y,z,a,b), ds;
> ideal I = -y7z3+x5y+a4bz, a4b+ab2x4-x9y4z7,
z5-7a3y+b4x-x8y4z3;
> module M = syz(I);
```

```
> ring Kxyz = 0, (x,y,z), dp;
> ideal I1 = imap(Rxyz, I);
> module M1 = imap(Rxyz, M);
> M1 = std(M1);
> module M2 = syz(I1);
> reduce(M2,M1);
_[1]=0
_[2]=x9y3z7*gen(1)-y6z3*gen(2)+x5*gen(2)
_[3]=x4y10z5*gen(3)-x3y10z*gen(2)+x4y3z7*gen(1)+gen(2)
_[4]=0
```

In the above code, the module `M` is the syzygies of the ideal I viewed as a module over the original ring. The module `M1` is then (a standard basis of) the reduction of the syzygies under the projection $R[[x, y, z]] \longrightarrow k[[x, y, z]]$, while the ideal `I1` is the image of the ideal I under the same map. Once the syzygies of `I1` are taken, given by the module `M2`, the command `reduce` determines whether `M2` is a submodule of `M1`. In particular, the output being zero indicates that the syzygy of the reduction of the ideal (here given by `M2`) is generated by the reduction of the original syzygies, i.e. `M1`. As we can see, the second and third syzygies can not be generated in this way. Consequently we may apply the third part of 2.1.11 to deduce that the module $S/\mathfrak{a}$ is not a flat R-module.

We now show that E-grade$((a, b)_S, S/\mathfrak{a}) \geq 2$ using `Macaulay2`.

```
R = QQ[a,b,x,y,z,MonomialOrder=>
{Weights=>{-1,-1,-1},GRevLex},Global=>false];
m = R^1/ideal(a,b);
M = cokernel | -y7z3+x5y+a4bz  a4b+ab2x4-x9y4z7
z5-7a3y+b4x-x8y4z3  |;
i12 : Hom(m,M)==0
o12 = true
i13 : Ext^11(m,M)==0
o13 = true
```

Consequently $S/\mathfrak{a}$ is in CohCM(R) but not in $\varinjlim \mathrm{CM}(R)$. □

The dual definable subcategories of both CohCM(R) and $\varinjlim \mathrm{CM}(R)$ can be described explicitly. Observe that for any module M and any $i \geq 0$ we have that

$$\begin{aligned} \mathrm{Ext}^i_R(k, M) = 0 &\iff \mathrm{Hom}_R(\mathrm{Ext}^i_R(k, M), E(k)) = 0, \\ &\iff \mathrm{Tor}_i^R(k, \mathrm{Hom}_R(M, E(k))) = 0, \text{ by 1.4.14(4).} \end{aligned}$$

where the first line holds as $E(k)$ is an injective cogenerator. In particular, we see that $M \in \mathrm{CohCM}(R)$ if and only if $M^\vee$ vanishes on the set of functors

$$\{\mathrm{Tor}_i^R(k, -) = 0 : 0 \leq i < d\}.$$

On the other hand, if N is an R-module that satisfies $\mathrm{Tor}_i^R(k, N) = 0$ for all $i < d$, then the isomorphism

$$\mathrm{Hom}_R(\mathrm{Tor}_i^R(k, N), E(k)) \simeq \mathrm{Ext}^i_R(k, \mathrm{Hom}_R(N, E(k)))$$

given in 1.4.14(2) shows that $N^\vee \in \mathrm{CohCM}(R)$. In particular, this shows that

$$\mathrm{CohCM}(R)^d = \{M \in \mathrm{Mod}R : \mathrm{Tor}_i^R(k, M) = 0 \text{ for all } i < d\}.$$

Similarly, the dual definable subcategory of $\varinjlim \mathrm{CM}(R)$ is $\varinjlim \mathrm{CM}(R)^d = \{M \in \mathrm{Mod}(R) : \mathrm{Ext}^1_R(R/(\boldsymbol{x}), M) = 0$ for all R-sequences $\boldsymbol{x}\}$. Since both $\varinjlim \mathrm{CM}(R)$ and $\mathrm{CohCM}(R)$ contain the flat R-modules, both $\varinjlim \mathrm{CM}(R)^d$ and $\mathrm{CohCM}(R)^d$ contain all the injective R-modules. Indeed, if E is an injective R-module, then $E^\vee$ lies in $\varinjlim \mathrm{CM}(R)$, and therefore $E^{\vee\vee} \in \varinjlim \mathrm{CM}(R)^d$. But E is a pure submodule of $E^{\vee\vee}$ and is therefore in $\varinjlim \mathrm{CM}(R)^d$. The same applies for $\mathrm{CohCM}(R)$.

Not every module in $\varinjlim \mathrm{CM}(R)^d$ can be realised as the dual of a module in $\varinjlim \mathrm{CM}(R)$, yet it will be a pure submodule of one (its double dual). When R is a complete Cohen-Macaulay ring, the modules in $\varinjlim \mathrm{CM}(R)^d$ that are isomorphic to a dual of a module in $\varinjlim \mathrm{CM}(R)$ can be described in more detail. If $N \in \varinjlim \mathrm{CM}(R)^d$ is a dual module then there is an $M = \varinjlim_I M_i \in \varinjlim \mathrm{CM}(R)$ with $M_i \in \mathrm{CM}(R)$ and $M^\vee \simeq N$. Then

$$N \simeq \mathrm{Hom}_R(\varinjlim_I M_i, E(k)) \simeq \varprojlim_I \mathrm{Hom}_R(M_i, E(k)).$$

But $\mathrm{Hom}_R(M_i, E(k)) \in \mathcal{A}(R)$, the class of Artinian modules. If $\mathcal{A}_0 := \mathcal{A} \cap \varinjlim \mathrm{CM}(R)^d$, then $\varprojlim \mathcal{A}_0$ is precisely the class of dual modules in $\varinjlim \mathrm{CM}(R)^d$, and any module in $\varinjlim \mathrm{CM}(R)$ is a pure submodule of a module in this class.

Example 2.1.12. Let us give a concrete example of the class $\mathcal{A}_0$ as described above. Let k be a field and consider the hypersurface ring $R = k[[x_1, \cdots, x_n]]/(f)$ for $f \in \mathfrak{m}^2$. By [35, 8.6], any Cohen-Macaulay R-module has a 2-periodic free resolution

$$\cdots \longrightarrow R^n \xrightarrow{\varphi} R^n \xrightarrow{\psi} R^n \xrightarrow{\varphi} R^n \longrightarrow M \longrightarrow 0.$$

Applying the Matlis dual shows that $M^\vee \simeq \mathrm{Ker}(\varphi^\vee : E(k)^n \longrightarrow E(k)^n)$. Using the results from 1.4.24, there are isomorphisms $\mathrm{End}_R(E(k)^n) \simeq M_n(\mathrm{End}_R(E(k))) \simeq M_n(R)$, therefore we can describe $\varphi^\vee \in \mathrm{End}_R(E(k)^n)$ by an $n \times n$ matrix with entries in R. Let $\boldsymbol{e} \in E(k)^n$ and set $g \in \mathrm{Hom}_R(R^n, E(k))$ to be the unique homomorphism corresponding to $\boldsymbol{e}$. The correspondence between $\mathrm{Hom}_R(R^n, E(k))$ and $E(k)^n$ is given by

$$f \mapsto (f_1(1), \cdots, f_n(1)),$$

where $f_j = f \circ \mathrm{inc}_j$ and $\mathrm{inc}_j : R \longrightarrow R^n$ is the canonical jth inclusion. The map $\varphi^\vee$ sends g to $g \circ \varphi$, and therefore the element of $E(k)^n$ corresponding to $\varphi^\vee(g)$ is

$$((g \circ \varphi)_1(1), \cdots, (g \circ \varphi)_n(1)).$$

Since the diagram

$$\begin{tikzcd} \mathrm{Hom}_R(R^n, E(k)) \arrow[r, "\varphi^\vee"] \arrow[d, "\simeq"] & \mathrm{Hom}_R(R^n, E(k)) \arrow[d, "\simeq"] \\ E(k)^n \arrow[r] & E(k)^n \end{tikzcd}$$

commutes, we see that the matrix corresponding to $\varphi^\vee$ is given by

$$(g_1(1), \cdots, g_n(1)) \mapsto ((g \circ \varphi)_1(1), \cdots, (g \circ \varphi)_n(1)).$$

Let the matrix corresponding to φ be given by $\Phi = (\alpha_{i,j})_{1 \le i,j \le n}$, where $\alpha_{i,j} \in R$. If $\mathbf{c}_j$ denotes the jth column of Φ, then $\varphi \circ \mathrm{inc}_j(1) = \mathbf{c}_j = \sum_{k=1}^n \alpha_{kj}\mathrm{inc}_k(1)$ by basic linear algebra. Therefore

$$\begin{aligned}(g \circ \varphi)_j(1) = g(\mathbf{c}_j) &= g\left(\sum_{k=1}^n \alpha_{kj}\mathrm{inc}_k(1)\right) \\ &= \sum_{k=1}^n \alpha_{kj} g_k(1)\end{aligned}$$

In other words, the matrix corresponding to $\varphi^\vee$ sends $g_j(1)$ to $\sum_{k=1}^n \alpha_{kj} g_k(1)$, but this is precisely the action of Φ^T acting on the vector $(g_1(1), \cdots, g_n(1))$. In particular, we see that if $M = \mathrm{Coker}(\Phi)$, then $M^\vee = \mathrm{Ker}(\Phi^T)$, where Φ^T is seen as acting on $E(k)^n$. Since $\mathcal{A}_0 = \{M^\vee : M \in \mathrm{CM}(R)\}$, every module in $\mathcal{A}_0$ can be realised in this way. $\square$

2.1.1 Extending the canonical dual

In 2.1.10 we assumed that R admits a canonical module, so we can consider how the canonical dual $(-)^* = \mathrm{Hom}_R(-, \Omega)$ acts on both $\mathrm{CohCM}(R)$ and $\varinjlim \mathrm{CM}(R)$ and see which properties extend from $\mathrm{CM}(R)$ to each of these categories. Recall that $(-)^\vee$ denotes the Matlis dual $\mathrm{Hom}_R(-, E(k))$.

Theorem 2.1.13. *Let $(R, \mathfrak{m}, k)$ be a complete Cohen-Macaulay ring admitting a canonical module Ω. Then $\mathrm{Hom}_R(-, \Omega)$ is an endofunctor on $\varinjlim \mathrm{CM}(R)$ and Ω is an injective object in $\varinjlim \mathrm{CM}(R)$.*

Proof. We show that for an R-sequence $\boldsymbol{x}$ and a limit Cohen-Macaulay module $M \in \varinjlim \mathrm{CM}(R)$ we have $\mathrm{Tor}_1^R(R/\boldsymbol{x}, \mathrm{Hom}_R(M, \Omega)) = 0$. Since R is a complete local ring we can apply Grothendieck local duality 1.4.25, so we have a chain of isomorphisms

$$\mathrm{Tor}_1(R/\boldsymbol{x}, \mathrm{Hom}(M, \Omega)) \simeq \mathrm{Tor}_1(R/\boldsymbol{x}, H^d_{\mathfrak{m}}(M)^\vee) \simeq \mathrm{Ext}^1(R/\boldsymbol{x}, H^d_{\mathfrak{m}}(M))^\vee,$$

where the final isomorphism follows from 1.4.14. Consequently it suffices to show that $\mathrm{Ext}^1(R/\boldsymbol{x}, H^d_{\mathfrak{m}}(M)) = 0$. Since $M \in \varinjlim \mathrm{CM}(R)$, we have $M = \varinjlim M_i$ with each $M_i \in \mathrm{CM}(R)$. Using the fact that local cohomology commutes with direct limits and $R/\boldsymbol{x}$ is finitely presented, we have

$$\mathrm{Ext}^1(R/\boldsymbol{x}, H^d_{\mathfrak{m}}(M)) \simeq \varinjlim \mathrm{Ext}^1(R/\boldsymbol{x}, H^d_{\mathfrak{m}}(M_i)).$$

Since $\mathrm{Hom}(M_i, \Omega) \in \mathrm{CM}(R)$ for every i, it follows that $\mathrm{Ext}^1(R/\boldsymbol{x}, H^d_{\mathfrak{m}}(M_i)) = 0$ by the above chain of isomorphisms. Consequently $\mathrm{Ext}^1(R/\boldsymbol{x}, H^d_{\mathfrak{m}}(M)) = 0$, proving the first claim. To show that Ω is an injective object, we note that for each limit Cohen-Macaulay module $M = \varinjlim M_i$ we have

$$\mathrm{Ext}^i(\varinjlim M_i, \Omega) \simeq \varprojlim \mathrm{Ext}^i(M_i, \Omega) = 0,$$

where the first isomorphism holds by 1.6.6 as Ω is a pure-injective R-module since $\Omega^{\vee\vee} \simeq \Omega$. □

We will later provide an alternative proof to this result. Let us consider the same question for $\mathrm{CohCM}(R)$.

Theorem 2.1.14. *Let $(R, \mathfrak{m}, k)$ be a complete Cohen-Macaulay ring of Krull dimension d. Then $\mathrm{Hom}_R(-, \Omega)$ is an endofunctor on $\mathrm{CohCM}(R)$ and Ω is an injective object in this category.*

Proof. It is enough to consider how $\mathrm{Hom}_R(-,\Omega)$ acts on modules M with $\text{E-dp}(M) = d$, since if $\text{E-dp}(M) = \infty$ we can use local duality to see immediately that $\mathrm{Hom}_R(M,\Omega) = 0$. By local duality, showing M^* is in $\mathrm{CohCM}(R)$ is equivalent to showing that $\text{E-dp}(H^d_{\mathfrak{m}}(M)^\vee) \geq d$, which is equivalent to showing $\text{T-codp}(H^d_{\mathfrak{m}}(M)) \geq d$ by 1.5.6. Since $H^d_{\mathfrak{m}}(M) \simeq M \otimes_R H^d_{\mathfrak{m}}(R)$, we show that $\mathrm{Tor}^R_i(k, M \otimes_R H^d_{\mathfrak{m}}(R)) = 0$ for $i < d$. In order to do this, we show that if $\text{E-dp}(M) = d$ there is a first quadrant spectral sequence

$$E_2^{p,q} = \mathrm{Tor}_p(M, \mathrm{Tor}_q(k, H^d_{\mathfrak{m}}(R))) \implies \mathrm{Tor}_{p+q}(k, M \otimes_R H^d_{\mathfrak{m}}(R)). \tag{2.1}$$

In [44, Theorem 10.59], for such a spectral sequence to exist it suffices to show that for every projective R-module P, we have $\mathrm{Tor}_i(M, H^d_{\mathfrak{m}}(R) \otimes P) = 0$ for all $i \geq 1$. Since R is local, all projective modules are free and as Tor commutes with direct sums it suffices to show $\mathrm{Tor}_i(M, H^d_{\mathfrak{m}}(R)) = 0$ for all $i \geq 1$. Yet

$$\mathrm{Tor}_i(M, H^d_{\mathfrak{m}}(R)) = 0 \iff \mathrm{Ext}^i(M, H^d_{\mathfrak{m}}(R)^\vee) = 0$$

and $H^d_{\mathfrak{m}}(R)^\vee \simeq \Omega$. But by 1.4.25, we have $\mathrm{Ext}^i(M,\Omega) = 0$ for $i \geq 1$. Therefore 2.1 exists for all modules with Ext-depth d. For any $q \geq 0$ there are isomorphisms

$$\mathrm{Tor}_q(k, H^d_{\mathfrak{m}}(R))^\vee \simeq \mathrm{Ext}^q(k,\Omega) \simeq \begin{cases} 0 & \text{if } q \neq d \\ k & \text{if } q = d, \end{cases}$$

since Ω is the canonical module. Since $E(k)$ is an injective cogenerator it follows that $E_2^{p,q}$ is entirely concentrated in the $q = d$ column, meaning that (2.1) collapses immediately. This gives isomorphisms

$$\mathrm{Tor}_p(M, \mathrm{Tor}_q(k, H^d_{\mathfrak{m}}(R))) \simeq \mathrm{Tor}_{p+q}(k, H^d_{\mathfrak{m}}(M))$$

for all $p, q \geq 0$. In particular, we see that $\mathrm{Tor}_i(k, H^d_{\mathfrak{m}}(M)) = 0$ for all $i < d$, since $\mathrm{Tor}_p(M, \mathrm{Tor}_q(k, H^d_{\mathfrak{m}}(R))) = 0$ when $p + q < d$. That Ω is injective in $\mathrm{CohCM}(R)$ is immediate from 1.4.25.

$\square$

Corollary 2.1.15. If $\text{E-dp}(M) = d$, then $\text{E-dp}(M^*) = d$.

Proof. By the reasoning at the start of the above proof, it suffices to show that $\text{T-codp}(H^d_{\mathfrak{m}}(M)) = d$. We showed that $\text{T-codp}(H^d_{\mathfrak{m}}(M)) \geq d$, so if we can show

$\mathrm{Tor}_d(k, H^d_{\mathfrak{m}}(M)) \neq 0$, we are done. There are isomorphisms

$$\mathrm{Tor}_p(M, \mathrm{Tor}_q(k, H^d_{\mathfrak{m}}(R))) \simeq \mathrm{Tor}_{p+q}(k, H^d_{\mathfrak{m}}(M))$$

with the left hand side zero whenever $q \neq d$. In particular

$$\mathrm{Tor}_d(k, H^d_{\mathfrak{m}}(M)) \simeq M \otimes \mathrm{Tor}_d(k, H^d_{\mathfrak{m}}(R)).$$

By 1.4.25, $H^d_{\mathfrak{m}}(R) \simeq \Omega^\vee$, so $\mathrm{Tor}_d(k, H^d_{\mathfrak{m}}(R)) \simeq \mathrm{Tor}_d(k, \Omega^\vee) \simeq \mathrm{Ext}^d(k, \Omega)^\vee \simeq k$, and therefore $M \otimes \mathrm{Tor}_d(k, H^d_{\mathfrak{m}}(R)) \simeq M \otimes k$. By assumption, $\text{E-dp}(M) = d$ so by 1.5.8 we know $\text{T-codp}(M) = 0$. Thus $M \otimes k \neq 0$, which shows the result. $\square$

As previously stated, the dual definable subcategory of $\mathrm{CohCM}(R)$ is $\mathrm{CohCM}(R)^d = \{M \in \mathrm{Mod}(R) : \text{T-codp}(M) \geq d\}$. The proof of the above theorem shows that $H^d_{\mathfrak{m}}(R) \otimes -$ is a functor $\mathrm{CohCM}(R) \longrightarrow \mathrm{CohCM}(R)^d$. However, this is very far from a duality. In fact, the functor $- \otimes H^d_{\mathfrak{m}}(R) \otimes H^d_{\mathfrak{m}}(R)$ is just the zero functor on $\mathrm{CohCM}(R)$, as we now show.

Lemma 2.1.16. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring with $\dim R = d$. If $\text{E-dp}(M) = d$, then $\text{E-dp}(H^d_{\mathfrak{m}}(M)) = 0$.

Proof. Since $\Gamma_{\mathfrak{m}} \circ \Gamma_{\mathfrak{m}} \simeq \Gamma_m$ and $\Gamma_{\mathfrak{m}}(E)$ is an injective R-module whenever E is injective, there is a Grothendieck spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{m}}(H^q_{\mathfrak{m}}(M)) \implies H^n_{\mathfrak{m}}(M)$$

for any R-module M. If we now assume $\text{E-dp}(M) = d$, then $H^q_{\mathfrak{m}}(M) = 0$ for all $q \neq d$, so $E_2^{p,q} = 0$ whenever $q \neq d$ which causes the spectral sequence to collapse. There are therefore isomorphisms $H^p_{\mathfrak{m}}(H^d_{\mathfrak{m}}(M)) \simeq H^{p+d}_{\mathfrak{m}}(M)$ for every M with $\text{E-dp}(M) = d$. Consequenlty $H^0_{\mathfrak{m}}(H^d_{\mathfrak{m}}(M)) \simeq H^d_{\mathfrak{m}}(M) \neq 0$ by assumption on M. $\square$

In particular, if $M \in \mathrm{CohCM}(R)$ with $\text{E-dp}(M) = d$, then $M \otimes H^d_{\mathfrak{m}}(R) \otimes H^d_{\mathfrak{m}}(R) \simeq H^d_{\mathfrak{m}}(H^d_{\mathfrak{m}}(M)) \simeq H^{2d}(M) = 0$.

2.2 A partial extension to Knörrer periodicity

Let R and S be Cohen-Macaulay rings and $F : \text{mod-}R \longrightarrow \text{mod-}S$ a functor that restricts to $F| : \mathrm{CM}(R) \longrightarrow \mathrm{CM}(S)$. We saw in §1.6.1 that F can be extended to a direct limit preserving functor $\overline{F} : \text{Mod-}R \longrightarrow \text{Mod-}S$ that is independent on the choice of directed system. In particular, if $M \in \varinjlim \mathrm{CM}(R)$ can be realised as $M = \varinjlim_I M_i$, with each $M_i \in \mathrm{CM}(R)$, then

$$\overline{F}(M) = \overline{F}(\varinjlim_I M_i) \simeq \varinjlim_I F(M_i) = \varinjlim_I F|(M_i) \in \varinjlim \mathrm{CM}(S),$$

hence F naturally extends to a functor $\varinjlim \mathrm{CM}(R) \longrightarrow \varinjlim \mathrm{CM}(S)$, which is simply the restriction of $\overline{F}$ to $\varinjlim \mathrm{CM}(R)$. It is not obvious that such a construction will yield a functor $\mathrm{CohCM}(R) \longrightarrow \mathrm{CohCM}(S)$. We will now consider a particular example of extending a functor between categories of Cohen-Macaulay modules to their direct limit closures by looking at the classical result of Knörrer periodicty.

Let $(S, \mathfrak{n})$ be a complete regular local ring and $0 \neq f \in \mathfrak{n}^2$ a regular element. Define $R = S/(f)$ to be the corresponding complete hypersurface ring, which is Gorenstein local. Following Knörrer, we define a new ring

$$R^\sharp = S[[z]]/(f + z^2).$$

As $R^\sharp$-modules, $R^\sharp/(z)$ and R are isomorphic, yielding a functor

$$(-)^\flat := R^\sharp/(z) \otimes_{R^\sharp} - : \mathrm{Mod}(R^\sharp) \longrightarrow \mathrm{Mod}(R)$$

that preserves direct limits and restricts to $\mathrm{mod}(R^\sharp) \longrightarrow \mathrm{mod}(R)$. If $M \in \mathrm{CM}(R^\sharp)$ we have

$$\mathrm{depth}_S(M/zM) = \mathrm{depth}_S(M) - 1 = \dim R,$$

so $(-)^\flat$ restricts to a functor $\mathrm{CM}(R^\sharp) \longrightarrow \mathrm{CM}(R)$. The functor $(-)^{\flat\flat} : \mathrm{CM}(R^{\sharp\sharp}) \longrightarrow \mathrm{CM}(R)$ is of more interest than $(-)^\flat$. One of the reasons for this is that there is a functor going in the other direction,

$$(-)^{\maltese} : \mathrm{CM}(R) \longrightarrow \mathrm{CM}(R^{\sharp\sharp}).$$

We will not give an explicit description of this functor, because it is enough for us to understand its properties in relation to $(-)^{\flat\flat}$. Many details about it, including an explicit description, can be found in §§8.2-8.3 of [35]. The following lemma gives one of the properties we need about $(-)^{\maltese}$.

Lemma 2.2.1. [35, Lemma 8.29] With the described setup, we have

$$(R^{\maltese})^{\flat\flat} \simeq R \oplus \Omega^1_R(R)$$

and

$$R^{\sharp\sharp} \simeq R^{\maltese} \oplus \Omega^1_{R^{\sharp\sharp}}(R^{\maltese}).$$

□

From the second isomorphism we see that $R^{\maltese}$ is a projective, and therefore free, $R^{\sharp\sharp}$-module. Comparing ranks shows that $R^{\maltese} \simeq R^{\sharp\sharp}$ and $\Omega^1_{R^{\sharp\sharp}}(R^{\maltese}) = 0$. Using the obvious notation, we therefore see that

$$(\maltese \circ \flat\flat)(R^{\sharp\sharp}) \simeq R^{\sharp\sharp}$$

and

$$(\flat\flat \circ \maltese)(R) \simeq R.$$

This gives an equivalence of categories $\operatorname{proj}(R) \longrightarrow \operatorname{proj}(R^{\sharp\sharp})$, which will then extend to an equivalence $\operatorname{Flat}(R) \longrightarrow \operatorname{Flat}(R^{\sharp\sharp})$.

In order to bring this equivalence into play, we invoke the following result, known as Knörrer periodicity.

Theorem 2.2.2. [51, Theorem 12.10] There is an equivalence of categories $\underline{\mathrm{CM}}(R) \simeq \underline{\mathrm{CM}}(R^{\sharp\sharp})$. □

Here $\underline{\mathrm{CM}}(R)$ and $\underline{\mathrm{CM}}(R^{\sharp\sharp})$ denote the projectively stable Cohen-Macaulay categories. These are defined as the following categories. The objects of $\underline{\mathrm{CM}}(R)$ are the same as the objects of $\mathrm{CM}(R)$, while

$$\mathrm{Hom}_{\underline{\mathrm{CM}}(R)}(M,N) = \mathrm{Hom}_R(M,N)/\mathcal{P}(M,N)$$

where

$$\mathcal{P}(M,N) = \{f : M \longrightarrow N : f \text{ factors through a finitely generated free module}\}.$$

We aim to extend Knörrer periodicity, at least a form of it, to $\varinjlim \mathrm{CM}(R)$. In order to do this, we need some more results about definable subcategories.

We will let fun-$R = (\mathrm{mod}(R), \mathbf{Ab})^{\mathrm{fp}}$ denote the abelian category of finitely presented functors from $\mathrm{mod}(R)$ into $\mathbf{Ab}$. By [39, Theorem 12.4.1] there are natural bijection between definable subcategories of $\mathrm{Mod}(R)$ and Serre subcategories of fun-R. We can describe this bijection: if $\mathcal{X}$ is a definable subcategory of $\mathrm{Mod}(R)$, then the corresponding Serre subcategory of fun-R is

$$\mathcal{S}_{\mathcal{X}} = \{F \in \text{fun-}R : \overrightarrow{F} X = 0 \text{ for all } X \in \mathcal{X}\},$$

while if $\mathcal{S}$ is a Serre subcategory of fun-R the category

$$\{M \in \mathrm{Mod}(R) : \overrightarrow{F} M = 0 \text{ for all } F \in \mathcal{S}\}$$

is clearly a definable subcategory by 1.6.10.

Suppose that $\mathcal{C} \subset \mathcal{D}$ are definable subcategories, then there is a reverse inclusion $\mathcal{S}_{\mathcal{D}} \subset \mathcal{S}_{\mathcal{C}}$ of Serre subcategories of fun-R. We can view $\mathcal{S}_{\mathcal{D}}$ as a Serre subcategory of the skeletally small abelian category corresponding to $\mathcal{S}_{\mathcal{C}}$, and therefore consider the Serre localisation $\mathcal{S}_{\mathcal{C}}/\mathcal{S}_{\mathcal{D}}$. This is a skeletally small abelian category, so there is a corresponding definable subcategory of some module category, called the *definable quotient* and introduced by Krause in [34]. It is the properties of this definable quotient that will enable us to consider Knörrer periodicity.

The categories $\mathrm{CM}(R)$ and $\mathrm{proj}(R)$ are covariantly finite in $\mathrm{mod}(R)$, so their direct limit closures are definable. As R is Gorenstein, we may identify $\varinjlim \mathrm{CM}(R)$ with $\mathrm{GFlat}(R)$, the category of Gorenstein flat R-modules. The inclusion $\mathrm{Flat}(R) \subset \mathrm{GFlat}(R)$ yields a definable quotient category $\mathcal{D}$, and by [34, Theorem 5.4] there is an equivalence $\mathcal{D}^{\mathrm{fp}} = \mathrm{CM}(R)/\mathrm{proj}(R) = \underline{\mathrm{CM}(R)}$. In particular, the definable quotient category $\mathcal{D} \simeq \varinjlim \underline{\mathrm{CM}(R)}$. Unfortunately it is not so straight forward to describe this category in its entirety. We can, however, say considerably more by restricting to pure injective modules.

Let $\mathrm{Flat}(R)_{PI}$ be the full subcategory of $\mathrm{Mod}(R)$ consisting of pure-injective flat modules. This is not the same as $\mathrm{Flat}_R \cap \mathrm{pinj}_R$, which is the set of indecomposable pure-injective flat modules. We similarly define $\mathrm{GFlat}(R)_{PI}$. By [34, Theorem 5.1] there is an equivalence

$$\mathrm{GFlat}(R)_{PI}/\mathrm{Flat}(R)_{PI} \simeq (\varinjlim \underline{\mathrm{CM}(R)})_{PI}.$$

Here $\mathrm{GFlat}(R)_{PI}/\mathrm{Flat}(R)_{PI}$ is the stable quotient category. Applying Knörrer periodicity, we see that there is an equivalence $\varinjlim \underline{\mathrm{CM}(R)} \simeq \varinjlim \underline{\mathrm{CM}(R^{\sharp\sharp})}$. Consequently we have

$$\mathrm{GFlat}(R)_{PI}/\mathrm{Flat}(R)_{PI} \simeq \mathrm{GFlat}(R^{\sharp\sharp})_{PI}/\mathrm{GFlat}(R^{\sharp\sharp})_{PI}.$$

We can go further. It is shown in [34, Cor. 6.3] that there is a homeomorphism of Ziegler spectra

$$\mathrm{Zg}\ \mathrm{GFlat}(R) \setminus \mathrm{Zg}\ \mathrm{Flat}(R) \simeq \mathrm{Zg}\ \varinjlim \underline{\mathrm{CM}(R)},$$

and likewise over $R^{\sharp\sharp}$. Yet we know $\mathrm{Zg}\ \varinjlim \underline{\mathrm{CM}(R)} \simeq \mathrm{Zg}\ \varinjlim \underline{\mathrm{CM}(R^{\sharp\sharp})}$ by 2.2.2, so we have homeomorphisms

$$\mathrm{Zg}\ \mathrm{GFlat}(R) \setminus \mathrm{Zg}\ \mathrm{Flat}(R) \simeq \mathrm{Zg}\ \mathrm{GFlat}(R^{\sharp\sharp}) \setminus \mathrm{Zg}\ \mathrm{Flat}(R^{\sharp\sharp}),$$

but since $\mathrm{Flat}(R) \simeq \mathrm{Flat}(R^{\sharp\sharp})$, the topological spaces $\mathrm{Zg}\ \mathrm{Flat}(R)$ and $\mathrm{Zg}\ \mathrm{Flat}(R^{\sharp\sharp})$ are in natural bijection. We can therefore deduce the following.

Theorem 2.2.3. Let R be as above, then there is a bijection of sets

$$\mathrm{GFlat}(R) \cap \mathrm{pinj}(R) \longrightarrow \mathrm{GFlat}(R^{\sharp\sharp}) \cap \mathrm{pinj}(R^{\sharp\sharp}).$$

Since $(-)^{\flat}$ is given by the functor $(R^{\sharp}/(z) \otimes_{R^{\sharp}} -) \simeq (R \otimes_{R^{\sharp}} -)$, it is clear that it preserves both direct limits and direct products. It is therefore an example of a *definable functor* (using the terminology of [37]), also called an *interpretation functor* in [39].

Such functors play a key role in discussing definable subcategories. If $\mathcal{C}$ and $\mathcal{D}$ are arbitrary definable categories (not necessarily over the same ring), let $\mathrm{Zg}(\mathcal{C})$ denote the part of the Ziegler spectrum corresponding to $\mathcal{C}$ (i.e. the set of indecomposable pure injective objects in $\mathcal{C}$ with closed sets given by definable subcategories of $\mathcal{C}$) and similarly for $\mathcal{D}$. If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a definable functor, then [37, 15.2] shows that there is an induced map $f : \mathrm{Zg}(\mathcal{C}) \longrightarrow \mathrm{Zg}(\mathcal{D})$ on topological spaces that is continuous and closed. In the case that F is full on pure injectives, then this induced map f is in fact a homeomorphism from the domain onto its image

$$f : \mathrm{Zg}(\mathcal{C}) \setminus \{M \in \mathrm{Zg}(\mathcal{C}) : F(C) = 0\} \longrightarrow \mathrm{Zg}(\mathcal{D}).$$

We therefore see that the functor $(-)^{\flat\flat} : \varinjlim \mathrm{CM}(R^{\sharp\sharp}) \longrightarrow \varinjlim \mathrm{CM}(R)$ induces a closed and continuous map on the topological spaces $\mathrm{Zg}\ \mathrm{GFlat}(R^{\sharp\sharp}) \longrightarrow \mathrm{Zg}\ \mathrm{GFlat}(R)$; so we can view $\mathrm{Zg}\ \mathrm{GFlat}(R^{\sharp\sharp})$ as a closed subset of $\mathrm{Zg}\ \mathrm{GFlat}(R)$. However, it is not so straightforward to say much more than that about the topological spaces: the functor $(-)^{\flat}$ is not necessarily full, even on pure-injectives, so we do not have the homeomorphism described above. However, since the map $R^{\sharp\sharp} \longrightarrow R$ is a ring epimorphism, it induces a closed inclusion $\mathrm{Zg}(R) \subseteq \mathrm{Zg}(R^{\sharp\sharp})$, by [38, Cor. 9].

Remark. Let us make a short aside into triangulated categories, which will not appear elsewhere in the thesis so we provide no introduction to the concepts. It is a classic result of Buchweitz that over an arbitrary Gorenstein ring the category gproj-R of finitely generated right Gorenstein projective modules is frobenius, with the projective modules acting as the projective-injective objects (see [13, §4]), hence the projectively stable category $\underline{\text{gproj-}R}$ is triangulated. If $R = S/(f)$ is a hypersurface as described above, we may identify $\underline{\text{gproj-}R}$ with $\underline{\mathrm{CM}(R)}$, and therefore by Knörrer periodicity there is an equivalence of triangulated categories

$$\underline{\mathrm{CM}(R)} \longrightarrow \underline{\mathrm{CM}(R^{\sharp\sharp})}.$$

As stated above, there is an equivalence of additive categories $\varinjlim \underline{\mathrm{CM}(R)} \longrightarrow \varinjlim \underline{\mathrm{CM}(R^{\sharp\sharp})}$, yet there is no reason for these to be triangulated. To return to the triangulated setting one would be better off considering the class $\varinjlim \mathrm{CM}(R) = \mathrm{GFlat}(R)$, or at least part of it, and seeing if there is a frobenius structure there. In general, this will not be the case as the following recent result shows.

Proposition 2.2.4. [17, 4.4] The following are equivalent over any ring A.

1. A is left perfect;
2. The category of left Gorenstein flat A-modules is frobenius;
3. Every cotorsion module is Gorenstein flat.

However, if A is any ring, then it is known that there is a frobenius subcategory of A-GFlat. Recall that an R-module C is *cotorsion* if $\mathrm{Ext}^1_A(F, C) = 0$ for all flat A-modules F.

Proposition 2.2.5. [28, 3.4] Let A be any ring. Then the exact subcategory A-GFlat$\,\cap$ A-Cot of A-Mod consisting of all Gorenstein flat modules that are cotorsion is a frobenius category with the projective-injectves being the flat cotorsion A-modules.

It is known that any pure injective module is cotorsion (see [24, 5.3.22] for a proof). In particular, if R is a hypersurface, there is an inclusion $\mathrm{GFlat}(R)_{PI}$ into the class $\mathrm{GFlat}(R) \cap \mathrm{Cot}(R)$. Not every pure-injective module is cotorsion (in such a ring one must have each prime ideal maximal, see [18, 3.5]) and the class of pure injective modules is rarely extension closed, so $\mathrm{GFlat}(R)_{PI}$ will not be frobenius in its own right. An area of further investigation may be to see how the classes $\mathrm{GFlat}(R) \cap \mathrm{Cot}(R)$ and $\mathrm{GFlat}(R^{\sharp\sharp}) \cap \mathrm{Cot}(R^{\sharp\sharp})$ relate and see whether the equivalence between $\mathrm{GFlat}(R)_{PI}$ and $\mathrm{GFlat}(R^{\sharp\sharp})_{PI}$ are contained in this relationship and how this will appear in the triangulated category induced by the frobenius category described above.

2.3 Ext-depth and balanced big Cohen-Macaulay modules

Let R be a d-dimensional Cohen-Macaulay ring, $\boldsymbol{y}$ a system of parameters for R and M a finitely generated R-module. Since local cohomology is invariant under radical, the functors $H^i_{\mathfrak{m}}(-)$ and $H^i_{(\boldsymbol{y})}(-)$ are naturally isomorphic for all $i \geq 0$; therefore we see that $H^i_{\mathfrak{m}}(M) = 0$ for all $i < d$, that is M is in $\mathrm{CM}(R)$, if and only if $\mathrm{grade}(\boldsymbol{y}, M) = d$, in other words the ideal $(\boldsymbol{y})$ contains an M-sequence of length d, which is therefore maximal as (y) is generated by exactly d-elements; in particular $\boldsymbol{y}$ is an M-sequence. Consequently the following are equivalent:

1. M is a non-zero module in $\mathrm{CM}(R)$;
2. Every system of parameters is an M-sequence;
3. One system of parameters is an M-sequence.

These equivalences fail when M is not finitely generated, but Hochster used the latter two equivalences (which do remain) to define the following class extending $\mathrm{CM}(R)$.

Definition 2.3.1. Let R be a Cohen-Macaulay ring. An R-module is a *balanced big Cohen-Macaulay module* if every system of parameters is an M-sequence.

We will let $\mathrm{bbCM}(R)$ denote the class of balanced big Cohen-Macaulay modules. These modules have been extensively studied using commutative algebra and homological methods (see the relevant sections in [48] and [12]). By considering Ext-depth, we can relate $\mathrm{bbCM}(R)$ to $\varinjlim \mathrm{CM}(R)$ when R admits a canonical module. Before doing this, let us make some observations. Since we have a chain of definable subcategories

$$\langle \mathrm{CM}(R) \rangle \subseteq \mathrm{CohCM}(R)$$

any module in $\langle \mathrm{CM}(R) \rangle$ will vanish on the functors that define $\mathrm{CohCM}(R)$, which are $\{\mathrm{Ext}^i_R(k, -) : i < d\}$. Consequently every element of $\langle \mathrm{CM}(R) \rangle$ will have Ext-depth either d or ∞. This is more obvious when R admits a canonical module, since then $\langle \mathrm{CM}(R) \rangle = \varinjlim \mathrm{CM}(R)$ and $\mathrm{Ext}^i_R(k, -)$ preserves direct limits. We can therefore partition these categories into two parts, one consisting of the modules with Ext-depth d, and one with those of Ext-depth ∞. Accordingly, we will define

$$\varinjlim \mathrm{CM}(R)_d = \{M \in \varinjlim \mathrm{CM}(R) : \text{E-dp}(M) = d\}$$

and

$$\varinjlim \mathrm{CM}(R)_\infty = \{M \in \mathrm{CohCM}(R) : \text{E-dp}(M) = \infty\}.$$

We use the same notation for the respective classes in $\langle \mathrm{CM}(R) \rangle$ and $\mathrm{CohCM}(R)$.

Proposition 2.3.2. Let R be a Cohen-Macaulay ring, then the classes $\langle \mathrm{CM}(R) \rangle_\infty$ and $\mathrm{CohCM}(R)_\infty$ are definable subcategories of $\mathrm{Mod}(R)$.

Proof. Let X be the collection of finitely presented functors defining $\langle \mathrm{CM}(R) \rangle$. Then $\langle \mathrm{CM}(R) \rangle_\infty = \{M : F(M) = 0 \text{ for all } F \in X \cup \{k \otimes -\}\}$ by 1.5.8. The same applies for $\mathrm{CohCM}(R)_\infty$. □

Let us now consider one way $\mathrm{CohCM}(R)_\infty$ sits inside $\mathrm{CohCM}(R)$.

Proposition 2.3.3. Let R be a Cohen-Macaulay ring and $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence of modules in $\mathrm{CohCM}(R)$. Then $M \in \mathrm{CohCM}(R)_\infty$ if and only if $L, N \in \mathrm{CohCM}(R)_\infty$. If R admits a canonical module, the same result holds for $\varinjlim \mathrm{CM}(R)_\infty$ in $\varinjlim \mathrm{CM}(R)$.

Proof. It is clear that $\mathrm{CohCM}(R)_\infty$ is extension closed, so we only show the other implication. Applying $\mathrm{Hom}_R(k, -)$ to the short exact sequence gives us an exact sequence

$$0 \longrightarrow \mathrm{Ext}^d_R(k, L) \longrightarrow \mathrm{Ext}^d_R(k, M) \longrightarrow \mathrm{Ext}^d_R(k, N) \longrightarrow \mathrm{Ext}^{d+1}_R(k, L).$$

If $M \in \mathrm{CohCM}(R)_\infty$, it is clear that $\mathrm{Ext}^d_R(k, L) = 0$, hence $L \in \mathrm{CohCM}(R)_\infty$. But then $\mathrm{Ext}^{d+1}_R(k, L) = 0$, so $\mathrm{Ext}^d_R(k, N) = 0$, giving $N \in \mathrm{CohCM}(R)_\infty$. Assume that L, M and N were now modules in $\varinjlim \mathrm{CM}(R)$. By considering the inclusion $\varinjlim \mathrm{CM}(R) \hookrightarrow \mathrm{CohCM}(R)$ it is clear that $L, N \in \mathrm{CohCM}(R)_\infty \iff M \in \mathrm{CohCM}(R)_\infty$. However, each of these modules is actually in $\varinjlim \mathrm{CM}(R)$, hence the result holds there.

□

Since both $\langle \mathrm{CM}(R) \rangle$ and $\langle \mathrm{CM}(R) \rangle_\infty$ are definable subcategories of $\mathrm{Mod}(R)$, they correspond to closed subsets of the Ziegler spectrum. Therefore in the subspace topology the indecomposable pure-injective modules in the complement $\langle \mathrm{CM}(R) \rangle \setminus \langle \mathrm{CM}(R) \rangle_\infty$ corresponds to an open subset of the Ziegler spectrum contained in $\langle \mathrm{CM}(R) \rangle$. As every element of $\langle \mathrm{CM}(R) \rangle$ has Ext-depth equal to d or ∞, the indecomposable pure-injective modules in this Ziegler-open subset are precisely the elements in $\langle \mathrm{CM}(R) \rangle$ that have Ext-depth equal to d. In the situation when R admits a canonical module, we can explicitly describe the complement $\varinjlim \mathrm{CM}(R) \setminus \varinjlim \mathrm{CM}(R)_\infty$, using what is essentially a corollary to 2.1.6.

Proposition 2.3.4. Let $(R, \mathfrak{m}, k)$ be a d-dimensional Cohen-Macaulay ring with a canonical module Ω. Then $\varinjlim \mathrm{CM}(R)_d$ coincides with the class of balanced big Cohen-Macaulay modules.

Proof. If M is a balanced big Cohen-Macaulay module it is clear that $M \in \varinjlim \mathrm{CM}(R)$ since every M-sequence is a weak M-sequence by 2.1.6. In particular, we have $\text{E-dp}(M) \geq d$. Note that for every system of parameters $(\boldsymbol{y})$ we have $R/(\boldsymbol{y}) \otimes_R M \simeq M/\boldsymbol{y}M \neq 0$, hence $\text{T-cogr}((\boldsymbol{y}), M) = 0$, so $\text{E-gr}((\boldsymbol{y}), M) < \infty$. Since $\text{E-gr}((\boldsymbol{y}), M) = \text{E-dp}(M)$, we see that $\text{E-dp}(M) = d$. The other direction is very similar: if $M \in \varinjlim \mathrm{CM}(R)_d$ then $\text{T-codp}(M) = 0$, but for every system of parameters $\boldsymbol{y}$ we have $\text{T-cogr}((\boldsymbol{y}), M) = \text{T-codp}(M) = 0$, hence $M/\boldsymbol{y}M \neq 0$, so M is an M-sequence. $\square$

We saw in 2.1.13 that $\mathrm{Hom}_R(-, \Omega)$ is an endofunctor on $\varinjlim \mathrm{CM}(R)$, but 1.4.25 shows us that for any $M \in \varinjlim \mathrm{CM}(R)_\infty$ we have $\mathrm{Hom}_R(M, \Omega) = 0$, so the image of $\mathrm{Hom}_R(-, \Omega)$ only depends on $\varinjlim \mathrm{CM}(R)_d$. We can see exactly how this functor behaves on this collection of modules.

Proposition 2.3.5. Let $(R, \mathfrak{m}, k)$ be a d-dimensional complete Cohen-Macaulay ring with canonical module Ω. If M is a balanced big Cohen-Macaulay R-module, the following hold:

1. $M^* = \mathrm{Hom}(M, \Omega)$ is also a balanced big Cohen-Macaulay module.
2. Ω is an injective cogenerator in $\mathrm{bbCM}(R)$.
3. If M does not have a direct summand of infinite depth, then the canonical morphism $M \longrightarrow M^{**}$ is injective.

Proof.

1. We know that $M^* \in \varinjlim \mathrm{CM}(R)$ by 2.1.13, but 2.1.15 shows that $\text{E-dp}(M^*) = d$, hence $M^* \in \varinjlim \mathrm{CM}(R)_d$, which is the class of balanced big Cohen-Macaulay modules.

2. This follows immediately from Grothendieck local duality 1.4.25.
3. The proof of this is essentially the same as the proof that $M \simeq M^{**}$ for $M \in \mathrm{CM}(R)$. Indeed, if $\boldsymbol{x}$ is an R-sequence, we may extend it to a system of parameters which is then an M-sequence as M is a balanced big Cohen-Macaulay module. If M does not have a direct summand of infinite Ext-depth, we can reduce to the case that $\dim R = 0$, as is done in [12, Theorem 3.3.10]. In this situation, $\Omega \simeq E(k)$, and then $M \longrightarrow M^{\vee\vee}$ is injective.

□

Let us illustrate how depth can be used to identify the balanced big Cohen-Macaulay modules.

Example 2.3.6. We consider the A_∞ curve singularity $R = k[[x, y]]/(x^2)$. This is a one-dimensional Gorenstein ring, so $\varinjlim \mathrm{CM}(R) = \mathrm{CohCM}(R) = \{M \in \mathrm{Mod}(R) : \mathrm{Hom}_R(k, M) = 0\}$. It was shown in [14] that this is one of two hypersurface curves that has countably many indecomposable maximal Cohen-Macaulay modules. These were classified in the same paper, and they are

1. the ring, R;
2. the ideals $I_j := (x, y^j)$, where $j \geq 1$;
3. the ideal $I_\infty := xR$.

Since R is a complete local ring, each of these is an indecomposable pure-injective R-module. The remaining indecomposable pure injective R-modules in $\varinjlim \mathrm{CM}(R)$ were classified by Puninski in [40]. They are

1. $Q = Q(R)$, the total quotient ring of R;
2. $\overline{R}$, the integral closure of R in Q;
3. the Laurent series $L := k((y))$, viewed as an R-module through the morphism $R \longrightarrow R/(x)$.

Let us now determine the Ext-depth of each of these indecomposable pure-injectives.

- Let us start with Q. Since R is a Gorenstein ring, Q is an injective R-module by [24, Theorem 9.3.3] and therefore $\mathrm{Ext}^i_R(k, Q) = 0$ for all $i \geq 1$, but since $R \in \mathrm{CohCM}(R)$ as well, it follows that $\mathrm{Ext}^i_R(k, Q) = 0$ for all i, hence $\text{E-dp}(Q) = \infty$.
- Let us now consider $L := k((y))$. The quotient $R \longrightarrow k[[y]]$ sends the maximal ideal $\mathfrak{m} = (x, y)$ of R to the maximal ideal (y) of $k[[y]]$. Consequently, the independence

theorem of local cohomology tells us that for every $k[[y]]$-module N there is an isomorphism of R-modules

$$H^i_{\mathfrak{m}}(N|_R) \simeq H^i_{(y)}(N)|_R.$$

In particular, there is an isomorphism $H^i_{\mathfrak{m}}(L) \simeq H^i_{(y)}(k((y)))|_R$. Since $L \in \varinjlim \mathrm{CM}(R)$, we know that $H^0_{\mathfrak{m}}(L) = 0$, and therefore $H^0_{(y)}(k((y))) = 0$ since the ring homomorphism is just factoring by x. But we also know that $k((y))$ is an injective $k[[y]]$ module for the same reason as in the case of Q, and therefore $H^i_{(y)}(k((y))) = 0$ for all i. It follows from the independence theorem that $\text{E-dp}(L) = \infty$.

- Lastly we consider $\overline{R}$. We show that $k \otimes_R \overline{R} \neq 0$, hence $\text{E-dp}(\overline{R}) = 1$. In [40, 2.1], Puninski shows that $y\overline{R} \subset \overline{R}$ with $y\overline{R}$ maximal. Thus $\overline{R}/y\overline{R} \simeq k$, the unique simple R-module. Since $\overline{R} \longrightarrow \overline{R}/y\overline{R} \longrightarrow 0$ is exact, we have $\overline{R} \otimes_R k \longrightarrow k \otimes_R k \longrightarrow 0$ exact. Since $k \otimes_R k \neq 0$, we cannot have $\overline{y} \otimes k = 0$. This shows the claim.

In particular, the only non-finitely generated indecomposable balanced big Cohen-Macaulay R-module is $\overline{R}$, while the indecomposable pure-injectives in $\varinjlim \mathrm{CM}(R)_\infty$ are $k((y))$ and Q. Puninski shows that the R-modules that have finite length over their endomorphism rings are precisely $k((y))$ and Q.

Remark. As we can see, when R has a canonical module the class $\mathrm{bbCM}(R)$ is not definable, as it is not even closed under direct summands: if $\varinjlim \mathrm{CM}(R)$ contains a module of infinite Ext-depth, say N, then the module $R \oplus N$ has Ext-depth d, so is a balanced big Cohen-Macaulay module, yet $N \notin \mathrm{bbCM}(R)$ by assumption. In the case that R is Gorenstein, the module N will always exist, since one can consider $E(R)$, which is flat. The definable closure of $\mathrm{bbCM}(R)$, however, is also $\varinjlim \mathrm{CM}(R)$. If one wished to eliminate all infinite depth direct summands, one could consider the stabilisation of $\varinjlim \mathrm{CM}(R)$ with respect to $\varinjlim \mathrm{CM}(R)_\infty$. We will consider this in more detail in a subsequent discussion.

Chapter 3

Categorical properties of $\mathbf{CohCM}(R)$ and $\varinjlim \mathbf{CM}(R)$

Having considered some ways in which $\mathrm{CohCM}(R)$ and $\varinjlim \mathrm{CM}(R)$ reflect the properties of $\mathrm{CM}(R)$, we will now consider them as categories in their own right and establish some intrinsic properties as well as how they relate to the entirety of the module category. Before doing this, we will introduce some further background information that will be relevant to our discussion.

3.1 Cotorsion pairs and Kaplansky classes

Definition 3.1.1. Let R be any ring and $\mathcal{F}$ a class of R-modules. If M is an R-module, we say that a morphism $\varphi : F \longrightarrow M$, where $F \in \mathcal{F}$ is an $\mathcal{F}$-*precover* of M if for any morphism $\psi : F' \longrightarrow M$ with $F' \in \mathcal{F}$ there is a morphism $F' \longrightarrow F$ such that

$$\begin{tikzcd} F' \arrow[d] \arrow[dr, "\psi"] & \\ F \arrow[r, "\varphi"'] & M \end{tikzcd}$$

commutes. If every R-module has an $\mathcal{F}$-precover, we say the class $\mathcal{F}$ is *precovering*. We say that an $\mathcal{F}$-precover $\varphi : F \longrightarrow M$ is an $\mathcal{F}$-*cover* if in the commutative diagram

$$\begin{tikzcd} F \arrow[d, "\alpha"'] \arrow[dr, "\varphi"] & \\ F \arrow[r, "\varphi"'] & M \end{tikzcd}$$

we have $\alpha \in \mathrm{Aut}(F)$. If every R-module has an $\mathcal{F}$-cover, we say that $\mathcal{F}$ is *covering*.

The dual notions are called *preenvelopes* and *envelopes*. As we can now see, a class $\mathcal{C} \in \text{mod}(R)$ is covariantly finite if and only if it is preenveloping, and the injective envelope of an R-module, as described previously, coincides with the notion here.

Definition 3.1.2. Let R be any ring and $\mathcal{A}$ a class of R-modules. We define

$$\mathcal{A}^{\perp} = \{M \in R\text{-Mod} : \text{Ext}^1_R(A, M) = 0 \text{ for all } A \in \mathcal{A}\}$$

and

$$^{\perp}\mathcal{A} = \{M \in R\text{-Mod} : \text{Ext}^1_R(M, A) = 0 \text{ for all } A \in \mathcal{A}\}.$$

Suppose that $\mathcal{F}$ is a class of R-modules and M is an R-module such that $\varphi : F \longrightarrow M$ is an $\mathcal{F}$-precover. We say that φ is *special* if it is surjective and $\text{Ker}(\varphi) \in \mathcal{F}^{\perp}$. We say that $\mathcal{F}$ is *special precovering* if every R-module has a special $\mathcal{F}$-precover.

The dual notion is a *special preenvelope*: if $\psi : M \longrightarrow G$ is a $\mathcal{G}$-preenvelope, it is special if ψ is injective and $\text{Coker}(\psi) \in {}^{\perp}\mathcal{G}$.

If $\mathcal{F}$ is a covering class, we can have monic envelopes that need not be special. We have, in fact, already encountered several examples of classes that are covering and preenveloping.

Proposition 3.1.3. Let $\mathcal{D} \subset R$-Mod be a definable subcategory. Then $\mathcal{D}$ is both covering and preenveloping.

Proof. These are [20, Theorem 2.7] and [39, Prop. 3.4.42] respectively. $\square$

The following definition introduces a notion that provides a good source of special precovers and preenvelopes.

Definition 3.1.4. Let R be any ring. We say that a pair of classes of R-modules $(\mathcal{A}, \mathcal{B})$ is a *cotorsion pair* if $\mathcal{A}^{\perp} = \mathcal{B}$ and $\mathcal{A} = {}^{\perp}\mathcal{B}$.

Example 3.1.5.

1. If R is any ring, then the pairs $(R\text{-Mod}, \mathcal{I}_0)$ and $(\mathcal{P}_0, R\text{-Mod})$ are cotorsion pairs. Here $\mathcal{I}_0$ denotes the class of injective modules, and $\mathcal{P}_0$ denotes the class of projective modules.

2. Let $\mathcal{F}$ denote the class of all flat R-modules. Then $(\mathcal{F}, \mathcal{EC})$ is a cotorsion pair. We call the elements of $\mathcal{EC}$ *Enochs cotorsion* modules.

We will see many more examples of cotorsion pairs. Let us see how cotorsion pairs relate to special precovers and preenvelopes.

Lemma 3.1.6. [29, Cor. 5.19] Let R be any ring and $(\mathcal{A}, \mathcal{B})$ a cotorsion pair of R-modules. If $\mathcal{A}$ is covering, it is special precovering, while if $\mathcal{B}$ is enveloping, it is special preenveloping. $\square$

The following result, known as Salce's lemma, shows that the precovering and preenveloping properties of a cotorsion pair are equivalent.

Lemma 3.1.7. [29, 5.20] Let $\mathcal{R}$ be any ring and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair of R-modules. The following are equivalent:

1. $\mathcal{A}$ is special precovering;
2. $\mathcal{B}$ is special preenveloping.

In this situation, we call the cotorsion pair *complete*. $\square$

Let us list some further properties of cotorsion pairs.

Definition 3.1.8. Let R be any ring and $\mathcal{C}$ be a class of R-modules.

1. Say that $\mathcal{C}$ is *resolving* if $\mathcal{P}_0 \subset \mathcal{C}$ and whenever $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is a short exact sequence of R-modules with $M, N \in \mathcal{C}$, we have $L \in \mathcal{C}$.
2. Say that $\mathcal{C}$ is *coresolving* if $\mathcal{I}_0 \subset \mathcal{C}$ and whenever $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is a short exact sequence of R-modules with $L, M \in \mathcal{C}$, we have $N \in \mathcal{C}$.

Lemma 3.1.9. Let R be any ring and $(\mathcal{A}, \mathcal{B})$ a cotorsion pair. The following are equivalent.

1. $\mathcal{A}$ is resolving;
2. $\mathcal{B}$ is coresolving;
3. $\operatorname{Ext}^i_R(A, B) = 0$ for all $i \geq 1$ and $A \in \mathcal{A}, B \in \mathcal{B}$.

In this case, we call the cotorsion pair *hereditary*. $\square$

Definition 3.1.10. Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a cotorsion pair.

1. We say $\mathfrak{C}$ is *perfect* if $\mathcal{A}$ is covering and $\mathcal{B}$ is enveloping.
2. We say $\mathfrak{C}$ is *closed* if $\mathcal{A}$ is closed under direct limits.

Let us now give a good source of cotorsion pairs.

Definition 3.1.11. Let R be a ring and $\mathcal{F}$ a class of R-modules. We say $\mathcal{F}$ is a *Kaplansky class* if there is a cardinal number κ such that for all $F \in \mathcal{F}$ and $x \in F$ there is a module $M \subseteq F$ such that $x \in M$, $M \in \mathcal{F}$, $F/M \in \mathcal{F}$ and $\mathrm{card}(M) \leq \kappa$.

The following result is due to Enochs and Lopez-Ramos.

Theorem 3.1.12. [23, Theorem 2.9] Let R be a ring and $\mathcal{F}$ a Kaplansky class of R-modules. If $\mathcal{F}$ is closed under extensions, direct limits and $\mathcal{P}_0 \subset \mathcal{F}$, then $(\mathcal{F}, \mathcal{F}^{\perp})$ is a perfect cotorsion pair of R-modules. $\square$

Let us provide some examples of cotorsion pairs that we will encounter, or have already encountered.

Example 3.1.13.

1. Let R be an n-Gorenstein ring. By 1.7.7 the class R-GFlat is extension closed, direct limit closed and contains the projectives. Thus (R-GFlat, $\mathcal{GC}$) is a perfect cotorsion pair. We call the modules in $\mathcal{GC}$ *Gorenstein cotorsion* modules.
2. Let R be an n-Gorenstein ring. Then (R-GProj, $\mathcal{I}_n$) is a cotorsion pair by 1.7.13.
3. Let R be an n-Gorenstein ring and $\mathcal{GI}$ denote the class of Gorenstein injective R-modules. Then ($\mathcal{I}_n$, R-GInj) is a cotorsion pair by 1.7.12.

3.2 Cotorsion pairs with CohCM(R) and $\langle$CM(R)$\rangle$

Let us now assume that $(R, \mathfrak{m}, k)$ is an arbitrary Cohen-Macaulay ring. Holm showed in [30, Theorem D] that when R admits a canonical module the pair

$$(\varinjlim \mathrm{CM}(R), \varinjlim \mathrm{CM}(R)^{\perp})$$

is a perfect, hereditary, closed cotorsion pair. We will show the corresponding result holds for CohCM(R) over any Cohen-Macaulay ring, and that we can extend said result by Holm to this case as well. The following result is an application of the Downward Löwenheim-Skolem theorem from model theory.

Proposition 3.2.1. Let R be any ring and M an R-module. If N is a submodule of M then there is a pure submodule P of M such that $N \subseteq P \subseteq M$ such that $\mathrm{card}(P) \leq \mathrm{card}(N) \cdot \mathrm{card}(R)$ if R is infinite, and with $\mathrm{card}(P) \leq \mathrm{card}(N) \cdot \aleph_0$ if R is finite. $\square$

An algebraic proof of the above result can be found at [22, Lemma 2.1.1]. Using the above result we can prove the following lemma.

Lemma 3.2.2. Let R be any ring and $\mathcal{D}$ a definable subcategory of R-modules. Then $\mathcal{D}$ is a Kaplansky class.

Proof. Let $\kappa = \mathrm{card}(R)$. Suppose $M \in \mathcal{D}$, $x \in M$ and consider the cyclic module Rx. By the above result there is a pure submodule $P \subseteq M$ such that $Rx \subseteq P \subseteq M$ such that $\mathrm{card}(P) \leq \kappa$. Since $\mathcal{D}$ is definable it is closed under pure submodules and pure quotients, so both P and M/P are in $\mathcal{D}$. Since κ was independent of our module, it suffices for all modules. This shows $\mathcal{D}$ is a Kaplansky class. $\square$

Corollary 3.2.3. Let R be a noetherian ring and $\mathcal{C}$ a covariantly finite subcategory of $\mathrm{mod}(R)$. Then $\mathcal{C}$ is a Kaplansky class.

Proof. Since $\mathcal{C}$ is covariantly finite, $\varinjlim \mathcal{C}$ is a definable subcategory, so is a Kaplansky class. Let κ be its corresponding cardinal. If $M \in \mathcal{C}$ and $x \in M$ we have a module P such that $x \in P$, $M/P, P \in \varinjlim \mathcal{C}$ and $\mathrm{card}(P) \leq \kappa$. As R is noetherian, both P and M/P are finitely generated, so lie in $(\varinjlim \mathcal{C})^{\mathrm{fp}} = \mathcal{C}$. Thus $\mathcal{C}$ is itself a Kaplansky class, whose cardinal number is also κ. $\square$

Corollary 3.2.4. Let R be any ring and $\mathcal{D}$ a definable subcategory. If $\mathcal{D}$ is extension closed and contains R, the pair $(\mathcal{D}, \mathcal{D}^{\perp})$ is a perfect cotorsion pair.

Proof. By the above, $\mathcal{D}$ is a Kaplansky class. If $R \in \mathcal{D}$, then $\mathcal{P}_0 \subseteq \mathcal{D}$ and since we assumed $\mathcal{D}$ is extension closed the result is immediate from 3.1.12. □

The following is essentially another corollary.

Theorem 3.2.5. Let $(R, \mathfrak{m}, k)$ be a d-dimensional Cohen-Macaulay ring. Then

$$(\mathrm{CohCM}(R), \mathrm{CohCM}(R)^{\perp})$$

is a perfect, hereditary, closed cotorsion pairs. If $\langle \mathrm{CM}(R) \rangle$ is extension closed, the same can be said of $(\langle \mathrm{CM}(R) \rangle, \langle \mathrm{CM}(R) \rangle^{\perp})$

Proof. By the preceeding corollary, we see that both $\mathfrak{C} = (\mathrm{CohCM}(R), \mathrm{CohCM}(R)^{\perp})$ and $(\langle \mathrm{CM}(R) \rangle, \langle \mathrm{CM}(R) \rangle^{\perp})$ are perfect, closed cotorsion pairs. To see that $\mathfrak{C}$ is hereditary, observe that it contains all free modules and if $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is an exact sequence of R-modules such that $\text{E-dp}(M), \text{E-dp}(N) \geq d$, then $\text{E-dp}(L) \geq d$. If $\langle \mathrm{CM}(R) \rangle$ is extension closed, then the same proof applies in this case. □

As a consequence of this, the class $\mathrm{CohCM}(R)$ is special precovering in $\mathrm{Mod}(R)$, so we can consider left resolutions and the corresponding dimension. If M is an R-module, we will let $\dim_{\mathrm{CohCM}(R)}(M)$ denote the minimal length of a left $\mathrm{CohCM}(R)$-resolution of M.

Proposition 3.2.6. Let R be a d-dimensional Cohen-Macaulay ring and M be an R-module. Then $\text{E-dp}(M) + \dim_{\mathrm{CohCM}(R)}(M) \geq \dim R$.

Proof. Assume that $\dim_{\mathrm{CohCM}(R)}(M) = n$ with $0 < n \leq d$ otherwise there is nothing to prove. Then there is an exact sequence

$$\mathcal{C} : 0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow M \longrightarrow 0$$

with each $C_i \in \mathrm{CohCM}(R)$. We can decompose $\mathcal{C}$ into n short exact sequences of the form

$$\mathcal{S}_i : 0 \longrightarrow \Omega^{i+1}(M) \longrightarrow C_i \longrightarrow \Omega^i(M) \longrightarrow 0$$

where $\Omega^i(M)$ is the image of the map $C_i \longrightarrow C_{i-1}$. In particular, we have $\Omega^n(M) \simeq C_n$ and $\Omega^0(M) \simeq M$. Applying the depth lemma to $\mathcal{S}_i$ gives the inequality

$$\text{E-dp}(\Omega^i(M)) + 1 \geq \text{E-dp}(\Omega^{i+1}(M)),$$

while the definition of $\dim_{\text{CohCM}(R)}$ yields

$$\dim_{\text{CohCM}(R)}(\Omega^{i+1}(M)) = \dim_{\text{CohCM}(R)}(\Omega^i(M)) - 1.$$

Combining these gives that

$$\text{E-dp}(\Omega^i(M)) + \dim_{\text{CohCM}(R)}(\Omega^i(M)) \geq \text{E-dp}(\Omega^{i+1}(M)) + \dim_{\text{CohCM}(R)}(\Omega^{i+1}(M))$$

for every i. Iterating this shows that

$$\text{E-dp}(M) + \dim_{\text{CohCM}(R)}(M) \geq \text{E-dp}(\Omega^n(M)) = d,$$

which proves the claim. □

Corollary 3.2.7. Let M be an R-module such that $\text{E-dp}(M) < \infty$. Then

$$\dim_{\text{CohCM}(R)}(M) \geq \text{T-codp}(M).$$

Proof. If $\text{E-dp}(M) < \infty$, then $\text{T-codp}(M) < \infty$ and $\text{E-dp}(M) + \text{T-codp}(M) \leq d$ by 1.5.8. But $\dim_{\text{CohCM}(R)}(M) + \text{E-dp}(M) \geq d$. Combing these inequalities gives the result. □

Let us now turn our attention to $\varinjlim \text{CM}(R)$ over an arbitrary Cohen-Macaulay ring. The following result provides a generic characterisation of the direct limit closure of certain classes of modules.

Proposition 3.2.8. [29, 8.40] Let R be any ring and $\mathcal{C}$ a class consisting of FP_2 modules such that $\mathcal{C}$ is closed under extensions, direct summands and $R \in \mathcal{C}$. Then $\varinjlim \mathcal{C} = {}^{\top}(\mathcal{C}^{\top})$ is a covering class.

Here $\mathcal{C}^{\top} = \{M \in R\text{-Mod} : \text{Tor}_1^R(C, M) = 0 \text{ for all } C \in \mathcal{C}\}$ and vice-versa for ${}^{\top}\mathcal{C}$. Returning to R being a Cohen-Macaulay ring, the class $\text{CM}(R)$ fits the hypotheses of the preceding result, hence $\varinjlim \text{CM}(R) = \text{CM}(R)^{\top\top}$ as R is commutative. One can deduce that $\varinjlim \text{CM}(R)$ is an extension closed covering class containing the projective modules, which is also closed under direct limits. We are able to use the above proposition to extend the cotorsion pair given by Holm in [30, Theorem D.].

Theorem 3.2.9. Let R be a Cohen-Macaulay ring. Then $(\varinjlim \mathrm{CM}(R), \varinjlim \mathrm{CM}(R)^{\perp})$ is a perfect cotorsion pair.

Proof. Since $\mathrm{CM}(R)$ is closed under finite direct sums the class $\varinjlim \mathrm{CM}(R)$ is closed under pure submodules by [39, 3.4.35] and pure quotients by [31, 3.1]. Since the proof of 3.2.2 only required closure under pure submodules and quotients, the same proof shows $\varinjlim \mathrm{CM}(R)$ is a Kaplansky class. Moreover, it is extension closed by 3.2.8. Indeed, if $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is a short exact sequence of R-modules with $L, N \in \varinjlim \mathrm{CM}(R) = {}^{\top}(\mathrm{CM}(R)^{\top})$, then for any $C \in \varinjlim \mathrm{CM}(R)^{\top}$ we have

$$\mathrm{Tor}_1^R(C, L) \longrightarrow \mathrm{Tor}_1^R(C, M) \longrightarrow \mathrm{Tor}_1^R(C, N)$$

exact, and the outer terms are zero. Moreover, it is clear that $\varinjlim \mathrm{CM}(R)$ is closed under direct limits and contains all the projective R-modules as it contains all the flat modules. Therefore by 3.1.12 the pair $(\varinjlim \mathrm{CM}(R), \varinjlim \mathrm{CM}(R)^{\perp})$ is a perfect cotorsion pair. $\square$

Remark. It is not clear that this is a hereditary cotorsion pair, since it it is not obvious that if $\varinjlim \mathrm{CM}(R)$ is resolving as we cannot deduce that $\mathrm{Tor}_2^R(C, M) = 0$ for $M \in \varinjlim \mathrm{CM}(R)$ and $C \in \mathrm{CM}(R)^{\top}$.

Let us give an example of the class $\mathrm{CM}(R)^{\top}$ over a Gorenstein local ring.

Lemma 3.2.10. Let R be a Gorenstein local ring. Then $\mathrm{CM}(R)^{\top} = \mathcal{I}_{<\infty}$.

Proof. Since R is Gorenstein $\varinjlim \mathrm{CM}(R) = \mathrm{GFlat}(R)$, which is also equal to $\mathrm{CM}(R)^{\top\top}$. But $\mathrm{GFlat}(R) = (\mathcal{I}_{<\infty})^{\top}$. But we also know that M has finite injective dimension if and only if $\mathrm{Tor}_1(M, F) = 0$ for all $F \in \mathrm{GFlat}(R)$, thus $\mathrm{CM}(R)^{\top} = \mathcal{I}_{<\infty}$. $\square$

3.3 Closure properties of CohCM(R) and $\varinjlim$ CM(R)

Let R be a Cohen-Macaulay ring. We will now consider some categorical properties of CohCM(R). Since CohCM(R) is a definable subcategory, it is closed under pure injective envelopes. However, using Matlis's results about injective modules over Noetherian commutative rings, we can obtain another closure result.

Proposition 3.3.1. Let $\dim R \geq 1$, then both CohCM(R) and CohCM$(R)_\infty$ are closed under injective hulls. In particular, they both have enough injectives.

Proof. Let $M \in$ CohCM(R) and consider its injective envelope $E(M)$ in Mod(R). By Matlis's results on the structure of injective modules 1.4.17, one has

$$E(M) \simeq \bigoplus_{p \in \mathrm{Spec(R)}} E(R/\mathfrak{p})^{(X_\mathfrak{p})},$$

where $\mathrm{card}(X_\mathfrak{p}) = \dim_{k(\mathfrak{p})} \mathrm{Hom}_R(R/\mathfrak{p}, M)_\mathfrak{p}$. Since $M \in$ CohCM(R) and $\dim R \geq 1$, we know $\mathrm{Hom}_R(R/\mathfrak{m}, M) = 0$, hence $\mathrm{card}(X_\mathfrak{m}) = 0$, so $E(k)$ is not a direct summand of $E(M)$. If $\mathfrak{p} \neq \mathfrak{m}$ is a prime ideal, then $E(R/\mathfrak{p}) \in$ CohCM(R), since if $\mathrm{Hom}_R(k, E(R/\mathfrak{p})) \neq 0$, there is an $f : k \longrightarrow E(R/\mathfrak{p})$ that factors through $E(k)$. Yet this cannot happen since $\mathrm{Hom}_R(E(k), E(R/\mathfrak{p})) = 0$ if $\mathfrak{p} \neq \mathfrak{m}$. Consequently $E(M) \in$ CohCM(R) and so CohCM(R) has enough injectives. The same proof shows CohCM$(R)_\infty$ has enough injectives. □

We will later show that there is an alternative way to show that these classes are closed under injective hulls. Note that if $\mathcal{D}$ is an arbitrary definable category containing R, then it contains every projective and flat module. One does not need R to be commutative for this property.

Let us turn our attention to some inverse limits. By definition, all the definable subcategories we have considered are closed under direct limits, but in general definable subcategories are not closed under inverse limits. We will show that this is the case for CohCM(R) under certain assumptions on the ring. But before doing this, we consider how the inverse limit closure of CM(R) relates to $\varinjlim$ CM(R) when the ring is complete.

Proposition 3.3.2. Let R be a complete Cohen-Macaulay ring. Then $\varprojlim$ CM(R) $\subseteq$ $\varinjlim$ CM(R).

Proof. Let $(M_i, f_{ij})_I$ be an inverse system of modules in $\mathrm{CM}(R)$ whose inverse limit is M. Since each M_i is finitely generated and R is complete, M_i is also Matlis reflexive by 1.4.24. Consequently we can view $(M_i, f_{ij})_I$ as the Matlis dual of a directed system $(N_i, g_{ij})_I$ in the dual definable subcategory $\varinjlim \mathrm{CM}(R)^d$, where $N_i = M_i^\vee$ and $g_{ij} = \mathrm{Hom}_R(f_{ij}, E(k))$. Since $\varinjlim \mathrm{CM}(R)^d$ is definable, the direct limit of the system $(N_i, g_{ij})_I$, which we denote by N, lies in $\varinjlim \mathrm{CM}(R)^d$, and therefore $N^\vee \in \varinjlim \mathrm{CM}(R)$. However

$$N^\vee = (\varinjlim_I N_i)^\vee \simeq \varprojlim_I N_i^\vee \simeq \varprojlim_I M_i = M,$$

so $M \in \varinjlim \mathrm{CM}(R)$. $\square$

This above result enables us to give an alternative proof of 2.1.13

Corollary 3.3.3. If R is a complete local ring with canonical module Ω, then $\mathrm{Hom}_R(M, \Omega) \in \varinjlim \mathrm{CM}(R)$ for all $M \in \varinjlim \mathrm{CM}(R)$.

Proof. If $M = \varinjlim_I M_i$ with each $M_i \in \mathrm{CM}(R)$, then

$$\mathrm{Hom}_R(\varinjlim_I M_i, \omega) \simeq \varprojlim_I \mathrm{Hom}_R(M_i, \Omega) \in \varprojlim \mathrm{CM}(R) \subseteq \varinjlim \mathrm{CM}(R),$$

by the above proposition. $\square$

We see, from the above proposition, that we have inclusions $\varinjlim \varprojlim \mathrm{CM}(R) \subseteq \varinjlim \mathrm{CM}(R)$ since $\varinjlim \mathrm{CM}(R)$ is closed under direct limits. However, as stated above, definable subcategories are not frequently closed under inverse limits. The following result, due to Bergman, highlights a way to construct modules that can be found in the inverse limit closure of a class of modules. We state a less general form of the original result, as we do not need the most general case.

Lemma 3.3.4. [9, Corollary 11] Let R be any ring and $\mathfrak{C}$ a class of R-modules that is closed under direct products and let $0 \longrightarrow M \longrightarrow C_0 \longrightarrow C_1$ be an exact sequence of R-modules with $C_0, C_1 \in \mathcal{C}$. Then M can be written as an inverse limit of a system of modules in $\mathcal{C}$ with injective morphisms.

Proposition 3.3.5. Let R be a Cohen-Macaulay ring of dimension at least 3. Then $\mathrm{CohCM}(R)$ is not closed under inverse limits.

Proof. Pick an R-module M such that E-dp$(M) = 2$. Then $M \notin \mathrm{CohCM}(R)$, but $\mu_i(\mathfrak{m}, M) = 0$ for $i = 0, 1$. Consequently the first two terms of a minimal injective resolution of M lie in CohCM(R), so applying the previous lemma we can realise M as an inverse limit of modules in CohCM(R). □

Example 3.3.6. Let $R = k[[x, y, z, w]]$ be a four-dimensional regular local ring. Then the module $R/(x)$ has Ext-depth equal to three and is not Cohen-Macaulay as an R-module. Consequently we can use the above result to obtain $R/(x)$ as an inverse limit of modules in CohCM(R). □

Despite this result, there are certain inverse systems of modules in CohCM(R) whose inverse limits lie in CohCM(R).

Until stated otherwise, we will assume that R is any ring. We call a sequence of R-modules

$$\mathcal{T}: \qquad \cdots \longrightarrow T_{i+1} \xrightarrow{t_i} T_i \longrightarrow \cdots \longrightarrow T_1 \xrightarrow{t_0} T_0$$

a *tower*. One can form an inverse system $(T_i, g_{ij})_{i,j<\omega}$ from a tower $\mathcal{T}$ as above by setting

$$g_{ij} = t_i \circ t_{i+1} \circ \cdots \circ t_{j-1} : T_j \longrightarrow T_i$$

whenever $i \leq j$ and $g_{ii} = \mathrm{Id}_{T_i}$.

Following the construction given in [29, §3.1], we define a map $\nabla_{\mathcal{T}} : \prod_{i<\omega} T_i \longrightarrow \prod_{i<\omega} T_i$ via

$$(\cdots, a_i, \cdots, a_0) \mapsto (\cdots, a_i - t_i(a_{i+1}), \cdots, a_0 - t_0(a_1)).$$

An element $(\cdots, a_i, \cdots, a_0)$ lies in the kernel of $\nabla_{\mathcal{T}}$ if and only if $a_i = t_i(a_{i+1})$ for all $i < \omega$; in particular, if $i \leq j < \omega$, then

$$g_{ij}(a_j) = (t_i \circ t_{i+1} \circ \cdots \circ t_{j-1})(a_j) = (t_i \circ \cdots t_{j-2})(a_{j-1}) = \cdots = t_i(a_{i+1}) = a_i,$$

so $\mathrm{Ker}\, \nabla_{\mathcal{T}} \subseteq \varprojlim T_i$. Since the reverse inclusion is clear, this inclusion is actually equality. If $0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{U} \longrightarrow \mathcal{V} \longrightarrow 0$ is a short exact sequence of inverse systems obtained from towers, then there is an exact sequence

$$0 \longrightarrow \varprojlim T_i \longrightarrow \varprojlim U_i \longrightarrow \varprojlim V_i \longrightarrow \mathrm{Coker}(\nabla_{\mathcal{T}})$$

from the Snake Lemma. Consequently, if $\mathrm{Coker}(\nabla_{\mathcal{T}}) = 0$, then the inverse limit functor will be exact on this short exact sequence of inverse systems. Sufficient conditions for the inverse system $\mathcal{T}$ are known to ensure this happens.

Definition 3.3.7. Let I be a set and $\mathcal{M} = (M_i, f_{ij})_{i,j\in I}$ an inverse system. We say that $\mathcal{M}$ satisfies the *Mittag-Leffler condition* if for every $i \in I$ there is a $j \in I$ with $i \leq j$ such that $\mathrm{Im}(f_{ij}) = \mathrm{Im}(f_{ik})$ for all $k \geq j$.

For any inverse system $(N_i, g_{ij})_I$ and $i, j, k \in I$ such that $i \leq j \leq k$, we have $g_{ik} = g_{ij} \circ g_{jk}$ and therefore $\mathrm{Im}(g_{ik}) \subseteq \mathrm{Im}(g_{ij})$. The system satisfies the Mittag-Leffler condition when this inclusion is (eventually) equality.

Lemma 3.3.8. [29, Lemma 3.6] Let $\mathcal{T}$ be an inverse system induced by a tower. If $\mathcal{T}$ satisfies the Mittag-Leffler condition then $\mathrm{Coker}(\nabla_{\mathcal{T}}) = 0$. □

We can prove the following result without much difficulty.

Proposition 3.3.9. Let R be a Cohen-Macaulay ring and $\mathcal{T}$ be a tower of modules in CohCM(R). If the associated inverse system $(T_i, f_{ij})_{i,j<\omega}$ satisfies the Mittag-Leffler condition then $\varprojlim T_i \in \mathrm{CohCM}(R)$. If R admits a canonical module, the corresponding result holds for $\varinjlim \mathrm{CM}(R)$.

Proof. Let us prove the case for $\varinjlim \mathrm{CM}(R)$ as the CohCM(R) proof is practically identical. If $\mathcal{T}$ is a tower with inverse system (T_i, f_{ij}), then there is an exact sequence

$$0 \longrightarrow \varprojlim T_i \longrightarrow \prod_{i<\omega} T_i \xrightarrow{\nabla_{\mathcal{T}}} \prod_{i<\omega} T_i \longrightarrow \mathrm{Coker}(\nabla_{\mathcal{T}}) \longrightarrow 0.$$

If the inverse system satisfies the Mittag-Leffler condition then, by the above, $\mathrm{Coker}(\nabla_{\mathcal{T}}) = 0$. Since $\varinjlim \mathrm{CM}(R)$ is definable

$$0 \longrightarrow \varprojlim T_i \longrightarrow \prod_{i<\omega} T_i \xrightarrow{\nabla_{\mathcal{T}}} \prod_{i<\omega} T_i \longrightarrow 0$$

is a short exact sequence with $\prod_{i<\omega} T_i \in \varinjlim \mathrm{CM}(R)$, it is now clear that $\varprojlim T_i \in \varinjlim \mathrm{CM}(R)$. □

The property of an inverse system satisfying the Mittag-Leffler condition is related to a relativisation of this property, originally introduced by P. Rothmaler in [43] in a model-theoretic approach.

Definition 3.3.10. Let R be any ring and $\mathcal{Q}$ be a class of R-modules. We say that an R-module M is *$\mathcal{Q}$-Mittag-Leffler* if for every collection $\{N_i\}_{i\in I}$ of modules in $\mathcal{Q}$ the canonical map

$$M \otimes \prod_{i\in I} N_i \longrightarrow \prod_{i\in I}(M \otimes N_i)$$

is injective. If $\mathcal{Q} = R$-Mod, we say that M is a *Mittag-Leffler module.*

Before relating these two Mittag-Leffler concepts we require some further definitions.

Definition 3.3.11. [3, 3.7] Let B be a right R-module.

1. We say a direct system $(M_i, f_{ij})_{i,j\in I}$ of right R-modules is *B-stationary* if the associated inverse system $(\mathrm{Hom}(M_i, B), \mathrm{Hom}(f_{ij}, B))_{i,j\in I}$ satisfies the Mittag-Leffler condition 3.3.7.
2. An R-module M is said to be *B-stationary* if it is a direct limit of a directed system of finitely presented R-modules $(M_i, f_{ij})_{ij\in I}$ that is B-stationary.
3. If $\mathcal{B}$ is a class of R-modules, then we say M is *$\mathcal{B}$-stationary* if M is B-stationary for every $B \in \mathcal{B}$.

The following result relates $\mathcal{Q}$-Mittag-Leffler modules and $\mathcal{B}$-stationary modules.

Theorem 3.3.12. [3, Theorem 6.6] Let $\mathcal{B}$ be a class of right R-modules closed under direct sums and let $\mathcal{Q}$ be a class of left R-modules. Assume that every finitely presented R-module F has a $\mathcal{B}$-preenvelope $f : F \longrightarrow B$ such that $B^+ \in \mathcal{Q}$ and $f \otimes Q$ is monic for every $Q \in \mathcal{Q}$. Then the following are equivalent for an R-module M.

1. M is $\mathcal{B}$-stationary,
2. M is $\mathcal{Q}$-Mittag-Leffler.

In particular, if $\mathcal{B}$ is a definable subcategory and $\mathcal{Q}$ is its dual definable subcategory, then every R-module F has a $\mathcal{B}$-preenvelope $f : F \longrightarrow B$ such that $B^+ \in \mathcal{Q}$. Consequently the only condition that needs to be checked is that $f \otimes Q$ is monic for each $Q \in \mathcal{Q}$.

Definition 3.3.13. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. We say $\mathcal{C}$ is of *finite type* if there is a set of finitely presented modules $\mathcal{S}$ such that $\mathcal{B} = \mathcal{S}^\perp$ and $\mathcal{A} = {}^\perp(\mathcal{S}^\perp)$.

The following result shows one way the monic condition in 3.3.12 can be obtained.

Lemma 3.3.14. [3, 9.4] Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair of finite type and set $\mathcal{S} = \mathcal{A} \cap \text{mod-}R$ and $\mathcal{C} = \mathcal{A}^{\top}$. If $f : N \longrightarrow M$ is a monomorphism with $M \in \varinjlim \mathcal{S}$ then $f \otimes C$ is a monomorphism for all $C \in \mathcal{C}$ if and only if $\mathrm{Coker}(f) \in \varinjlim \mathcal{S}$.

This lemma can be used to prove the following result.

Theorem 3.3.15. [3, 9.5] Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair of finite type. Set $\mathcal{S} = \mathcal{A} \cap \text{mod-}R$ and $\mathcal{C} = \mathcal{M}^{\top}$. The following are equivalent for an R-module M.

1. M is $\mathcal{B}$-stationary,
2. M is $\mathcal{C}$-Mittag-Leffler.

Moreover, every $A \in \mathcal{A}$ is $\mathcal{C}$-Mittag-Leffler.

Let us now assume that R is an n-Gorenstein ring. We can relate the above theorem to understand $\mathcal{GF}$-Mittag-Leffler modules.

Proposition 3.3.16. Let R be n-Gorenstein. Then every module in $\mathcal{I}_n$ is $\mathcal{GF}$-Mittag-Leffler. Moreover, an R-module is $\mathcal{GF}$-Mittag-Leffler if and only if it is $\mathcal{GI}$-stationary.

Proof. Since R is n-Gorenstein, the classes $\mathcal{I}_n$ and $\mathcal{P}_n$ coincide. We showed in 3.1.13 that $\mathcal{I}_n^{\perp} = \mathcal{GI}$, which is a class closed under direct sums and therefore by [8, 4.1] it follows that $(\mathcal{I}_{\leq n}, \mathcal{GI})$ is a cotorsion pair of finite type. As stated in 1.7.7, the class $(\mathcal{I}_n)^{\top} = \mathcal{GF}$. We are therefore in the set up of 3.3.15, and the result follows immediately. $\square$

Chapter 4

Cotilting with Cohen-Macaulay modules

4.1 Tilting and cotilting classes

Let R be an arbitrary ring.

Definition 4.1.1. An R-module T is *tilting* if

1. $T \in \mathcal{P}_{<\infty}$;
2. $\mathrm{Ext}^i_R(T, T^{(\kappa)}) = 0$ for all cardinal numbers κ and $1 \leq i < \omega$;
3. There is an exact sequence $0 \longrightarrow R \longrightarrow T_0 \longrightarrow \cdots \longrightarrow T_r \longrightarrow 0$ where $T_i \in \mathrm{Add}(T)$ for all $0 \leq i \leq r$.

If the projective dimension of T is at most n, we say that T is *n-tilting.*

Classical tilting theory is ubiquitous, but we will not use it. Introductions can be found in almost every introductory textbook on the representation theory of associative algebras. Associated to a tilting module is a *tilting class*, which is given by

$$T^{\perp_\infty} := \{M \in R\text{-Mod} : \mathrm{Ext}^i_R(T, M) = 0 \text{ for all } 1 \leq i < \omega\},$$

and we say that two tilting modules are equivalent if they induce the same tilting class. If T is an n-tilting module, we call the associated class an n-tilting class.

The dual notion to tilting is cotilting, which we now introduce.

Definition 4.1.2. An R-module C is *cotilting* if

1. $C \in \mathcal{I}_{<\infty}$;
2. $\mathrm{Ext}^i_R(C^\kappa, C) = 0$ for all cardinal numbers κ and $1 \leq i < \omega$;
3. There is an exact sequence $0 \longrightarrow C_r \longrightarrow \cdots \longrightarrow C_0 \longrightarrow E \longrightarrow 0$ where E is an injective cogenerator in R-Mod and $C_i \in \mathrm{Prod}(C)$ for each $0 \leq i \leq r$.

If the injective dimension of C is at most n, we say that C is *n-cotilting.*

Associated to a cotilting module C is a *cotilting class* given by

$$^{\perp_\infty}C := \{M \in R\text{-Mod} : \mathrm{Ext}^i_R(M, C) = 0 \text{ for all } 1 \leq i < \omega\},$$

and we say that two cotilting modules are equivalent if they induce the same cotilting class. If C is n-cotilting, we say the associated class in an n-cotilting class.

Given an arbitrary class of modules it is possible to determine whether or not it is a tilting or cotilting class. Before stating the corresponding result, we introduce the following two concepts.

Definition 4.1.3. Let $\mathcal{C}$ be a class of R-modules and $n \geq 1$. An R-module M is an *n-submodule* in $\mathcal{C}$ if there is an exact sequence

$$0 \longrightarrow M \longrightarrow C_0 \longrightarrow C_1 \longrightarrow \cdots \longrightarrow C_{n-1}$$

with $C_i \in \mathcal{C}$ for all $0 \leq i \leq n-1$. Conversely, say M is an *n-image* in $\mathcal{C}$ if there is an exact sequence

$$C_0 \longrightarrow \cdots \longrightarrow C_{n-2} \longrightarrow C_{n-1} \longrightarrow M \longrightarrow 0$$

with $C_i \in \mathcal{C}$ for all $0 \leq i \leq n-1$.

We are now in a position to state necessary and sufficient conditions for a class to be tilting, respectively cotilting. Both results are taken from [7].

Theorem 4.1.4. Let R be a ring and $\mathcal{C}$ a class of R-modules. The following are equivalent.

1. $\mathcal{C}$ is n-tilting,
2. $\mathcal{C}$ is definable, coresolving and $^{\perp}\mathcal{C} \subseteq \mathcal{P}_n$;
3. $\mathcal{C}$ is definable, coresolving and $\mathcal{C}$ is closed under n-images.

The dual result for cotilting is as follows.

Theorem 4.1.5. Let R be a ring and $\mathcal{C}$ a class of R-modules. The following are equivalent.

1. $\mathcal{C}$ is n-cotilting;
2. $\mathcal{C}$ is definable, resolving and $\mathcal{C}^{\perp} \subseteq \mathcal{I}_n$;
3. $\mathcal{C}$ is definable, resolving and closed under n-submodules.

Example 4.1.6. We have already encountered some tilting and cotilting classes. [4, 3.4] showed that a ring is Gorenstein if and only if the class of Gorenstein injective modules is tilting if and only if the class of Gorenstein flat modules is cotilting.

If $\mathcal{T}$ is an n-tilting class induced by a tilting module T, it is definable by the above theorem. We may then consider the dual definable subcategory $\mathcal{T}^d$. It transpires (see [29, 15.2]) that this is an n-cotilting class induced by the cotilting module $C = T^+$. In general the converse is not true. Recall the definition of a class of finite type from 3.3.13, and consider the dual condition: we say that a class $\mathcal{C}$ is of *cofinite type* if there is a set of FP_∞, or compact modules, $\mathcal{S}$ such that $\mathcal{S}^{\top} = \mathcal{C}$. The dual of a class of finite type is a class of cofinite type, and it is known that a class is finite type if and only if it is tilting (see [29, 13.46]). Moreover, a class is of cofinite type if and only if it is the dual of a class of finite type (see [29, 15.18]), and therefore any cofinite type class is cotilting. However, unlike the tilting case, there are cotilting classes that are not of cofinite type, so not every cotilting class is the dual of a tilting class. However, over commutative Noetherian rings every cotilting class is of cofinite type, so there is a duality between tilting and cotilting classes given by considering the dual definable subcategories, see [29, 16.21]. We will use this later.

We will be more concerned with cotilting than tilting, so we turn our attention towards more of the background around cotilting classes.

Definition 4.1.7. Let R be a ring and $\mathcal{C}$ a cotilting class induced by the cotilting module C. For any $i \geq 0$ define

$$\mathcal{C}_{(i)} = \{M \in R\text{-Mod} : \operatorname{Ext}^j_R(M, C) = 0 \text{ for all } j > i\} = {}^{\perp_{>i}}C.$$

It is clear that there are inclusions $\mathcal{C}_{(0)} \subseteq \mathcal{C}_{(1)} \subseteq \cdots$. Moreover $\mathcal{C}_{(0)} = \mathcal{C}$ and if $\mathcal{C}$ is n-cotilting then $\mathcal{C}_{(n+k)} = \mathcal{C}_{(n)} = R$-Mod for all $k \geq 0$. The following result says that each $\mathcal{C}_{(i)}$ has an additional structure.

Proposition 4.1.8. [29, 15.13] Let R be a ring and C an n-cotilting module for some $n < \omega$. If $\mathcal{C}$ is the cotilting class induced by C, then $\mathcal{C}_{(i)}$ is an $(n-i)$-cotilting class for all $i \leq n$.

The following result is clear from the above definition.

Lemma 4.1.9. Let $\mathcal{C}$ be a cotilting class and $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ a short exact sequence with $M \in \mathcal{C}$. Then $L \in \mathcal{C}$ if and only if $N \in \mathcal{C}_{(1)}$.

Proof. Applying $\mathrm{Hom}_R(-, C)$ to the short exact sequence shows that $\mathrm{Ext}^i_R(L, C) = 0$ for all $i > 0$ if and only if $\mathrm{Ext}^i_R(N, C) = 0$ for all $i > 1$. □

Let us now turn our attention to the case when R is a commutative Noetherian ring. Cotilting classes over such rings were completely classified in [5]. As a notational point, if M is an R-module we will let $\Omega^{-i}(M)$ denote the *i-th cosyzygy* of M in a minimal injective resolution, that is

$$\Omega^{-i}(M) = \mathrm{Im}(E^i \longrightarrow E^{i+1}).$$

The following result of Bass enables us to relate the associated primes of such cosyzygies to the Bass invariants of M.

Lemma 4.1.10. [29, 16.8] Let $\mathfrak{p} \in \mathrm{Spec(R)}$ and $M \in \mathrm{Mod}(R)$. Then for each $i \geq 0$

$$\mathfrak{p} \in \mathrm{Ass}\, \Omega^{-i}(M) \iff \mathfrak{p} \in \mathrm{Ass}\, E^i \iff \mu_i(\mathfrak{p}, M) \neq 0,$$

where $0 \longrightarrow M \longrightarrow E^\bullet$ is a minimal injective resolution of M. □

Recall the following condition on subsets of Spec(R).

Definition 4.1.11. Let $X \subset \mathrm{Spec(R)}$. Say X is *generalisation closed* if for $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{p} \in X$ implies $\mathfrak{q} \in X$.

Generalisation closed subsets play a large part in the classification of cotilting classes due to the following construction.

Definition 4.1.12. Let $n \geq 1$. A sequence $\boldsymbol{X} = (X_0, \cdots, X_{n-1})$ of subsets of Spec(R) is called *characteristic* if

1. X_i is generalisation closed for all i;

2. $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-1}$;
3. $\operatorname{Ass} \Omega^{-i}(R) \subseteq X_i$ for all $i < n$.

Associated to a characteristic sequence $\boldsymbol{X}$ is a class of modules

$$\mathcal{C}_{\boldsymbol{X}} = \{M \in \operatorname{Mod}(R) : \operatorname{Ass} \Omega^{-i}(M) \subseteq X_i \text{ for all } i < n\}.$$

Using 4.1.10, we can restate this class in terms of Bass invariants:

$$\mathcal{C}_{\boldsymbol{X}} = \{M \in \operatorname{Mod}(R) : \mu_i(\mathfrak{p}, M) = 0 \text{ for all } i < n \text{ and } \mathfrak{p} \in \operatorname{Spec(R)} \setminus X_i\}.$$

In other words, the minimal injective resolutions of modules in $\mathcal{C}_{\boldsymbol{X}}$ are of the form

$$0 \longrightarrow \bigoplus_{\mathfrak{p} \in X_0} E(R/\mathfrak{p})^{(\mu_0(\mathfrak{p},M))} \longrightarrow \bigoplus_{\mathfrak{p} \in X_1} E(R/\mathfrak{p})^{(\mu_1(\mathfrak{p},M))} \longrightarrow \cdots$$
$$\cdots \longrightarrow \bigoplus_{\mathfrak{p} \in X_{n-1}} E(R/\mathfrak{p})^{(\mu_{n-1}(\mathfrak{p},M))} \longrightarrow \bigoplus_{\mathfrak{p} \in \operatorname{Spec(R)}} E(R/\mathfrak{p})^{(\mu_n(\mathfrak{p},M))} \longrightarrow \cdots$$

The following result shows why characteristic classes are of importance.

Lemma 4.1.13. [29, 16.18] Let $n \geq 1$ and $\boldsymbol{X} = (X_0, \cdots, X_{n-1})$ be a characteristic sequence. Then $\mathcal{C}_{\boldsymbol{X}}$ is n-cotilting. □

On the other hand, if $\mathcal{C}$ is an n-cotilting class, then we can consider $\operatorname{Ass} \mathcal{C}_{(i)} = \bigcup_{M \in \mathcal{C}_{(i)}} \operatorname{Ass} M$ for all $i < n$. The following result says that this is a reasonable thing to do.

Lemma 4.1.14. [29, 16.17] Let $\mathcal{C}$ be an n-cotilting class for $n \geq 1$. Then

$$(\operatorname{Ass} \mathcal{C}_{(0)}, \operatorname{Ass} \mathcal{C}_{(1)}, \cdots, \mathcal{C}_{(n-1)})$$

is a characteristic sequence. □

These two results help prove the following classification of cotilting classes over R.

Theorem 4.1.15. [29, 16.19] Let $n \geq 1$. Then there is a bijection between the n-cotilting classes in $\operatorname{Mod}(R)$ and characteristic sequences. The mutually inverse maps are

$$\mathcal{C} \mapsto (\operatorname{Ass} \mathcal{C}_{(0)}, \operatorname{Ass} \mathcal{C}_{(1)}, \cdots, \mathcal{C}_{(n-1)})$$

and

$$\boldsymbol{X} \mapsto \mathcal{C}_{\boldsymbol{X}}.$$

4.1.1 Determining a cotilting module

Having considered cotilting classes, we will turn our attention to cotilting modules, namely how to construct a cotilting module for a given cotilting class. The following process is due to Trlifaj, Stovicek and Herbera in [47]. Given a generalisation closed set $Y \subset \mathrm{Spec(R)}$ consider the class of injective modules

$$\mathcal{E}(Y) = \mathrm{Add}\{E(R/\mathfrak{p}) : \mathfrak{p} \in Y\}.$$

The following lemma states some facts about this class.

Lemma 4.1.16. [47, 3.2, 3.3] Let Y be generalisation closed and $\mathcal{E}(Y)$ as above.

1. $\mathcal{E}(Y)$ is definable, extension closed and is both covering and enveloping.
2. The following are equivalent:
   (a) $\mathrm{Ass}\, R \subseteq Y$;
   (b) each $\mathcal{E}(Y)$-precover of a projective module is injective;
   (c) each $\mathcal{E}(Y)$-preenvelope of an injective module is surjective.

□

Now, given a sequence $Y_0 \subseteq Y_1 \subseteq \cdots$ of generalisation closed subsets, for any injective module E and $i \geq 0$ we can construct a complex

$$0 \longrightarrow C \longrightarrow E_0 \xrightarrow{\varphi_0} E_1 \longrightarrow \cdots \longrightarrow E_{i-1} \xrightarrow{\varphi_{i-1}} E_i \xrightarrow{\varphi_i} E \longrightarrow 0$$

such that φ_i is an $\mathcal{E}(Y_i)$-precover of $E(R/\mathfrak{p})$, φ_{j-1} is an $\mathcal{E}(Y_{j-1})$-precover of $\mathrm{Ker}(\varphi_j)$ for all $j \leq i$ and $C = \mathrm{Ker}(\varphi_0)$.

In relation to characteristic sequences, we have the following result.

Proposition 4.1.17. [47, 3.5] If $\mathrm{Ass}\, \Omega^{-i}(R) \subseteq Y_i$ for each $i \geq 0$ then the above complex is exact. □

Now, if $\mathcal{C}$ is an n-cotilting class with characteristic sequence $\boldsymbol{X}_{\mathcal{C}} = (X_0, \cdots, X_{n-1})$, then the properties of the preceeding proposition hold, as well as those in 4.1.16. For each $\mathfrak{p} \in X_0$ define $C(\mathfrak{p}) = E(R/\mathfrak{p})$, while if $i \geq 0$ and $\mathfrak{p} \in X_{i+1} \setminus X_i$ define $C(\mathfrak{p})$ to be the module that arises in the exact sequence

$$0 \longrightarrow C(\mathfrak{p}) \longrightarrow E_0 \xrightarrow{\varphi_0} E_1 \longrightarrow \cdots$$

$$\cdots \longrightarrow E_{i-1} \xrightarrow{\varphi_{i-1}} E_i \xrightarrow{\varphi_i} E(R/\mathfrak{p}) \longrightarrow 0$$

where in this case we can assume that φ_i is an $\mathcal{E}(Y_i)$-cover of E and φ_{j-1} is an $\mathcal{E}(Y_{j-1})$-cover of $\mathrm{Ker}(\varphi_j)$ for all $j \leq i$. If one defines the module

$$C := \prod_{\mathrm{Spec(R)}} C(\mathfrak{p}),$$

one obtains the following result.

Theorem 4.1.18. [47, 4.12] If $\mathcal{C}$ is a cotilting class with characteristic sequence $\boldsymbol{X}_{\mathcal{C}}$, then the module C as defined above is a cotilting module that induces $\mathcal{C}$.

4.2 The Cohen-Macaulay case

Let us now turn our attention to the case when $(R, \mathfrak{m}, k)$ is a d-dimensional Cohen-Macaulay ring and to begin with assume that R admits a canonical module Ω_R. As previously stated, when R is a Gorenstein local ring, then $\mathrm{GFlat}(R)$ is a cotilting class, and $\mathrm{GFlat}(R) = \varinjlim \mathrm{CM}(R)$. We will now show that this result extends to the more general situation we have been considering.

Theorem 4.2.1. Let R be a d-dimensional Cohen-Macaulay ring admitting a canonical module.

(1) The class $\varinjlim \mathrm{CM}(R)$ is d-cotilting,

(2) The characteristic sequence corresponding to $\varinjlim \mathrm{CM}(R)$ is

$$(H_{(0)}, H_{(1)}, \cdots, H_{(d-1)}),$$

where $H_{(i)} = \{\mathfrak{p} \in \mathrm{Spec(R)} : \mathrm{ht}\,\mathfrak{p} \leq i\}$.

(3) The corresponding sequence of cotilting classes is

$$(\mathcal{L}_0, \mathcal{L}_1, \cdots, \mathcal{L}_{d-1}),$$

where

$$\mathcal{L}_i = \{M \in \mathrm{Mod}(R) : \mathrm{Tor}_j^R(R/(\boldsymbol{x}), M) = 0 \text{ for all } j > i \text{ and } R\text{-sequences } \boldsymbol{x}\}.$$

(4) There is a balanced big Cohen-Macaulay d-cotilting module inducing $\varinjlim \mathrm{CM}(R)$.

Proof.

(1) We have already seen that $\varinjlim \mathrm{CM}(R)$ is definable and it is resolving by 2.1.6, so by 4.1.5 it suffices to show that it is closed under d-submodules. Suppose $0 \longrightarrow X \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_{d-1}$ is an exact sequence with $M_i \in \varinjlim \mathrm{CM}(R)$ for all $0 \leq i \leq d-1$. Since the functor $Z : \mathrm{Mod}(R) \longrightarrow \mathrm{Mod}(R \ltimes \Omega_R)$ is exact, the sequence

$$0 \longrightarrow Z(X) \longrightarrow Z(M_0) \longrightarrow \cdots \longrightarrow Z(M_{d-1}) \tag{4.1}$$

is exact in $\mathrm{Mod}(R \ltimes \Omega_R)$. By 2.1.6.(5), each $Z(M_i)$ is a Gorenstein flat $R \ltimes \Omega_R$-module, hence (4.1) is a d-submodule in $\mathrm{GFlat}(R \ltimes C)$. Yet this is a d-cotilting class by 4.1.6, hence $Z(X)$ is also a Gorenstein flat $R \ltimes C$-module, so $X \in \varinjlim \mathrm{CM}(R)$ by 2.1.6.(5).

(2) Let $\boldsymbol{X} = (X_0, \cdots, X_{d-1})$ denote the characteristic sequence corresponding to $\varinjlim \mathrm{CM}(R)$, so $\varinjlim \mathrm{CM}(R) = \mathcal{C}_{\boldsymbol{X}}$. We will show that $X_i = H_{(i)}$ for all $i < d$ by considering the Bass invariants of the modules in $\varinjlim \mathrm{CM}(R)$. First off, we show that if $M \in \varinjlim \mathrm{CM}(R)$ has $\mu_i(\mathfrak{p}, M) \neq 0$ for any prime ideal $\mathfrak{p}$, then there is a maximal Cohen-Macaulay R-module M_0 such that $\mu_i(\mathfrak{p}, M) \neq 0$, so the characteristic sequence depends entirely on $\mathrm{CM}(R)$. To see this, let $M = \varinjlim_J M_j$ be an element of $\varinjlim \mathrm{CM}(R)$ with each $M_j \in \mathrm{CM}(R)$. For every $\mathfrak{p} \in \mathrm{Spec(R)}$ and $i \geq 0$ there are isomorphisms

$$\mathrm{Ext}^i_R(R/\mathfrak{p}, M)_\mathfrak{p} \simeq \varinjlim_J \mathrm{Ext}^i_R(R/\mathfrak{p}, M_j)_\mathfrak{p} = 0$$

as $R/\mathfrak{p}$ is finitely generated and localisation preserves direct limits. We therefore see that if $\mu_i(\mathfrak{p}, M) \neq 0$, there must be a $j \in J$ such that $\mu_i(\mathfrak{p}, M_j) \neq 0$, which shows the claim. Therefore, assume that $M \in \mathrm{CM}(R)$. If $\mathfrak{p} \notin \mathrm{Supp}\, M$, then $M_\mathfrak{p} = 0$, so $\mu_i(\mathfrak{p}, M) = 0$ for all $i \geq 0$ since

$$\mathrm{Ext}^i_R(R/\mathfrak{p}, M)_\mathfrak{p} \simeq \mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) = 0.$$

On the other hand, if $\mathfrak{p} \in \mathrm{Supp}\, M$ then $M_\mathfrak{p}$ is a maximal Cohen-Macaulay $R_\mathfrak{p}$-module by [12, 2.1.3(b)], and $\mathrm{depth}_{R_\mathfrak{p}} = \dim R_\mathfrak{p} = \mathrm{ht}\, \mathfrak{p}$. Yet, also by [12, 2.1.3(b)], $\mathrm{grade}(\mathfrak{p}, M) = \mathrm{depth}\, M_p$, so $\mathrm{Ext}^i_R(R/\mathfrak{p}, M) = 0$ for all $i < \mathrm{ht}\, \mathfrak{p}$, hence $\mu_i(\mathfrak{p}, M) = 0$ for all $i < \mathrm{ht}\, \mathfrak{p}$. In particular, if $N \in \varinjlim \mathrm{CM}(R)$ then $\mu_i(\mathfrak{p}, N) = 0$ for all $i < \mathrm{ht}\, \mathfrak{p}$, hence $X_i \subseteq H_{(i)}$. In order to show that $X_i = H_{(i)}$ it suffices to show that for every $\mathfrak{p} \in H_i$ there is a module in $\varinjlim \mathrm{CM}(R)$ such that $E(R/\mathfrak{p})$ is a direct summand of the ith term of its minimal injective resolution. As the canonical module Ω_R is faithful it is supported everywhere, but $(\Omega_R)_\mathfrak{p} \simeq \Omega_{R_\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec(R)}$ by [12, 3.3.5]. In particular, if $\mathrm{ht}\, \mathfrak{p} = i$, we see that $\mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), (\Omega_R)_\mathfrak{p}) \neq 0$, hence $\mu_i(\mathfrak{p}, \Omega_R) \neq 0$. This proves the claim.

(3) We proceed by induction on i. When $i = 0$ the result is immediate from 2.1.6. For induction assume that

$$\mathcal{L}_i = \{M \in \mathrm{Mod}(R) : \mathrm{Tor}^R_j(R/(\boldsymbol{x}), M) = 0 \text{ for all } R\text{-sequences } \boldsymbol{x} \text{ and } j > i\}.$$

Let M be an R-module and consider the short exact sequence

$$0 \longrightarrow \Omega(M) \longrightarrow P \longrightarrow M \longrightarrow 0 \tag{4.2}$$

where P is projective. Suppose $M \in \mathcal{L}_{i+1}$. Since $P \in \varinjlim \mathrm{CM}(R)$ it is also in $\mathcal{L}_i$, so by 4.1.9 we see $\Omega(M) \in \mathcal{L}_i$ as well. By dimension shifting, for every R-sequence $\boldsymbol{x}$ and $k \geq 1$ there is an isomorphism

$$\mathrm{Tor}_k^R(R/(\boldsymbol{x}), \Omega(M)) \simeq \mathrm{Tor}_{k+1}^R(R/(\boldsymbol{x}), M),$$

in particular we see that $\mathrm{Tor}_{i+1+\lambda}^R(R/(\boldsymbol{x}), M) = \mathrm{Tor}_{i+\lambda}^R(R/(\boldsymbol{x}), \Omega(M)) = 0$ for every $\lambda > 0$ by the induction hypothesis. For the reverse inclusion, if $\mathrm{Tor}_j^R(R/(\boldsymbol{x}), M) = 0$ for all $j > i+1$ and R-sequences $\boldsymbol{x}$, we can again apply $R/(\boldsymbol{x}) \otimes_R -$ to (4.2) and by a similar dimension shifting argument we see that $\mathrm{Tor}_j^R(R/(\boldsymbol{x}), \Omega(M)) = 0$ for all $j > i$, that is $\Omega(M) \in \mathcal{L}_i$ by the inductive hypothesis. We may therefore apply 4.1.9 to see that $M \in \mathcal{L}_{i+1}$, which proves the claim.

(4) To show the last claim, we note that cotilting module inducing $\varinjlim \mathrm{CM}(R)$ will be in $\varinjlim \mathrm{CM}(R)$, so it suffices to show that there is a d-cotilting module C with Ext-depth d. Let

$$C := \prod_{\mathfrak{p} \in \mathrm{Spec(R)}} C(\mathfrak{p})$$

be the cotilting module inducing $\varinjlim \mathrm{CM}(R)$ formed by the construction given by Trlifaj, Stovicek and Herbera. Since

$$\text{E-dp}(C) = \min\{\text{E-dp}(C(\mathfrak{p})) : \mathfrak{p} \in \mathrm{Spec(R)}\}$$

it suffices to show that $\text{E-dp}(C(\mathfrak{p})) = d$ for some prime ideal $\mathfrak{p}$. The maximal ideal $\mathfrak{m}$ provides such an ideal. Indeed, let

$$0 \longrightarrow C(\mathfrak{m}) \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_{d-1} \longrightarrow E(k) \longrightarrow 0$$

be the exact sequence corresponding to $E(k)$ as given in 4.1.17. One can decompose this into d short exact sequences

$$0 \longrightarrow C(\mathfrak{m}) \longrightarrow E_0 \longrightarrow K_1 \longrightarrow 0$$
$$0 \longrightarrow K_1 \longrightarrow E_1 \longrightarrow K_2 \longrightarrow 0$$
$$\cdots$$
$$0 \longrightarrow K_{d-1} \longrightarrow E_{d-1} \longrightarrow E(k) \longrightarrow 0.$$

Since $\text{E-dp}(E(k)) = 0$, it follows that $\text{E-dp}(K_{d-1}) = 1$, and so $\text{E-dp}(K_{d-2}) = 2$ and so on. Consequently $\text{E-dp}(C(\mathfrak{m})) = d$, so $C(\mathfrak{m})$ is a balanced big Cohen-Macaulay module.

□

Remark. It follows that the associated primes of the class of balanced big Cohen-Macaulay modules are the minimal primes. This result, however, was already known due to the following result due to R. Sharp.

Proposition 4.2.2. [46, 2.1] Let R be a Cohen-Macaulay ring with canonical module. If M is a balanced big Cohen-Macaulay module, then

$$\mathrm{Ass}(M) \subseteq \{\mathfrak{p} \in \mathrm{Spec(R)} : \dim R/\mathfrak{p} = \dim R\}$$

Since R is a Cohen-Macaulay local ring, this is precisely the class of minimal primes.

Corollary 4.2.3. $\varinjlim \mathrm{CM}(R)$ is closed under injective hulls. In particular, it has enough injectives.

Proof. [29, 16.15] says that any cotilting class over a commutative Noetherian ring is closed under injective hulls. □

In the case that R is a regular local ring, we have seen that $\varinjlim \mathrm{CM}(R) = \mathrm{GFlat}(R)$, and therefore the Bass invariants of such Gorenstein flat modules are described by 4.2.1. However, it is possible to do better than this, and using the above we are able to determine the Bass invariants of Gorenstein flat modules over any commutative Noetherian Gorenstein ring. Notice that from the equivalent definitions of Gorenstein flat, it is clear that if R is a commutative Gorenstein ring, then M is a Gorenstein flat R-module if and only if $M_{\mathfrak{p}}$ is Gorenstein flat for every $\mathfrak{p} \in \mathrm{Spec(R)}$.

Corollary 4.2.4. Let R be a commutative Gorenstein ring. The following are equivalent.

(1) M is a Gorenstein flat R-module,

(2) For all $\mathfrak{p} \in \mathrm{Spec(R)}$, the Bass invariant $\mu_i(\mathfrak{p}, M) = 0$ for all $i < \mathrm{ht}\,\mathfrak{p}$.

Proof.

(1) $\Rightarrow$ (2). If $M \in \mathrm{GFlat}(R)$, then $M_{\mathfrak{p}} \in \mathrm{GFlat}(R_{\mathfrak{p}})$ for every prime ideal $\mathfrak{p} \in \mathrm{Spec(R)}$. Now, for each $\mathfrak{p}$, there is an equality $\mu_i(\mathfrak{p}, M) = \mu_i(\mathfrak{p}_{\mathfrak{p}}, M_{\mathfrak{p}})$ for every $i \geq 0$ by [24, 9.2.1]. By 4.2.1 we know that $\mu_i(\mathfrak{p}_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$ for all $i < \mathrm{ht}\,\mathfrak{p}_{\mathfrak{p}}$, but since $\mathrm{ht}\,\mathfrak{p}_{\mathfrak{p}} = \mathrm{ht}\,\mathfrak{p}$, it follows that $\mu_i(\mathfrak{p}, M) = 0$ for all $i < \mathrm{ht}\,\mathfrak{p}$.

(2) $\Rightarrow$ (1). Let M be as in the statement and $\mathfrak{p} \in \mathrm{Spec(R)}$. We will show that $M_\mathfrak{p}$ is a Gorenstein flat $R_\mathfrak{p}$-module. Indeed, if $\mathfrak{q} \subset \mathfrak{p}$, the equality $\mu_i(\mathfrak{q}, M) = \mu_i(\mathfrak{q}_\mathfrak{p}, M_p)$ obtained from [24, 9.2.1] shows that $\mu_i(\mathfrak{q}_\mathfrak{p}, M_\mathfrak{p}) = 0$ for all $i < \operatorname{ht} \mathfrak{q}$. Since the prime ideals of $R_\mathfrak{p}$ coincide with the primes in Spec(R) contained in $\mathfrak{p}$ and $R_\mathfrak{p}$ is a Gorenstein local ring, we can use 4.2.1 to see that $R_\mathfrak{p}$ is a Gorenstein flat $R_\mathfrak{p}$-module, and hence this holds for all $\mathfrak{p}$ in Spec(R). Consequently M is a Gorenstein flat R-module due to Gorenstein flatness being a local property.

□

Remark. One can deduce the Bass invariants for Gorenstein flat modules immediately from what Christensen calls the *AB Formula for Gorenstein Flat dimension* in [16, 2.3.13]. This states that if R is a Cohen-Macaulay local ring with a canonical module, then if $M \in \mathrm{Mod}(R)$ has finite Gorenstein flat dimension there is an equality

$$\operatorname{Gfd}_R M = \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} : \mathfrak{p} \in \mathrm{Spec(R)}\}.$$

Since every flat module is Gorenstein flat, over a Gorenstein ring the Bass invariants of flat modules will satisfy the conclusions of 4.2.4. This is indeed the case, since the Bass invariants for flat modules have been described by Xu, as seen in the following result.

Theorem 4.2.5. [50, 5.1.5] Let R be a commutative Noetherian ring. The following are equivalent.

(1) R is Gorenstein;

(2) For any flat module F and $\mathfrak{p} \in \mathrm{Spec(R)}$, we have $\mu_i(\mathfrak{p}, F) = 0$ for all $i \neq \operatorname{ht} \mathfrak{p}$;

(3) An R-module is flat if and only if its Bass invariants are as in (2).

In particular, we can see that the flat modules are precisely the Gorenstein flat modules M such that $\mu_i(\mathfrak{p}, M) = 0$ for all $i > \operatorname{ht} \mathfrak{p}$. Indeed, in 4.2.4 nothing was said about $\mu_i(\mathfrak{p}, -)$ for $i > \operatorname{ht} \mathfrak{p}$.

Now, suppose that R is a commutative Gorenstein ring such that the classes of flat and Gorenstein flat modules coincide (such as a regular local ring). For every $\mathfrak{p}$ in Spec(R) the localisation $R_\mathfrak{p}$ is a Gorenstein local ring with canonical module $R_\mathfrak{p}$. Consequently

$$\operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), R_\mathfrak{p}) = \begin{cases} k(\mathfrak{p}) & \text{if } i = \operatorname{ht}\mathfrak{p}, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we see that $\mu_i(\mathfrak{p}, R) = 0$ for all $i \neq \operatorname{ht}\mathfrak{p}$, and therefore if F is any Gorenstein flat R-module, we see that $\mu_i(\mathfrak{p}, F) = 0$ for all $i \neq \operatorname{ht}\mathfrak{p}$. In particular, in this situation 4.2.4 completely agrees with Xu's result.

For each $i \geq 0$ define

$$\mathcal{D}_i = \{M \in \operatorname{Mod}(R) : \text{E-dp}(M) \geq i\},$$

so $\mathcal{D}_d = \operatorname{CohCM}(R)$, and $\mathcal{D}_{d+i} = \operatorname{CohCM}(R)_\infty$ for all $i > 0$. We can consider a corresponding result to 4.2.1 for the class $\mathcal{D}_d$.

Theorem 4.2.6. Let R be an arbitrary Cohen-Macaulay ring of dimension d.

(1) $\operatorname{CohCM}(R)$ is d-cotilting.

(2) The characteristic sequence for $\operatorname{CohCM}(R)$ is $(\mathrm{pSpec(R)}, \mathrm{pSpec(R)}, \cdots, \mathrm{pSpec(R)})$.

(3) The sequence of cotilting classes corresponding to $\operatorname{CohCM}(R)$ is

$$(\mathcal{D}_d, \mathcal{D}_{d-1}, \cdots, \mathcal{D}_1).$$

(4) Set $\mathcal{E}_0 = \operatorname{Add}\{E(R/\mathfrak{p}) : \mathfrak{p} \in \mathrm{pSpec(R)}\}$. Then a cotilting module inducing $\operatorname{CohCM}(R)$ is

$$\Omega^d_{\mathcal{E}_0}(E(k)),$$

dth syzygy of a minimal resolution of $E(k)$ with respect to the class $\mathcal{E}_0$. Moreover, this module has Ext-depth equal to d.

Proof.

(1) It is obvious that $\mathcal{D}_d$ is definable and it is resolving since it contains the projective modules and is closed under kernels of epimorphisms by 1.5.9. Let

$$0 \longrightarrow X \xrightarrow{f_0} M_0 \xrightarrow{f_1} M_1 \longrightarrow \cdots \longrightarrow M_{d-2} \xrightarrow{f_{d-1}} M_{d-1}$$

be an exact sequence with $M_i \in \mathcal{D}_d$ for $0 \leq i < d$. There are then d short exact sequences

$$0 \longrightarrow \operatorname{im} f_i \longrightarrow M_i \longrightarrow \operatorname{coker} f_i \longrightarrow 0$$

and applying the functor $\mathrm{Hom}_R(k,-)$ to these shows that $\operatorname{depth} \operatorname{im} f_i \geq \operatorname{depth} \operatorname{coker} f_i + 1$ for all i, as $\operatorname{depth} M_i \geq d$. However, $\operatorname{coker} f_i \simeq \operatorname{im} f_{i+1}$ for all $0 \leq i < d-1$, and therefore

$$\operatorname{depth} X = \operatorname{depth} \operatorname{im} f_0 \geq \operatorname{depth} \operatorname{im} f_1 + 1 \geq \cdots \geq \operatorname{depth} \operatorname{coker} f_{d-1} + d \geq d.$$

In particular, we see that $X \in \mathcal{D}_d$, showing that $\mathcal{D}_d$ is d-cotilting.

(2) Let $\boldsymbol{X} = (X_0, \cdots, X_{d-1})$ be the characteristic sequence for $\mathcal{D}_d$. If $\mathfrak{p} \in \mathrm{Spec(R)}$, note that any homomorphism $k \longrightarrow E(R/\mathfrak{p})$ factors through $E(k)$, giving a non-zero homomorphism $E(k) \longrightarrow E(R/\mathfrak{p})$. But by [24, 3.3.8(4)], this can only happen if $\mathfrak{p} = \mathfrak{m}$, hence for every $\mathfrak{p} \in \mathrm{Spec(R)}$ the indecomposable injective $E(R/\mathfrak{p})$ lies in $\mathcal{D}_d$ and $E(k) \notin \mathcal{D}_d$. But since $\operatorname{Ass} E(R/\mathfrak{p}) = \mathfrak{p}$, it follows that $\mathrm{pSpec(R)} = X_0$. By definition of depth, it follows that for every $i < d$ and $M \in \mathcal{D}_d$ we have $\mu_i(\mathfrak{m}, M) = 0$, so $X_i = \mathrm{pSpec(R)}$ for all $1 \leq i \leq d-1$.

(3) Similar to the corresponding result in 4.2.1 we will proceed by induction. The $n = 0$ case is clear, so assume that $(\mathcal{D}_d)_{(i)} = \mathcal{D}_{d-i}$ for some $i < d-1$. For any R-module we have the short exact sequence

$$0 \longrightarrow \Omega(M) \longrightarrow P \longrightarrow M \longrightarrow 0 \tag{4.3}$$

with P projective which gives isomorphisms $\mathrm{Ext}^j_R(k, M) = \mathrm{Ext}^{j+1}_R(k, \Omega(M))$ for every $j > 0$. Suppose $M \in (\mathcal{D}_d)_{(i+1)}$, then it follows from (4.3) and 4.1.9 that $\Omega(M) \in (\mathcal{D}_d)_{(i)} = \mathcal{D}_{d-i}$, hence $\operatorname{depth} \Omega(M) \geq d - i$. But by the above isomorphisms it follows that $\operatorname{depth} M \geq d-(i+1)$, so $M \in \mathcal{D}_{d-(i+1)}$. On the other hand if $M \in \mathcal{D}_{d-(i+1)}$ then $\operatorname{depth} \Omega(M) \geq d-i$, so $\Omega(M) \in (\mathcal{D}_d)_{(i)}$ giving $M \in (\mathcal{D}_d)_{(i+1)}$ by 4.1.9.

(4) A cotilting module that generates $\mathcal{D}_d$ will be of the form

$$C \simeq \prod_{\mathrm{Spec(R)}} C(\mathfrak{p}),$$

where $\mathcal{C}(\mathfrak{p}) := E(R/\mathfrak{p})$ if $\mathfrak{p} \in \operatorname{Ass} \mathcal{D}_d = \mathrm{pSpec(R)}$ and $C(\mathfrak{m})$ arises in the following exact sequence

$$0 \longrightarrow C(\mathfrak{p}) \longrightarrow E_0 \xrightarrow{\varphi_0} E_1 \longrightarrow \cdots$$

$$\cdots \longrightarrow E_{i-1} \xrightarrow{\varphi_{d-2}} E_i \xrightarrow{\varphi_{d-1}} E(k) \longrightarrow 0.$$

Here $\varphi_{d-1} : E_{d-1} \longrightarrow E(k)$ is an $\mathcal{E}_0$-cover, $\varphi_i : E_i \longrightarrow E_{i+1}$ is an $\mathcal{E}_0$-cover of $\ker \varphi_{i+1}$ for all $i \leq d-2$ and $C(\mathfrak{m}) = \ker \varphi_0$. Yet this is the same as saying that $C(\mathfrak{m})$ is the dth syzygy of a minimal $\mathcal{E}_0$ resolution of $E(k)$. Now

$$\operatorname{Ext}^i_R\left(-, \prod_{\mathfrak{p}\in\mathrm{pSpec(R)}} E(R/\mathfrak{p})\right) = 0$$

for every $i \geq 1$ as the product of injective modules is injective, these modules are redundant in defining the cotilting class. Consequently the only module that contributes to inducing $\mathcal{D}_d$ from the above construction is $\Omega^d_{\mathcal{E}_0}(E(k))$. The statement about Ext-depth follows from the proof of 4.2.1(4).

□

Corollary 4.2.7. The class $\{M \in \mathrm{Mod}(R) : \text{T-codp}(M) \geq d\}$ is d-tilting.

Proof. Said class is the dual definable subcategory of $\mathrm{CohCM}(R)$, and [29, 16.21] shows that every cotilting class is of cofinite type. Therefore it follows that this class is d-tilting. □

If we assume that R is a Cohen-Macaulay ring with canonical module, then $\varinjlim \mathrm{CM}(R)$ is the smallest d-cotilting class containing $\mathrm{CM}(R)$, while $\mathrm{CohCM}(R)$ is the largest. Therefore, by considering the corresponding characteristic sequences, we have essentially classified all d-cotilting classes containing $\mathrm{CM}(R)$. In particular, we see that whenever $\dim R \geq 2$ that $\mathrm{CohCM}(R)$ and $\varinjlim \mathrm{CM}(R)$ do not coincide and that there are cotilting, and therefore definable, classes between them. This gives an alternative proof to 2.1.10.

Chapter 5

The one dimensional case

Let us assume that R is a one-dimensional Cohen-Macaulay ring. As proved in 2.1.10, if R admits a canonical module the classes $\varinjlim \mathrm{CM}(R) = \mathrm{CohCM}(R)$. However, the existence of a canonical module was not necessary for this proof of that statement, and therefore over any one-dimensional Cohen-Macaulay ring we have $\mathrm{CohCM}(R) = \varinjlim \mathrm{CM}(R)$. We therefore get the following as an easy corollary.

Proposition 5.1.1. Let R be a one-dimensional Cohen-Macaulay ring. Then $\mathrm{CM}(R)$ is covariantly finite in $\mathrm{mod}(R)$.

Moreover, requiring just the vanishing of $\mathrm{Hom}_R(k, -)$, as opposed to Ext functors, endows $\varinjlim \mathrm{CM}(R)$ with some properties that do not appear in the higher dimensional case. For example, it is clear that $\varinjlim \mathrm{CM}(R)$ is closed under submodules, which does not hold for $\varinjlim \mathrm{CM}(R)$ nor $\mathrm{CohCM}(R)$ in higher dimensions. Moreover, we showed that whenever R has dimension at least three the class $\mathrm{CohCM}(R)$ is not closed under arbitrary inverse limits. Yet the natural isomorphism $\mathrm{Hom}_R(k, \varprojlim_I M_i) \simeq \varprojlim_I \mathrm{Hom}_R(k, M_i)$ shows that $\varinjlim \mathrm{CM}(R)$ is closed under inverse limits. In particular, the fact that the class is closed under submodules endows that class $\varinjlim \mathrm{CM}(R)_\infty$ with a much richer structure than usual.

Theorem 5.1.2. With R as above, $\varinjlim \mathrm{CM}(R)_\infty$ is a Grothendieck abelian category.

Proof. In order to show $\varinjlim \mathrm{CM}(R)_\infty$ is abelian, [44, Prop. 5.92] states that it suffices to prove that $\varinjlim \mathrm{CM}(R)_\infty$ is closed under direct sums, contains a zero object and if $f : M \longrightarrow N$ is a morphism in $\varinjlim \mathrm{CM}(R)_\infty$, then both $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ lie in

$\varinjlim \mathrm{CM}(R)_\infty$. Clearly the first two hold. Suppose $f : M \longrightarrow N$ is a morphism in $\varinjlim \mathrm{CM}(R)_\infty$. There are two associated short exact sequences $\mathcal{S}_1 : 0 \longrightarrow \mathrm{Ker}\ f \longrightarrow M \longrightarrow \mathrm{Im}\ f \longrightarrow 0$ and $\mathcal{S}_2 : \mathrm{Im}\ f \longrightarrow N \longrightarrow \mathrm{Coker}\ f \longrightarrow 0$, from which it is clear that both $\mathrm{Ker}\ f$ and $\mathrm{Im}\ f$ are elements of $\varinjlim \mathrm{CM}(R)$. Applying 2.3.3 to $\mathcal{S}_1$ we see that $\mathrm{Ker}\ f$ and $\mathrm{Im}\ f$ are both in $\varinjlim \mathrm{CM}(R)_\infty$. Applying $\mathrm{Hom}_R(k, -)$ to $\mathcal{S}_2$ shows that $\mathrm{Coker}\ f$ also has infinite Ext-depth. This shows that $\varinjlim \mathrm{CM}(R)_\infty$ is closed under kernels and cokernels, so is an Abelian category. We now show that this abelian category is Grothendieck. Since $\varinjlim \mathrm{CM}(R)_\infty$ is definable, it is closed under coproducts, so is cocomplete, and products, so is complete. Suppose

$$\{0 \longrightarrow L_i \longrightarrow M_i \longrightarrow N_i \longrightarrow 0\}_I$$

is a directed system of short exact sequence with terms in $\varinjlim \mathrm{CM}(R)_\infty$, then it is also a directed system in $\mathrm{Mod}R$ whose direct limit is the short exact sequence $\mathbb{S} : 0 \longrightarrow \varinjlim L_i \longrightarrow \varinjlim M_i \longrightarrow \varinjlim N_i \longrightarrow 0$. Yet all three terms of this exact sequence lie in $\varinjlim \mathrm{CM}(R)_\infty$, so $\mathbb{S}$ is actually short exact sequence in $\varinjlim \mathrm{CM}(R)_\infty$. Lastly, we have to show that $\varinjlim \mathrm{CM}(R)_\infty$ contains a generator. Since $\varinjlim \mathrm{CM}(R)_\infty$ is definable, there is a set of objects $\mathcal{X}$ such that every object in $\varinjlim \mathrm{CM}(R)_\infty$ can be realised as the direct limit of a directed system in $\mathcal{X}$ (this is a consequence of the Downwards Löwenheim-Skolem theorem, see [39, §18.1.4] for more details). The module $G = \bigoplus_{X \in \mathcal{X}} X$ acts as a generator for $\varinjlim \mathrm{CM}(R)_\infty$. Indeed, let M be a module in $\varinjlim \mathrm{CM}(R)_\infty$ and $(X_i, f_{i,j})_{i,j \in I}$ a directed system in $\mathcal{X}$ with direct limit M. By properties of direct limits, there is a pure epimorphism in $\varinjlim \mathrm{CM}(R)_\infty$

$$\bigoplus_{i \in I} X_i \longrightarrow M.$$

There is then a projection $G^{(I)} \longrightarrow \bigoplus_I X_i$, we may compose with π to obtain the required surjection $G^{(I)} \longrightarrow M$. $\square$

We can also consider homomorphisms between modules in $\varinjlim \mathrm{CM}(R)_\infty$.

Proposition 5.1.3. If $M, N \in \varinjlim \mathrm{CM}(R)_\infty$, then $\mathrm{Hom}(M, N) \in \varinjlim \mathrm{CM}(R)_\infty$.

Proof. By Hom-Tensor adjunction, we have isomorphisms $\mathrm{Hom}(k, \mathrm{Hom}(M, N)) \simeq \mathrm{Hom}(k \otimes M, N) = 0$ since M has infinite Tor-codepth; therefore $\mathrm{Hom}(M, N) \in \varinjlim \mathrm{CM}(R)$, so all that is needed is to show $\mathrm{Ext}^1(k, \mathrm{Hom}(M, N)) = 0$. To this end, there is a short exact sequence $0 \longrightarrow N \longrightarrow E \longrightarrow \Omega^{-1}(N) \longrightarrow 0$. Since $\mu_0(\mathfrak{m}, N) = 0$, both N and E have infinite Ext-depth, and therefore so does $\Omega^{-1}(N)$. Applying the functor $\mathrm{Hom}(M, -)$ gives the exact sequence

$$0 \longrightarrow \mathrm{Hom}(M, N) \xrightarrow{\alpha} \mathrm{Hom}(M, E) \xrightarrow{\beta} \mathrm{Hom}(M, \Omega^{-1}(N)) \xrightarrow{\gamma} \mathrm{Ext}^1(M, N) \longrightarrow 0.$$

In particular, we have another short exact sequence

$$0 \longrightarrow \mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(M, E) \longrightarrow \mathrm{Coker}(\alpha) \longrightarrow 0$$

which yields the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(k, \mathrm{Hom}(M, N)) \longrightarrow \mathrm{Hom}(k, \mathrm{Hom}(M, E)) \longrightarrow \mathrm{Hom}(k, \mathrm{Coker}(\alpha)) \longrightarrow \\ \longrightarrow \mathrm{Ext}^1(k, \mathrm{Hom}(M, N)) \longrightarrow \mathrm{Ext}^1(k, \mathrm{Hom}(M, E)) \longrightarrow \cdots \end{aligned}$$

As previously shown, $\mathrm{Hom}(k, \mathrm{Hom}(M, N)) = 0 = \mathrm{Hom}(k, \mathrm{Hom}(M, E))$ and we see that $\mathrm{Ext}^1(k, \mathrm{Hom}(M, E)) \simeq \mathrm{Hom}(\mathrm{Tor}_1(k, M), E) = 0$ as M has infinite Tor-codepth. Consequently we have an isomorphism
$\mathrm{Hom}(k, \mathrm{Coker}(\alpha)) \simeq \mathrm{Ext}^1(k, \mathrm{Hom}(M, N))$. But $\mathrm{Coker}(\alpha)(= \mathrm{Hom}(M, E)/\mathrm{Ker}(\beta) \simeq \mathrm{Im}(\beta)) \subset \mathrm{Hom}(M, \Omega^{-1}(N))$, and since $\mathrm{Hom}(M, \Omega^{-1}(N)) \in \varinjlim \mathrm{CM}(R)$, we see that $\mathrm{Hom}(k, \mathrm{Coker}(\alpha)) = 0$, which is what we wanted to show. □

Before stating similar thing for tensor products, we show that $\varinjlim \mathrm{CM}(R)_\infty$ contains an injective cogenerator.

Corollary 5.1.4. The injective module $\bigoplus_{\mathrm{ht}\,\mathfrak{p}=0} E(R/\mathfrak{p})$ is an injective cogenerator in $\varinjlim \mathrm{CM}(R)_\infty$.

Proof. Since $\dim R = 1$ we have $\mathrm{pSpec(R)} = \{\mathfrak{p} \in \mathrm{Spec(R)} : \mathrm{ht}\,\mathfrak{p} = 0\}$ and for any $M \in \varinjlim \mathrm{CM}(R)_\infty$ we see

$$E(M) = \bigoplus_{\mathrm{pSpec(R)}} E(R/\mathfrak{p})^{(X_\mathfrak{p})}.$$

In particular, there will be a non-zero morphism into $\bigoplus_{\mathrm{pSpec(R)}} E(R/\mathfrak{p})$. □

We now show $\varinjlim \mathrm{CM}(R)_\infty$ is closed under tensor product.

Proposition 5.1.5. Let $M, N \in \varinjlim \mathrm{CM}(R)_\infty$, then $M \otimes N \in \varinjlim \mathrm{CM}(R)_\infty$.

Proof. By 1.5.8, it is enough to show that $k \otimes (M \otimes N) = 0 = \mathrm{Hom}(k, M \otimes N)$. The first equality is obvious since M and N have infinite Tor-codepth. For the second, we have $\mathrm{Hom}(k, M \otimes N) = 0$ if and only if $\mathrm{Hom}(k, M \otimes N)^\vee = 0$ if and only if $k \otimes \mathrm{Hom}(M \otimes N, E(k)) = 0$. But $\mathrm{Hom}(M \otimes N, E(k)) \simeq \mathrm{Hom}(M, N^\vee)$ and since $N \in \varinjlim \mathrm{CM}(R)_\infty$ we also have $N^\vee \in \varinjlim \mathrm{CM}(R)_\infty$ since $\varinjlim \mathrm{CM}(R)_\infty^d = \varinjlim \mathrm{CM}(R)_\infty$. But by the previous proposition, we know that $\mathrm{Hom}(M, N^\vee) \in \varinjlim \mathrm{CM}(R)_\infty$, and therefore $k \otimes \mathrm{Hom}(M, N^\vee) = 0$. $\square$

There is another property that is specific to the one-dimensional case. In general, both $\varinjlim \mathrm{CM}(R)$ and $\varinjlim \mathrm{CM}(R)_\infty$ are extension closed subcategories of $\mathrm{Mod}(R)$, so we can view them as exact categories in their own right, and $\varinjlim \mathrm{CM}(R)_\infty \subset \varinjlim \mathrm{CM}(R)$ like a Serre subcategory by 2.3.3. In the abelian setting one can localise by Serre subcategories, but the class $\varinjlim \mathrm{CM}(R)$ is not abelian. However, in the dimension one case, we are still able to perform a certain type of localisation. In order to do this, we need the following definition.

Definition 5.1.6. [10, Def. 2.15] Let $\mathcal{D}$ be an exact category. An exact full subcategory $\mathcal{C} \subset \mathcal{D}$ is *left filtering* if every morphism $X \longrightarrow Y$ in $\mathcal{D}$, with $X \in \mathcal{C}$, factors through an admissible monomorphism $X' \hookrightarrow Y$, with $X' \in \mathcal{C}$:

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \\ & X' \arrow[u, hook] \end{tikzcd}$$

Lemma 5.1.7. When $\dim R = 1$, $\varinjlim \mathrm{CM}(R)_\infty$ is left filtering in $\varinjlim \mathrm{CM}(R)$.

Proof. Let $f : X \longrightarrow Y$ be a morphism in $\varinjlim \mathrm{CM}(R)$ with $X \in \varinjlim \mathrm{CM}(R)_\infty$. Then there are two short exact sequences of R-modules

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow X \longrightarrow \mathrm{Im}(f) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Im}(f) \longrightarrow Y \longrightarrow \mathrm{Coker}(f) \longrightarrow 0.$$

Since $\mathrm{CohCM}(R)$ is closed under submodules, we see that $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ lie in $\mathrm{CohCM}(R)$, and therefore $\mathrm{Ker}(f) \hookrightarrow X \twoheadrightarrow \mathrm{Im}(f)$ is a conflation in $\varinjlim \mathrm{CM}(R)_\infty$ by applying 2.3.3. In particular, $\mathrm{Ext}^1(k, \mathrm{Im}(f)) = 0$ in $\mathrm{Mod}(R)$. Applying the functor $\mathrm{Hom}(k, -)$ to the second exact sequence and then applying the depth lemma shows that $\mathrm{Coker}(f) \in \varinjlim \mathrm{CM}(R)$, so $\mathrm{Im}(f) \hookrightarrow Y \twoheadrightarrow \mathrm{Coker}(f)$ is a conflation in $\varinjlim \mathrm{CM}(R)$. Therefore $f : X \longrightarrow Y$ through the admissible monomorphism $\mathrm{Im}(f) \hookrightarrow Y$. $\square$

Remark. It is straightforward to show that any Serre subcategory of an abelian category is left filtering, and the above just modifies this result. The assumption of $\dim R = 1$ is necessary for this result: if $\dim R > 1$, then in general $\varinjlim \mathrm{CM}(R)$ will not be closed under submodules, so one cannot usually form the conflation $\mathrm{Ker}(f) \hookrightarrow X \twoheadrightarrow \mathrm{Im}(f)$ in $\varinjlim \mathrm{CM}(R)$, let alone $\varinjlim \mathrm{CM}(R)_\infty$.

The following result, due to M. Schlichting, shows how left-filtering can be used to localise with respect to an exact category.

Lemma 5.1.8. [45, 1.13] Let $\mathcal{B}$ be an exact category and $\mathcal{A} \subset \mathcal{B}$ an extension closed subcategory. Let Σ denote the collection of admissible epimorphisms in $\mathcal{B}$ with kernel in $\mathcal{A}$. Then Σ admits a calculus of left fractions in $\mathcal{B}$ if and only if $\mathcal{A}$ is left filtering in $\mathcal{D}$.

If one lets $\mathcal{B}[\Sigma^{-1}]$ denote the localisation of $\mathcal{B}$ at Σ, Schlichting shows that this category does not necessarily inherit the structure of an exact category. Before describing this localisation in more detail, we recall some background information on localisations, which is taken from [33, §7].

Definition 5.1.9. If $\mathcal{C}$ is a category and $\mathcal{S}$ is a collection of morphisms in $\mathcal{C}$, a *localisation* is a category $\mathcal{C}_\mathcal{S}$ and a functor $Q : \mathcal{C} \longrightarrow \mathcal{C}_\mathcal{S}$ such that

1. for every $s \in \mathcal{S}$, $Q(s)$ is an isomorphism;
2. if $\mathcal{A}$ is another category and $F : \mathcal{C} \longrightarrow \mathcal{A}$ is a functor such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$, then there is a functor $F_\mathcal{S} : \mathcal{C}_\mathcal{S} \longrightarrow A$ such that $F = F_\mathcal{S} \circ \mathcal{Q}$.
3. if $\mathcal{A}$ is another category and $F_1, F_2 : \mathcal{C}_\mathcal{S} \longrightarrow \mathcal{A}$ are functors, then there is a bijection
$$\mathrm{Hom}_{(\mathcal{C}_\mathcal{S},\mathcal{A})}(F_1, F_2) \longrightarrow \mathrm{Hom}_{(\mathcal{C},\mathcal{A})}(F_1 \circ Q, F_2 \circ Q).$$
Here $(\mathcal{C}_\mathcal{S}, \mathcal{A})$ is the functor category.

A class $\mathcal{S}$ of morphisms in $\mathcal{C}$ is said to be a *left multiplicative system* if

1. every isomorphism in $\mathcal{C}$ lies in $\mathcal{S}$;
2. $\mathcal{S}$ is closed under composition;
3. given two morphisms $f : X \longrightarrow Y$ and $t : Y' \longrightarrow Y$ with $t \in \mathcal{S}$, then there is an object X' and morphisms $s : X' \longrightarrow X$ and $g : X' \longrightarrow Y'$ with $s \in \mathcal{S}$ such that $t \circ g = f \circ s$;

4. given two morphisms $f, g : X \longrightarrow Y$ and a map $t : Y \longrightarrow Z$, with $t \in \mathcal{S}$ such that $t \circ f = t \circ g$, then there is an $s : W \longrightarrow X$ in $\mathcal{S}$ such that $f \circ s = g \circ s$.

A left multiplicative system $\mathrm{CM}(S)$ in a category $\mathcal{C}$ induces a localisation $Q : \mathcal{C} \longrightarrow \mathcal{C}_{\mathcal{S}}$, the construction of which can be found in [33]. We note that the functor Q is essentially surjective, and use this to describe morphisms in $\mathcal{C}_{\mathcal{S}}$. The objects of $\mathcal{C}_{\mathcal{S}}$ are the same as those of $\mathcal{C}$, while a morphism $f : Q(A) \longrightarrow Q(B)$ in $\mathcal{C}_{\mathcal{S}}$ is given by an equivalence class of triples (X, s, g), where

$$A \xleftarrow{\ s\ } X \xrightarrow{\ g\ } B$$

with $s \in \mathcal{S}$. The equivalence relation is given as follows: say $(X, s, g) \sim (Y, t, h)$ if there is a commutative diagram

$$\begin{tikzcd}
& X \arrow[dl, "s"'] \arrow[dr, "g"] & \\
A & Z \arrow[l, dashed, "u"] \arrow[r, dashed, "j"'] \arrow[u, dashed] \arrow[d, dashed] & B \\
& Y \arrow[ul, "t"] \arrow[ur, "h"'] &
\end{tikzcd}$$

with $u \in \mathcal{S}$. Since under Q every element of $\mathcal{S}$ becomes an isomorphism, we see that $f = Q(g) \circ Q(s)^{-1}$.

Returning to the original situation, Schlichting's result shows that Σ is a left multiplicative system, and therefore the localisation can be realised using the above calculus of left fractions. For brevity, we will let $Q : \varinjlim \mathrm{CM}(R) \longrightarrow \mathcal{A}$ denote the localisation of $\varinjlim \mathrm{CM}(R)$ with respect to Σ.

Proposition 5.1.10. Let $M \in \varinjlim \mathrm{CM}(R)_\infty$, then $Q(M) \simeq 0$ in $\mathcal{A}$.

Proof. If $M \in \varinjlim \mathrm{CM}(R)_\infty$, then the zero map $M \longrightarrow 0$ is an admissible epimorphism with kernel in $\varinjlim \mathrm{CM}(R)_\infty$. By [49, 10.3.10], it follows that $Q(M) \simeq 0$ in $\mathcal{A}$. □

If R is a complete one-dimensional Cohen-Macaulay ring, there are several interactions between the localisation $Q : \varinjlim \mathrm{CM}(R) \longrightarrow \mathcal{A}$ and some more familiar functors, such as the canonical dual. Indeed, we have seen that if $\text{E-dp}(M) = 1$, then $\text{E-dp}(M^*) = 1$ as well, hence the image of the canonical dual lies within the class of modules with Ext-depth one, which is precisely the class $\mathrm{bbCM}(R)$, which we will view as a full subcategory of $\mathrm{Mod}(R)$. Now, by local duality the collection of modules M such that $M^* = 0$ is precisely the class $\varinjlim \mathrm{CM}(R)_\infty$. In particular, if $s : M \longrightarrow N$ is an admissible epimorphism in $\varinjlim \mathrm{CM}(R)$ with kernel in $\varinjlim \mathrm{CM}(R)_\infty$, then by the exactness of $\mathrm{Hom}_R(-, \Omega)$ on $\varinjlim \mathrm{CM}(R)$ there is an isomorphism $M^* \simeq N^*$, hence s^* is an isomorphism. By the property of the localisation $Q : \varinjlim \mathrm{CM}(R) \longrightarrow \mathcal{A}$, we have the following commutative diagram of categories.

$$\begin{tikzcd}
\varinjlim \mathrm{CM}(R) \arrow[r, "(-)^*"] \arrow[d, "Q"] & \mathrm{bbCM}(R) \\
\mathcal{A} \arrow[ur, swap, "\widetilde{(-)^*}"] &
\end{tikzcd}$$

We therefore see that the canonical duality factors through the localisation $Q : \varinjlim \mathrm{CM}(R) \longrightarrow \mathcal{A}$.

Chapter 6

Potential future work

Through the thesis several avenues have opened that lend themselves to further research. We will look at some of these possible questions and how they relate to the above work.

Silting and infinite depth modules

We have seen that the classes $\varinjlim \mathrm{CM}(R)$ and $\mathrm{CohCM}(R)$ are cotilting, while their dual definable categories are tilting. The notions of silting and cosilting are weaker than tilting and cotilting, so there will be at least as many cosilting classes containing $\mathrm{CM}(R)$ as the cotilting classes described above. Using recent developments in the theory of silting and cosilting over commutative rings, it may be possible to completely describe all the (co)silting classes containing the class $\mathrm{CM}(R)$. Another possible application of silting is in relation to infinite depth modules over a one-dimensional Cohen-Macaulay ring. In this setting the class $\varinjlim \mathrm{CM}(R)_\infty$ is a definable Grothendieck abelian category, so is an extension closed bireflective subcategory of $\mathrm{Mod}(R)$, see [2, 5.2]. There is a close relationship between silting classes and such subcategories. Indeed, given any minimal silting R-module, one obtains an extension closed bireflective subcategory of $\mathrm{Mod}(R)$, see [2, 5.11]. However, not every extension-closed bireflective subcategory arises this way, so there is no certainty that $\varinjlim \mathrm{CM}(R)_\infty$ does. There are cases when it does, however, namely over one dimensional regular local rings (that is discrete valuation rings), which are hereditary. Investigating which Cohen-Macaulay rings this extends to is an area for further investigation. One may also be able to generalise these beyond the Cohen-Macaulay case, by considering $\mathfrak{a}$ torsion and torsion free modules for arbitrary ideals $\mathfrak{a}$ of a Noetherian commutative ring.

Homological properties of balanced big Cohen-Macaulay modules

If R is Cohen-Macaulay local ring with a canonical module, we have seen how to identify the balanced big Cohen-Macaulay modules by considering depth. However, there is an ambiguity about how one can define the category of balanced big Cohen-Macaulay modules, in particular the morphisms between them. One can simply view $\mathrm{bbCM}(R)$ as a full subcategory of $\mathrm{Mod}R$, as we have done above; this replicates how one defines the morphisms in $\mathrm{CM}(R)$. However, doing this will give morphisms that factor through a module of infinite Ext-depth, something that cannot happen in $\mathrm{mod}R$ as there are no finitely generated modules of infinite Ext-depth. Consequently one may wish to consider the naive quotient category

$$\underline{\varinjlim \mathrm{CM}(R)} := \varinjlim \mathrm{CM}(R) / \varinjlim \mathrm{CM}(R)_\infty,$$

in which there are no infinite Ext-depth modules and no morphisms factoring through infinite Ext-depth modules. In particular, one may be able to think of the canonical dual on such a category: if $f : M \longrightarrow N$ is a morphism in $\varinjlim \mathrm{CM}(R)$ with $M, N \notin \varinjlim \mathrm{CM}(R)_\infty$, then if there is a factorisation

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "g"'] & & N \\ & X \arrow[ur, "h"'] & \end{tikzcd}$$

with $X \in \varinjlim \mathrm{CM}(R)_\infty$, then $\mathrm{Hom}(f, \Omega) = \mathrm{Hom}(g, \Omega) \circ \mathrm{Hom}(h, \Omega)$. Yet $\mathrm{Hom}(g : \Omega) : \mathrm{Hom}(X, \Omega) \longrightarrow \mathrm{Hom}(M, \Omega)$ is the zero map, since $\mathrm{Hom}(X, \Omega) = 0$ by local duality, so $\mathrm{Hom}(f, \Omega) = 0$. Therefore by the universal property of the stabilisation the canonical dual factors through $\underline{\varinjlim \mathrm{CM}(R)}$.

Relative homological algebra and connections with triangulated categories

We have already seen some applications in relative homological algebra by considering dimensions with respect to $\mathrm{CohCM}(R)$ and $\mathrm{GFlat}(R)$-Mittag-Leffler modules.

There are, however, many more applications that arise from the classes $\mathrm{CohCM}(R)$ and $\varinjlim \mathrm{CM}(R)$. For example, if $\mathcal{Q}$ is a class of modules, one can consider the class of Gorenstein $\mathcal{Q}$-flat modules, those modules that arise as $\mathrm{Ker}(F^0 \longrightarrow F^1)$ for a $(\mathcal{Q} \otimes -)$-acyclic acyclic complex of flat modules. Depending on the properties of the class $\mathcal{Q}$ (such as being definable, or part of a cotorsion or duality pair), one can consider different applications to the class of Gorenstein $\mathcal{Q}$-flat modules . In [25] closure properties of this class are considered, while in [17] connections are made to triangulated categories through homotopy categories of totally acyclic complexes. I would be interested in looking at these applications in relation to $\varinjlim \mathrm{CM}(R)$ (in particular) and $\mathrm{CohCM}(R)$, and seeing how they relate to the class of balanced big Cohen-Macaulay modules. Another angle of enquiry is looking into properties that $\mathrm{CM}(R)$ inherits from the class $\varinjlim \mathrm{CM}(R)$. For example, 2.1.6 shows that $\mathrm{CM}(R)$ is covariantly finite in $\mathrm{mod}R$. This in some sense provides a dual result to the classic result of Auslander and Buchweitz, that $(\mathrm{CM}(R), \mathcal{I}^{fg}_{<\infty})$ is a complete cotorsion pair in $\mathrm{mod}R$. It seems unlikely that $\mathrm{CM}(R)$ will be special preenveloping in $\mathrm{mod}R$, nor that it appears in the right side of a cotorsion pair, but the properties of these envelopes may be worth investigating.

Bibliography

[1] Anderson, D.D. and Winders, M. *Idealization of a module*. Journal of Commutative Algebra, Vol. 1, No. 1, 2009. pp. 3-56.

[2] Angeleri Hügel, L. *On the abundance of silting modules*. Surveys in Representation Theory of Algebras, Contemporary Mathematics, AMS, 2018. pp. 1-24.

[3] Angeleri Hügel, L. and Herbera, D. *Mittag-Leffler conditions on modules*. Indiana University Mathematics Journal, Vol. 57, No. 5, 2008. pp. 2459-2517

[4] Angeleri Hügel, L., Herbera, D. and Trlifaj, J. *Tilting modules and Gorenstein rings*. Forum Mathematicum, Vol. 18, Issue 2, 2006. pp. 211-229

[5] Angeleri Hügel, L. Popíšil, D. Šťovíček, J. and Trlifaj, J. *Tilting, cotiliting, and spectra of commutative Noetherian rings*. Transactions of the American Mathematical Society, Vol. 366, No. 7, 2014. pp. 3487-3517.

[6] Atiyah, M.F. and MacDonald, I.G. *An Introduction to Commutative Algebra*. Addison-Wesley, 1969.

[7] Bazzoni, S. *When are definable classes tilting and cotilting classes?* Journal of Algebra, Vol. 320, Issue 12, 2008. pp. 4281-4299.

[8] Bazzoni, S. and Herbera, D. *Cotorsion pairs generated by modules of bounded projective dimension*, Israel Journal of Mathematics, Vol. 174, Issue 1, 2009. pp. 119-160.

[9] Bergman, G. *Every module is an inverse limit of injectives.* Proceedings of the American Mathematical Society, Vol. 141, No. 4, 2013. pp. 1177-1183.

[10] Braunling, O., Groechenig, M. and Wolfson, J. *Tate objects in Exact categories*. Moscow Mathematical Journal, Vol. 16, Issue 3, 2016. pp. 433-504.

[11] Brodmann, M.P. and Sharp, R.Y. *Local Cohomology: An Algebraic Introduction with Geometric Applications* Second Edition. Cambridge University Press, 2013.

[12] Bruns, W. and Herzog, J. *Cohen-Macaulay rings* Revised edition. Cambridge University Press, 1998.

[13] Buchweitz, R.O. *Maximal Cohen-Macaulay modules and Tate-Cohomology over Gorenstein rings*. Unpublished manuscript, (1986).

[14] Buchweitz, R.O., Greuel, G.M. and Schreyer, F.O. *Cohen-Macaulay modules on hypersurface singularities II*. Inventiones Mathematicae, Vol. 88, Issue 1, 1987. pp. 165-182.

[15] Bühler, T. *Exact categories*. Expositiones Mathematicae, Vol. 28, Issue 1, 2010. pp. 1-69.

[16] Christensen, L.W. *Gorenstein Dimensions*, Springer, 2000.

[17] Christensen, L.W., Estrada, S. and Thompson, P. *Homotopy categories of totally acyclic complexes with applications to the flat-cotorsion theory*, (preprint), arXiv:1812.04402

[18] Couchot, F. *Commutative rings whose cotorsion modules are pure-injective*. Palestine Journal of Mathematics, Vol. 5, Special issue 1, 2016. pp. 81-89.

[19] Crawley-Boevey, W. *Locally finitely presented additive categories*, Communications in Algebra, Vol. 22, Issue 5, 1994. pp. 1641-1674.

[20] Crivei, S., Prest, M. and Torrecillas, B. *Covers in finitely accessible categories*. Proceedings of the American Mathematical Society, Vol 138, No. 4, 2010. pp. 1213-1221.

[21] Decker, W. and Lossen, C. *Computing in Algebraic Geometry: A quick start using Singular*. Springer, 2006.

[22] Enochs, E.E. and Lopez-Ramos, J.A. *Gorenstein Flat Modules*, Nova Science Publishers, 2001.

[23] Enochs, E.E. and Lopez-Ramos, J.A. *Kaplansky classes*, Rendiconti del Seminario Matematico della Universit di Padova, Vol. 107, 2002. pp. 567-79.

[24] Enochs, E.E. and Jenda, O.M.G. *Relative Homological Algebra* De Gruyter, 2000.

[25] Estrada, S., Iacob, A. and Perez, M.A. *Model structures and relative Gorenstein flat modules and chain complexes*, (preprint) arXiv:1709.00658. (2018)

[26] Fossum, R.M., Griffith, P.A. and Reiten, I. *Trivial Extensions of Abelian Categories.* Springer, 1975.

[27] Foxby, H.-B. *Gorenstein modules and related modules.* Mathematica Scandinavica, Vol. 31, No. 2, 1972. pp. 267-284.

[28] Gillespie, J. *The flat stable module category of a coherent ring*, Journal of Pure and Applied Algebra, Vol. 221, Issue 8, 2017. pp. 2025-2031.

[29] Göbel, R. and Trlifaj, J. *Approximations and Endomorphism Algebras of Modules* Second Revised and Extended Edition. De Gruyter, 2012.

[30] Holm, H. *The structure of balanced big Cohen-Macaulay modules over Cohen-Macaulay rings.* Glasgow Mathematical Journal, Vol. 59, Issue 3, 2017. pp. 459-561.

[31] Holm, H. and Jørgensen, P. *Cotorsion pairs induced by duality pairs.* Journal of Commutative Algebra, Vol. 1, No. 4, 2009. pp. 621-633.

[32] Iyengar, S. et al. *Twenty Four Hours of Local Cohomology.* American Mathematical Society, 2007.

[33] Kashiwara, M. and Schapira, P. *Categories and Sheaves.* Springer, 2006.

[34] Krause, H. *Exactly definable categories.* Journal of Algebra, Vol. 201, Issue 2, 1998. pp.456-492.

[35] Leuschke, G. J. and Wiegand, R. *Cohen-Macaulay Representations.* American Mathematical Society, 2012.

[36] Matsumura, H. *Commutative Ring Theory.* Cambridge University Press, 1987.

[37] Prest, M. *Definable Additive Categories: Purity and Model Theory.* Memoirs of the AMS, Vol. 210, No. 987, (2011)

[38] Prest, M. *Epimorphisms of rings, interpretations of modules and strictly wild algebras.* Communications in Algebra, Vol 24, Issue 2, 1994. pp. 517-531.

[39] Prest, M. *Purity, Spectra and Localisation.* Cambridge University Press, 2009

[40] Puninski, G. *The Ziegler spectrum and Ringel's quilt of the A-infinity plane curve singularity.* Algebra and Representation Theory, Vol. 21, Issue 2, 2018. pp. 419-446.

[41] Reiten, I. *The converse to a theorem of Sharp on Gorenstein modules.* Proceedings of the American Mathematical Society, Vol. 32, No. 2, 1972. pp. 417-420.

[42] Ren, W. *Gorenstein projective modules and Frobenius extensions.* Science China Mathematics, Vol. 61, Issue 7, 2018. pp. 1175-1186.

[43] Rothmaler, P. *Mittag-Leffler modules and positive atomicity*, Habilitationsschrift, Kiel. (1994)

[44] Rotman, J. *An Introduction to Homological Algebra*, Second edition. Springer, 2009.

[45] Schlichting, M. *Delooping the K-theory of exact categories.* Topology, Vol,43, Issue 5, 2004. pp. 1089-1103.

[46] Sharp, R. Y. *Cohen-Macaulay properties for balanced big Cohen-Macaulay modules.* Mathematical Proceedings of the Cambridge Philosophical Society, Vol. 90, Issue 2, 1981. pp. 229-238.

[47] Šťovíček, J. Trlifaj, J. and Herbera, D. *Cotilting modules over commutative Noetherian rings.* Journal of Pure and Applied Algebra, Vol. 218, Issue 9, 2014. pp. 1696-1711.

[48] Stroker, J.R. *Homological Questions in Local Algebra.* Cambridge University Press, 1990.

[49] Weibel, C. *An Introduction to Homological Algebra*. Cambridge University Press, 1994.

[50] Xu, J. *Flat Covers of Modules*. Springer, 1996.

[51] Yoshino, Y. *Cohen-Macaulay Modules over Cohen-Macaulay rings*. Cambridge University Press, 1990.