Definable categories and monoidal categories

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Abstract Definable categories are axiomatisable additive categories. They appear as definable subcategories of module categories, equivalently as the categories of exact functors on some small abelian category. We give an exposition of their structure and their model theory from an essentially intrinsic point of view. We recall the anti-equivalence between definable categories and small abelian categories and we describe a monoidal version of this due to Wagstaffe.

1 Introduction

Definable categories first appeared in the model theory of modules as additive axiomatisable subcategories of module categories (see, for example, [20, §2.6]). These subcategories were given a purely algebraic characterisation in [5, §2.3] as being those closed under direct products, directed colimits and pure submodules (and isomorphism, which we generally assume of the subcategories we discuss). Definable categories have a rich structure and they have been appearing in a variety of contexts. This is a largely expository paper about certain aspects of these categories: their model theory; some ways in which they resemble module categories (which they include); and how, if there is a monoidal structure, it interacts with the definability structure.

There is a third characterisation of definable categories, namely as the categories of exact (additive, as are all functors in this paper) functors on skeletally small abelian categories, [14, §4]). More precisely, if we fix a skeletally small abelian category $\mathcal{A}$, then the category $\mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ of exact functors on it is a typical definable category. Let us state the equivalence of these definitions: we say that an additive category is

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definable[1] if it can be obtained in the following equivalent ways (see Section 2 for purity and Theorem 2 for axiomatisability).

Theorem 1 *The following are equivalent for an additive category* $\mathcal{D}$*:*
(i) $\mathcal{D}$ *is equivalent to a subcategory of* Mod-R*, where* R *is a skeletally small preadditive category, which is closed in* Mod-R *under direct products, directed colimits and pure submodules;*
(ii) $\mathcal{D}$ *is an additive axiomatisable subcategory of a category of the form* Mod-R*;*
(iii) $\mathcal{D}$ *is equivalent to the category* $\mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ *of exact functors on a skeletally small abelian category* $\mathcal{A}$*.*

From the point of view of the third equivalent, a module is none other than an exact functor on a small abelian category. In so far as this paper adopts this viewpoint, it can be seen as a continuation of [27], where the main theme is that a module is intrinsically given as a faithful exact functor on a canonically associated small abelian category. In that paper we also briefly described the anti-equivalence between the 2-category $\mathbb{ABEX}$ of small abelian categories and exact functors and the 2-category $\mathbb{DEF}$ of definable additive categories with interpretation functors. Here we will say some more about that before describing the monoidal version of this correspondence, due to Wagstaffe [33, 3.2.1], [34, 1.1].

The material in this paper is very general so, in order to anchor ideas, we present some illustrative examples.

Example 1

Let $\mathcal{DIV}$ be the category of divisible abelian groups. This is a definable subcategory of **Ab**, as is clear from condition (ii) of Theorem 1, suitable axioms being those of the form $\forall x \exists y \, (x = ny)$ for $n \geq 2$. The equivalent condition (i) is also easy enough to check but what about condition (iii)? It is not immediately clear what the relevant abelian category $\mathcal{A}$ should be but general theory, see [26, 7.2], gives that it is the opposite of the category of finitely generated abelian groups (note that, in [27, Ex. 8.5], the "opposite" has been omitted). The action of $\mathcal{DIV}$ on $(\text{mod-}\mathbb{Z})^{\mathrm{op}}$, that is, the equivalence $\mathcal{DIV} \to \mathrm{Ex}((\text{mod-}\mathbb{Z})^{\mathrm{op}}, \mathbf{Ab})$ is given on objects by $D \mapsto (-, D)$ for $D \in \mathcal{DIV}$ (note that $(-, D)$ is exact since D is an injective $\mathbb{Z}$-module).

A divisible abelian group is a $\mathbb{Z}$-module but, in the example above, we saw the same group appearing as a right mod-$\mathbb{Z}$-module where, by an $\mathcal{A}$**-module**, we mean an additive functor from $\mathcal{A}$ to **Ab** if $\mathcal{A}$ is a skeletally small pre additive category. Thus a module M, regarded as an object of a definable category, is a module over many rings. For another example, if M is an R-module then it is also a module over any ring Morita equivalent to R. If, however, we require that $R = \mathcal{A}$ is chosen to be abelian and that M be a faithful exact functor, then $\mathcal{A}$ is unique to equivalence, see Theorem 26 below, and hence $\mathcal{A}$ is the canonical ring over which M is a module

[1] The notion of definable category has been extended beyond the additive case, see [15], [16], but, in this paper, all our categories are additive so we don't usually say "definable additive category".

(and $\mathcal{A}$ contains, in some sense, cf. [27, 5.2], [21], all the other rings over which M is a faithful module). We refer to $\mathcal{A} = \mathcal{A}(M)$ as the **functor category** of M and set $\mathrm{fun}(M) = \mathcal{A}$. Similarly, if $\mathcal{D}$ is a definable category, then the, unique-to-equivalence small abelian category $\mathcal{A}$ such that $\mathrm{Ex}(\mathcal{A}, \mathbf{Ab}) = \mathcal{D}$ is denoted $\mathrm{fun}(\mathcal{D})$.[2]

A small choice

If $\mathcal{A}$ is a skeletally small abelian category, then so is its opposite $\mathcal{A}^{\mathrm{op}}$ so we have a choice of how to represent a definable category - as co- or contra-variant functors of a skeletally small abelian category. In [27] we chose the former representation and we will do the same here. But note that various other papers use the contravariant representation.

K-categories

If K is a commutative ring then, by a K-category, we mean a category enriched in Mod-K. More explicitly, to say that $\mathcal{A}$ is a K-category means that every hom group of $\mathcal{A}$ is endowed with the structure of a K-module and $fk = kf$ whenever $f \in (A, B)$ and $k \in K$. In that case, $\mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ has an induced structure as a K-category and is equivalent to the category $\mathrm{Ex}(\mathcal{A}, \text{Mod-}K)$ of exact K-linear functors on $\mathcal{A}$ and is a typical Mod-K-**valued definable category**. Specifically, if $F : \mathcal{A} \to \mathbf{Ab}$ is an additive functor then, because multiplication by $k \in K$ is a natural transformation from F to itself, one sees that each FA, for $A \in \mathcal{A}$ has the induced structure of a K-module and then F is K-linear, that is, F can be regarded as a functor from $\mathcal{A}$ to Mod-K. So the results here, which we present for target category **Ab** apply as well to K-categories. Indeed, if $\mathcal{G}$ is a locally finitely presented Grothendieck abelian category and $\mathcal{A}$ is skeletally small abelian, then $\mathrm{Ex}(\mathcal{A}, \mathcal{G})$ is a definable category ([25, §3.3]).

We mention that further development in the direction of definable categories and enriched categories is carried out in [8] (in the additive context) and [16] (in the non-additive context).

If $\mathcal{A}$ is a skeletally small abelian category, then $\mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ is contained as a **definable subcategory** of the full module category $\mathcal{A}\text{-Mod} = (\mathcal{A}, \mathbf{Ab})$, meaning that the inclusion preserves direct products and directed colimits. That's easy to see because, if $M : \mathcal{A} \to \mathbf{Ab}$ is an $\mathcal{A}$-module, then M is exact if, for every composable pair $A \xrightarrow{f} B \xrightarrow{g} C$ of morphisms in $\mathcal{A}$ which is exact (i.e. $\ker(g) = \mathrm{im}(f)$), we have that $\ker(Mg) = \mathrm{im}(Mf)$. That can easily be written down as the condition

[2] The category $\mathrm{fun}(\mathcal{D})$ can be defined as the category of functors (additive, as always in this paper) from $\mathcal{D}$ to **Ab** which commute with direct products and directed colimits, see Section 4. It also has a model-theoretic definition as the category of interpretation functors from $\mathcal{D}$ to **Ab**, that is, functors given by pp-definable sorts, see [23, Chpt. 25].

that M satisfies certain sentences in the language of $\mathcal{A}$-modules (see Example 5 and Theorem 2 in the next section).

Intermediate between $\mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ and $\mathcal{A}$-Mod we have the (definable sub)category $\mathrm{Lex}(\mathcal{A}, \mathbf{Ab})$ of left exact functors on $\mathcal{A}$. That category may be identified with the Ind-completion, $\mathrm{Ind}(\mathcal{A})$ of $\mathcal{A}^{\mathrm{op}}$ which is naturally embedded into $\mathrm{Lex}(\mathcal{A}, \mathbf{Ab})$ by $A \in \mathcal{A} \mapsto (A, -) : \mathcal{A} \to \mathbf{Ab}$. An advantage of $\mathrm{Lex}(\mathcal{A}, \mathbf{Ab}) = \mathrm{Ind}(\mathcal{A}^{\mathrm{op}})$ is that it is locally coherent Grothendieck category and the embedded copy of $\mathcal{A}^{\mathrm{op}}$ is equivalent to its full subcategory of finitely presented objects ([30, Prop. 2]). Moreover, $\mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ is the definable subcategory of $\mathrm{Lex}(\mathcal{A}, \mathbf{Ab})$ consisting of the absolutely pure = fp-injective objects, see [23, Chpt. 11]. Thus, the definable category $\mathcal{D} = \mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ and (the opposite of) its associated abelian category $\mathcal{A}$ are brought together into the same category if we work inside $\mathrm{Lex}(\mathcal{A}, \mathbf{Ab}) = \mathrm{Ind}(\mathcal{A}^{\mathrm{op}})$.

For a comparison of these categories and the relations between them, see [23, Chpt. 11]. For a comparison of the model-theoretic languages (all equivalent) which result from the choices of which category to embed $\mathcal{D} = \mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ into, see [24].

Example 2

In the case of the category $\mathcal{DIV}$ of divisible abelian groups, see Example 1, we have $\mathcal{DIV}$ represented as a definable subcategory of the category of left modules over $(\text{mod-}\mathbb{Z})^{\mathrm{op}}$ which we are regarding as a ring with many objects. The intermediate category $\mathrm{Lex}((\text{mod-}\mathbb{Z})^{\mathrm{op}}, \mathbf{Ab}) = \mathrm{Ind}(\text{mod-}\mathbb{Z})$ is the category of abelian groups. That is itself a definable subcategory of the much larger category $(\text{mod-}\mathbb{Z})^{\mathrm{op}}$-Mod. (If we were dealing with the definable category of torsionfree = flat abelian groups rather than the divisible ones, then the corresponding abelian category would be, [26, 7.1], $(\text{mod-}\mathbb{Z})^{\mathrm{op}}$, so the intermediate category would be the Ind-completion of $(\text{mod-}\mathbb{Z})^{\mathrm{op}}$.)

Example 3

The category $\mathcal{DIVTORS}$ of divisible torsion abelian groups (that is, direct sums of copies of the various Prüfer groups $\mathbb{Z}_{p^\infty}$) is not a definable subcategory of **Ab** - for instance because it is not closed under products, but it is a definable category. In particular, one may check that the product in $\mathcal{DIVTORS}$ of objects M_i is given by taking the torsion submodule of the product $\prod_i M_i$ in **Ab**. That is shown for instance in [27, Ex. 8.7], where it is also shown that, with **Fin** denoting the category of finite abelian groups, the functor category of $\mathcal{DIVTORS}$, $\mathrm{fun}(\mathcal{DIVTORS}) = \mathbf{Fin}^{\mathrm{op}}$. In this case, the intermediate category between $\mathcal{DIVTORS}$ and $\mathbf{Fin}^{\mathrm{op}}$-Mod is the category of torsion abelian groups. (In the dual case, that of reduced divisible abelian groups, see [27, 8.7], the functor category is **Fin** and the intermediate category $\mathrm{Ind}(\mathbf{Fin}^{\mathrm{op}})$ is the opposite of the category of profinite abelian groups.)

2 Model theory in definable categories

There are many introductory accounts of the model theory of modules (pp formulas, pp-types etc.) but these are based on the view of a module as an abelian group with a ring acting on it as endomorphisms. Here we reformulate some of the basic definitions to fit directly with the view of a module as an exact functor on a small abelian category. We don't, however, carry this too far forward; we just say how it can be done. For those who know the usual approach, we add explanatory comments.

2.1 Pp formulas, pp-types and pure-injectives

Elements

A definable category $\mathcal{D} = \mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ might have no nonzero finitely presented objects; therefore the usual way of defining *elements* of an object of a category, as morphisms from a finitely presented object, can't be applied internally to $\mathcal{D}$. Regarding $M \in \mathcal{D}$ as an exact functor on $\mathcal{A}$ we can say that an **element** of M **of sort** $A \in \mathcal{A}$ is simply an element of $M(A)$. We can bring this closer to the first idea of "element" by noting that each representable functor $(A, -)$ is in $\mathrm{Lex}(\mathcal{A}, \mathbf{Ab})$, so then we have $\mathcal{A}^{\mathrm{op}}$ sitting inside $\mathrm{Ind}(\mathcal{A}) = \mathrm{Lex}(\mathcal{A}, \mathbf{Ab})$ as the subcategory of finitely presented objects[3] while $\mathcal{D}$ sits definably within the same category of functors on $\mathcal{A}$. Then, by the Yoneda Lemma, we have a natural isomorphism $((A, -), M) \simeq M(A)$, so we may, alternatively, say that an **element** of $M \in \mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ **of sort** $A \in \mathcal{A}$ is a morphism $a : (A, -) \to M$ and that is what we will do here.

Explanation

If R is a ring in the usual sense, that is a ring with one object (a 1-sorted ring), and if M is a right R-module, then we have the canonical identification of the elements $a \in M$ with the morphisms $f : R_R \to M$ via $f \mapsto a = f(1)$. This of course is the basis of the above definition of "element" but the latter is wider since we allow morphisms from arbitrary finitely presented objects[4].

Example 4

Continuing Example 3, where we have $\mathcal{DIVTORS}$ and its functor category $\mathcal{A} = \mathbf{Fin}^{\mathrm{op}}$, we have both $\mathbf{Fin}$ and $\mathcal{DIVTORS}$ sitting inside $\mathrm{Lex}(\mathbf{Fin}^{\mathrm{op}}, \mathbf{Ab})$, and so the sorts of a torsion divisible abelian group D are of the form (A, D) as A ranges

[3] It sits as the finitely generated projectives within $(\mathcal{A}, \mathbf{Ab}) = \mathcal{A}$-Mod which is another possible context to use.

[4] or arbitrary finitely generated projective objects if we were using $\mathcal{A}$-Mod, rather than $\mathrm{Ind}(\mathcal{A})$, as the context.

over finite abelian groups. Decomposing A as a direct sum of indecomposables $\mathbb{Z}_{p^k}$ we have a finite sequence $(p_1^{k_1}, \ldots, p_n^{k_n})$ of prime powers and so see that the elements of D of sort A are n-tuples of elements $(a_1, \ldots, a_n)$ where the order of a_i is a divisor of $p_i^{k_i}$. This does give all the sorts, up to isomorphism, in the usual model-theoretic, quotient of pp-definable subgroups sense: for instance the elements of D of the quotient sort $xp^3 = 0/xp^2 = 0$ is isomorphic to the solution set of the formula $xp = 0$ (because D is divisible) which is the sort $((\mathbb{Z}_p, -), D)$ of D (the general point is that $\mathcal{A} = \mathbf{Fin}^{\mathrm{op}}$ is abelian so already contains each quotient of sorts as a sort).

A simplification, seen in the example above, with respect to the usual model-theoretic approach is that we don't need to treat n-tuples of elements for $n > 1$ because $\mathcal{A}$ is additive, so an n-tuple $(a_1, \ldots, a_n)$ with $a_i : (A_i.-) \to M$ can be regarded as a single element of sort $A_1 \oplus \cdots \oplus A_n$.

pp formulas

Suppose that A is an object of $\mathcal{A}$. A **pp formula** for elements of sort A is a morphism $\rho : B \to A$ in $\mathcal{A}$. If $M \in \mathcal{D} = \mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ and $a \in M$ is of sort A, that is, $a \in ((A, -), M)$, then a **satisfies the pp formula** ρ (in M) if a factors through $(\rho, -) : (A, -) \to (B, -)$. We refer to such a formula as being **of sort** A or as having **free variable of sort** A.

Explanation

Because we are using the maximal language (see [24]) for $\mathcal{D}$, the distinction between pp formulas and pp-pairs = pp-sorts disappears, as commented at the end of Example 4. The point is that $\mathcal{A}$ can be regarded as the category, usually denoted $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D})$, of pp-pairs and pp-definable maps for $\mathcal{D}$, see [23, 22.2]. These sorts - the objects of $\mathcal{A}$ - are given by pairs $\phi(\overline{x})/\psi(\overline{x})$ of pp formulas (in the usual sense) and we then identify a pp formula $\phi(\overline{x})$ with the pp-pair $\phi(\overline{x})/(\overline{x} = 0)$. So to say that an element $a : (A, -) \to M$ of M of sort A satisfies a formula ϕ, written $M \models \phi(a)$ or $a \in \phi(M)$, is to say that a factors through the map $(A, -) \to ((\phi(x)/(x = 0), -)$ induced by the canonical inclusion of $\phi(x)/(x = 0)$ into A (that inclusion exists since we are assuming that ϕ is a formula which applies to elements of sort A, that is, whose free variable is of sort A).

We haven't insisted that ρ be monic. To do so would give a more direct translation of the usual notion of a formula of a given sort but we haven't seen a strong reason to make the restriction.

It might be noted that the above definition yields only divisibility formulas; to see why that is enough, that is, why every pp formula is equivalent in the language based on $\mathcal{A}$ to a divisibility formula, see Example 5 below.

Example 5

Let $\mathcal{D} = \mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ and regard $\mathcal{D}$ as a subcategory of $\mathcal{A}$-Mod. To show that $\mathcal{D}$ is a definable subcategory, we must show how to express the condition of exactness, so suppose that $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence in $\mathcal{A}$. The requirement that $D \in \mathcal{D}$ be exact at, say, B is the condition that, for every $b : (B, -) \to D$ with $b(g, -) = 0$, there exist $a : (A, -) \to D$ with $b = a(f, -)$. This is an implication between pp formulas, namely

$$\forall x_B \left((x_B(g, -) = 0) \to (\exists x_A\, (x_B = x_A(f, -))) \right),$$

where subscripts to variables show their sorts, and hence $\mathcal{D}$ is definable by Theorem 2 below.

We expressed the pp formulas above in the usual way, so how are these reformulated following the definition above? The exact sequence in $\mathcal{A}$ gives the (non-exact) sequence $0 \to (C, -) \xrightarrow{(g,-)} (B, -) \xrightarrow{(f,-)} (A, -) \to 0$ in $\mathcal{A}$-Mod. The pp formula $\exists x_A\, (x_B = x_A(f, -))$ is the requirement that a morphism from $(B, -)$ to an $\mathcal{A}$-module factor through $(f, -)$ so, according to the above definition, is precisely the morphism $f : A \to B$ in $\mathcal{A}$. The other pp formula $x_B(g, -) = 0$ is not of the same form but, because the objects of $\mathcal{D}$ are exact on $\mathcal{A}$, it is equivalent, *on* $\mathcal{D}$, to the requirement that x_B factor through $(f, -)$.

Pure embeddings

We say that a morphism $f : M \to N$ in $\mathcal{D}$ is a **pure embedding** if, whenever an element $a : (A, -) \to M$ is such that $fa : (A, -) \to N$ satisfies a pp formula $\rho : B \to A$, then already a satisfies that formula. That is, if fa factors through $(\rho, -) : (A, -) \to (B, -)$, then already a factors through $(\rho, -)$.

In a module category, any pure embedding between finitely presented modules is split; in fact, in any finitely presentable category, the pure embeddings are the directed colimits of split embeddings [1, 2.30].

Pp-types

The **pp-type** of an element $a \in M$ of sort A, that is, $a : (A, -) \to M$, is

$$\mathrm{pp}^M(a) = \{\rho : B \to A \text{ in } \mathcal{A} : a \text{ factors through } (\rho, -) : (A, -) \to (B, -)\}.$$

Note that if ρ and ρ' are in this set, then a will factor through the pushout of $(\rho, -)$ and $(\rho', -)$ which, since $\mathcal{A}$ is abelian, is $(\rho'', -)$ for $\rho'' : B'' \to A$ the pullback of ρ and ρ' in $\mathcal{A}$, hence $\mathrm{pp}^M(a)$ is closed under pullback. Also, if $\rho : B \to A$ factors through $\rho' : B' \to A$, and if $\rho \in \mathrm{pp}^M(a)$, then $\rho' \in \mathrm{pp}^M(a)$. Thus, a pp-type, regarded, via $\rho \mapsto (\rho, -)$, as a set of maps in $\mathcal{A}^{\mathrm{op}}$ which is embedded in $\mathrm{Lex}(\mathcal{A}, \mathbf{Ab})$, is a consistent collection of factorisation requirements on a map, or element, a.

If $A \in \mathcal{A}$ then a **pp-type** of sort A is a filter in the slice category $\mathcal{A}/A$ where this is pre-ordered by $\rho \leq \rho'$ iff ρ factors through ρ'. It is the case, see [22, 4.1.4, also 3.3.6], that every such, abstractly defined, pp-type is **realised**, that is, is of the form $\mathrm{pp}^M(a)$ for some element a of sort A of some object M in $\mathcal{D}$.

A pp-type, of sort A, is **neg-isolated** by a pp formula ψ (also of sort A) if p is maximal, among pp-types of sort A, with respect to not containing ψ.

Explanation

The fact that pp-types are closed under pullback is essentially the fact that they (in the usual definition) are closed under intersection, and the other part of pp-types being a filter is the usual closure under implication.

Formulas and types with parameters

We also consider pp formulas and pp-types with parameters (i.e. formulas that name elements of a module). We say that an element $a : (A, -) \to M$ of M of sort A **satisfies** the formula B **with parameter** $c : (C, -) \to M$ if there is a morphism $\rho : B \oplus C \to A$ such that a factors through $(\rho, -) : (A, -) \to (B, -) \oplus (C, -)$ along a morphism of the form $(b : (B, -) \to M) \oplus c$. We identify the data (B, c) as a **formula with parameters**.

A **pp-type with parameters** is a set p of pp formulas all of the same sort, A say, with all parameters from some $M \in \mathcal{D}$ and such that any finite subset of p is realised in M (equivalently, every formula in the closure of p under pullback is satisfied in M). If M is purely embedded in $N \in \mathcal{D}$, then we say that $a : (A, -) \to N$ **realises** p if a satisfies every formula in p.

Explanation

As with the usual route to defining formulas with parameters, we may consider the formula $B \oplus C$ - so with a free variable of sort B and one of sort C - then we fix the value of the C-variable to be a specific element of sort C in a specific object of $\mathcal{D}$.

Definable subcategories again

A definable subcategory of a module category is any axiomatisable additive subcategory of a module category, Theorem 1, but we can say what the defining axioms look like.

Theorem 2 *(see [22, 3.4.7]) A subcategory $\mathcal{D}$ of* Mod-R *is a definable subcategory iff $\mathcal{D}$ can be defined by a set of axioms of the form $\forall\overline{x}(\phi(\overline{x} \to \psi(\overline{x}))$ where ϕ and ψ are pp formulas for R-modules. Definable subcategories of $\mathcal{D}$ can be cut out by adding further axioms of this form.*

In case $\mathcal{D}$ is definably embedded as $\mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ *in the category* $\mathrm{Lex}(\mathcal{A}, \mathbf{Ab})$ *or in the category of $\mathcal{A}$-modules (so the axioms defining $\mathcal{D}$ are those expressing exactness - see Example 5 above), then definable subcategories of $\mathcal{D}$ are obtained by adding axioms of the form $(A, -) = 0$ for some objects A in $\mathcal{A}$.*

Example 6

The definable subcategory $\mathcal{DIV}$ of divisible abelian groups is defined within **Ab** by axioms of the form $\forall x\,(x = x \to (\exists y\,(ny = x)))$ for $n \geq 2$. Definable subcategories of $\mathcal{D}$ are obtained by adding further axioms of the form $\forall x\,(xp = 0 \to x = 0)$ for p from some set of primes. If we regard $\mathcal{DIV}$ as definably embedded as $\mathrm{Ex}(\mathbb{Z}\text{-mod}^{\mathrm{op}}, \mathbf{Ab})$, then definable subcategories of $\mathcal{DIV}$ are obtained by adding axioms of the form $(\mathbb{Z}_p, -) = 0$.

Pure-injectives

An object $M \in \mathcal{D}$ is **pure-injective** if every pp-type with parameters from M is already realised in M. Equivalently, see [22, 4.3.11], M is pure-injective iff it is injective over pure embeddings (in $\mathcal{D}$ or, by the fact of the equivalence, in any category containing $\mathcal{D}$ as a definable subcategory).

There is an approach to pure-injectivity which uses that there is an embedding, given by $M \to (-\otimes_R M)$ on objects, of R-Mod into the category (mod-R, **Ab**) of left modules over the finitely presented right R-modules. This induces an equivalence between the pure-injectives in the first category and the injectives of the second. For more on this, see [11, 7.12], [22, 12.1.6].

If M is any module then its **pure-injective hull** is the minimal pure-injective into which M purely embeds. This exists (for instance via the embedding appearing in the above paragraph and the existence of injective hulls in Grothedieck categories) and is unique up to isomorphism over M. For example, the pure-injective hull of the localisation $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at a prime p is the p-adic integers $\overline{\mathbb{Z}_{(p)}}$ and, in general, the pure-injective hull of a module is obtained by adding realisations of pp-types with parameters from M.

Every neg-isolated pp-type is realised in an indecomposable pure-injective, see [22, 4.3.52, 4.3.49], in which case we refer to that indecomposable pure-injective as being **neg-isolated**. In the above-mentioned comparison between pure-injectives and injectives in the associated functor category this condition on the corresponding indecomposable injective is that it be the injective hull of a simple object (see [22, 5.3.45]).

Example 7

Divisible abelian groups are injective, so every object of $\mathcal{DIV}$ is pure-injective. The indecomposables are the Prüfer groups $\mathbb{Z}_{p^\infty}$ and the group of rationals $\mathbb{Q}$.

Each Prüfer group is neg-isolated in $\mathcal{DIV}$: to see, at least plausibly, that this is the case, consider the pp-type in $\mathbb{Z}_{p^\infty}$ of an element a of order p. That pp-type contains all divisibility formulas $n|x$ for $n \geq 2$ plus the formula $px = 0$ and there is no more information that can be added to a pp-description of an element other than the formula $x = 0$. So the pp-type of a in $\mathbb{Z}_{p^\infty}$ is neg-isolated by the formula $x = 0$. An algebraic argument is to use the criterion, see [22, 5.3.48], which is that an indecomposable pure-injective N is neg-isolated in a definable category $\mathcal{D}$ iff, whenever N is a direct summand of a direct product $\prod_i N_i$ of pure-injectives in $\mathcal{D}$, it must be a direct summand of one of the N_i.

On the other hand, $\mathbb{Q}$ is, by that criterion, not neg-isolated in $\mathcal{D}$. (Of course $\mathbb{Q}$ is neg-isolated in the definable subcategory of torsionfree divisible abelian groups.)

Example 8

The dual (in the sense of elementary duality, [9], see [22, §3.4.2]) definable category to $\mathcal{DIV}$ is the category $\mathcal{TF}$ of torsionfree = flat abelian groups. This has, for its indecomposable pure-injectives, the p-adic integers $\overline{\mathbb{Z}_{(p)}}$ for p prime, together with $\mathbb{Q}$. Each of these is neg-isolated in $\mathcal{TF}$.

2.2 The Ziegler spectrum

Theorem 3 *([32, Cor. 4 to Thm. 4], see [22, 4.3.21]; [35, 4.8], see [22, 5.1.4]) If $\mathcal{D}$ is a definable subcategory of a module category, then the pure-injective hull of every module in $\mathcal{D}$ is contained in $\mathcal{D}$.*

Furthermore, $\mathcal{D}$ is generated, as a definable subcategory, by the indecomposable pure-injectives in $\mathcal{D}$.

It is the case, see [22, 4.3.38], that there is, up to isomorphism, just a set of indecomposable pure-injectives in any definable category $\mathcal{D}$; let $\mathrm{pinj}(\mathcal{D})$ denote a set of representatives. The **Ziegler spectrum** of $\mathcal{D}$, $\mathrm{Zg}(\mathcal{D})$, is the set $\mathrm{pinj}(\mathcal{D})$ topologised with a basis of open sets of the form

$$(\phi/\psi) = \{N \in \mathrm{pinj}(\mathcal{D}) : \phi(N)/\psi(N) \neq 0\}.$$

Theorem 4 *If $\mathcal{D}$ is a definable category, then there is a natural bijection between the definable subcategories of $\mathcal{D}$ and the closed subsets of its Ziegler spectrum $\mathrm{Zg}(\mathcal{D})$, given by:*
$\mathcal{D}' \mapsto \mathcal{D}' \cap \mathrm{pinj}(\mathcal{D})$ when $\mathcal{D}'$ is a definable subcategory of $\mathcal{D}$ and

$X \mapsto \langle X \rangle$ when X is a closed subset of $\mathrm{Zg}(\mathcal{D})$, where $\langle - \rangle$ denotes the definable subcategory generated by X, that is, the smallest definable subcategory of $\mathcal{D}$ which contains X.

Example 9

The indecomposable pure-injectives in $\mathcal{DIV}$ are described in Example 7. Each Prüfer group $\mathbb{Z}_{p^\infty}$ is an isolated = open point of $\mathrm{Zg}(\mathcal{DIV})$ since this is the only indecomposable pure-injective which contains a non-zero element of order p, that is, the open set $(px = 0/x = 0)$ contains just $\mathbb{Z}_{p^\infty}$. On the other hand, it can be checked, see Example 10 below, that $\mathbb{Q}$ is in the closure of each Prüfer group, so every nonempty closed set contains $\mathbb{Q}$. So the nonempty closed sets of $\mathrm{Zg}(\mathcal{DIV})$ are of the form $X \cup \{\mathbb{Q}\}$ where X is any, possibly empty, set of Prüfer groups. Hence each definable subcategory of $\mathcal{DIV}$ consists of the groups of the form $\bigoplus_{p \in X} \mathbb{Z}_{p^\infty}^{(\kappa_p)} \oplus \mathbb{Q}^{(\kappa_0)}$ where the κ_p can be any cardinals ≥ 0, for some such X.

Frames

A **complete Heyting algebra**, or **frame**, is a complete lattice in which meet distributes over infinite join. If T is a topological space, then the lattice $\mathcal{O}(T)$ of open subsets of T is a frame and any frame of this form is said to be **spatial**. A continuous map $T \to T'$ of topological spaces induces a morphism $\mathcal{O}(T') \to \mathcal{O}(T)$ in the category of complete Heyting algebras. The category of frames is the opposite of that category, so has the same objects but the morphisms ‘go in the same direction as continuous maps’ (see [12, Chpt. II, §1.1]). Elementary duality, induces an isomorphism of frames between the right and left Ziegler spectrum of any ring R, see [22, §5.4]. In all known cases, this is induced by a homeomorphism between these spaces but, at least currently, for general rings we have only a “homeomorphism at the level of topology", that is, an isomorphism of frames.

2.3 Ultraproducts

This applies to any type of first-order structure but we will describe reduced products and ultraproducts just for R-modules, where R is a, possibly many-sorted, ring.

A **filter** on a set I is a set, $\mathcal{F}$, of subsets of I such that $I \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$, $J, K \in \mathcal{F}$ implies $J \cap K \in \mathcal{F}$, and $J \in \mathcal{F}$ and $J \subseteq J' \subseteq I$ implies $J' \in \mathcal{F}$. An **ultrafilter** on I is a filter $\mathcal{U}$ which is maximal with respect to set-inclusion among filters on I, equivalently which satisfies the property that, for every $J \subseteq I$, either $J \in \mathcal{U}$ or $I \setminus J \in \mathcal{U}$.

Suppose that $(M_i)_{i \in I}$ is an I-indexed set of R-modules and let $\mathcal{F}$ be a filter on I. Note that the products $\prod_{i \in J} M_i$ for $J \in \mathcal{F}$, together with the natural projection maps $\pi_{JK} : \prod_{i\in J} M_i \to \prod_{i\in K} M_i$ for $J \supseteq K$, $J, K \in \mathcal{F}$, form a directed system of modules. The **reduced product** $\prod_i M_i/\mathcal{F}$ is the directed colimit of that directed system. If $\mathcal{F}$ is an ultrafilter, then we refer to this as an **ultraproduct**. If all $M_i = M$ then we use the terms **reduced power** and **ultrapower** and write $M^I/\mathcal{F}$.

Remark 5 Every definable subcategory is closed under direct products and directed colimits, hence under reduced products.

There is an elementwise description of reduced powers as follows. The elements, of any given sort, of a reduced product $\prod_i M_i/\mathcal{F}$ are of the form $(a_i)_i/\sim$ where a_i is an element, of that sort, of M_i, and the equivalence relation $\sim$ on the elements of $\prod_{i\in I} M_i$ of that sort is given by $(a_i)_i \sim (b_i)_i$ iff $\{i \in I : a_i = b_i\} \in \mathcal{F}$. That is, two elements, of the same sort, in the full product are equivalent iff they agree on a large set of coordinates where “large" means “in $\mathcal{F}$". That this is an equivalence relation follows from the definition of filter. So the reduced product is a quotient structure of $\prod_{i\in I} M_i$. The R-module structure on this quotient (rather collection of quotients, one for each sort) is defined in the obvious way, pointwise, and can be checked to be well-defined and to give the same structure as the directed-colimit definition.

Theorem 6 *(Łos' Theorem) Suppose that M_i, $i \in I$ are R-modules and that $\mathcal{F}$ is a filter on I. Set $M^* = \prod_i M_i/\mathcal{F}$.*
(a) If $\phi(\overline{x})$ is a pp formula and $\overline{a} = (a^1, \dots, a^n) \in (M^*)^n$, with $a^k = (a_i^k)_i/\sim$,[5] then $M^* \models \phi(\overline{a})$ iff $\{i \in I : M_i \models \phi(a_i^1, \dots, a_i^n)\} \in \mathcal{F}$.
(b) Suppose that $\mathcal{F}$ is an ultrafilter. Then, if $\sigma(\overline{x})$ is any formula for R-modules and $\overline{a} = (a^1, \dots, a^n) \in (M^*)^n$, with $a^k = (a_i^k)_i/\sim$, then $M^* \models \sigma(\overline{a})$ iff $\{i \in I : M_i \models \sigma(a_i^1, \dots, a_i^n)\} \in \mathcal{F}$.

That is, a pp formula, possibly with parameters, is true in the reduced product iff it is true on a “large" set of coordinates, and the same is true for general formulas - these may include “or",“not" and universal quantifiers $\forall$ - if we have an ultraproduct.

Example 10

Take M to be the Prüfer group $\mathbb{Z}_{p^\infty}$, take I to be the set of integers $n \geq 1$ and let $\mathcal{U}$ be any ultrafilter (existence by Zorn's Lemma) which contains the filter of cofinite subsets of I. Let M^* be the corresponding ultrapower $M^I/\mathcal{U}$. For each $i \geq 1$ choose an element $a_i \in \mathbb{Z}_{p^\infty}$ of order p^i. Consider the element $a = (a_1, a_2, \dots, a_i, \dots)/\sim$. Then a has infinite order since, given any n, the set of its coordinates i which satisfy $p^n a_i = 0$ is finite, so the complementary set of coordinates is in $\mathcal{U}$ and we have $p^n a \neq 0$ by Łos' theorem. Also by Łos' Theorem, M^* is divisible (formulas include **sentences** - formulas which have no free variables - such as $\forall x\, n|x$), so we deduce that M^* splits off a(t least one) copy of $\mathbb{Q}$. It follows then, by Remark 5, that $\mathbb{Q}$ is in the definable subcategory generated by $\mathbb{Z}_{p^\infty}$.

[5] Note that the equivalence relation $\sim$ might vary with k since the elements a^k might belong to different sorts.

2.4 Purity in definable categories

There are many equivalent ways of defining purity (pure embeddings, pure-exact sequences,...) in a module category (see e.g. [22, §§2.1.1, 2.1.3] and this purity restricts to any definable subcategory because, if M belongs to a definable subcategory $\mathcal{D}$ and M' is a pure submodule, then both M' and M/M' are in $\mathcal{D}$ (e.g. [22, 2.1.17]). Every definable category embeds as a definable subcategory of many module categories, so one may ask whether these induced purities on a given definable category agree. Indeed, they do agree, because there is an intrinsic definition of purity in definable categories which agrees with any such induced purity. Namely, we say that a sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in a definable category $\mathcal{D}$ is **pure exact** if some ultrapower $0 \to A^I/\mathcal{U} \xrightarrow{f^I/\mathcal{U}} B^I/\mathcal{U} \xrightarrow{g^I/\mathcal{U}} C^I/\mathcal{U} \to 0$ is split exact, in which case f is said to be a **pure embedding** and g a **pure epimorphism**.

This works because, from the perspective of $\mathcal{D}$ being a definable subcategory of a module category, if we have any exact sequence whose terms lie in $\mathcal{D}$ and which is pure in the module category, then any ultrapower of that sequence is again pure and it is a theorem from model theory, see [22, §4.2.5] for outline and references, that there is a choice of I and $\mathcal{U}$ which is such that every module of the form $M^I/\mathcal{U}$ is pure-injective, making the corresponding ultrapower sequence split.

2.5 The Downwards Löwenheim-Skolem Theorem

The Downwards Löwenheim-Skolem Theorem, which we will state as Theorem 7, just for modules, says that any first-order structure is a directed union of small elementary substructures, where “small" means of cardinality bounded by the size of the language for that structure. If R is a ring, possibly with many objects, then, by the cardinality $|R|$ of R we mean the cardinality of the set of elements, that is, morphisms if we regard R as a small preadditive category, in R. If $\mathcal{A}$ is a skeletally small abelian category and $\mathcal{D} = \mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$, then we can replace $\mathcal{A}$ by any small equivalent subcategory, for instance by a skeletal version of $\mathcal{A}$ and a suitable language for $\mathcal{D}$ would be based on that ring with many objects.

Since we're taking the view that a module can be regarded as a module over many rings, there is some further ambiguity as to what is meant by the cardinality of a module. Let us say that the cardinality of an R-module M is the cardinality of the union, over all objects A of R, of the set $M(A) = ((A, -), M)$ of elements of M of sort A: $|M| = \left|\bigcup_{A \in R} ((A, -), M)\right|$. So, if R is 1-sorted, then $|M|$ has the usual meaning (the cardinality of the underlying set of M). If we regard M as a(n exact) $\mathrm{fun}(M)$-module then, unless M is of cardinality $< |R| + \aleph_0$, the cardinality of M will not have changed because all the extra sorts of M are (definable) sections of some finite power M^n of M and the number of such sorts is no more than $|R| + \aleph_0$. So, in fact, there is little ambiguity in the meaning of $|M|$.

We say that a submodule M_0 of a module M is an **elementary submodule** if, for every tuple $\overline{a}$ of elements of M and every formula σ with free variables matching $\overline{a}$ in number and sorts, we have $M \models \sigma(\overline{a})$ iff $M_0 \models \sigma(\overline{a})$. In particular, since this applies to pp formulas, an elementary submodule is a pure submodule.

Theorem 7 *Suppose that M is an R-module and that A is a subset of M. Then there is an elementary submodule M_0 of M, containing A, of cardinality no more than* $\max(|A|, |R|, \aleph_0)$.

Elementary submodules are pure submodules, so we have the following corollary.

Corollary 8 *If $\mathcal{D}$ is a definable subcategory of* Mod-R*, then every object D in $\mathcal{D}$ is a directed union $D = \bigcup_i D_i$ of elementary submodules with each $D_i \in \mathcal{D}$ and* $|D_i| \leq |R| + \aleph_0$.

2.6 Definable and elementary equivalence

Two R-modules, M and N, are **elementarily equivalent** if they satisfy the same sentences in the language of R-modules.

Remark 9 By Łos' Theorem, every module is elementarily equivalent to each of its ultrapowers.

The next result, basic in the model theory of modules, follows directly from Baur's pp-elimination of quantifiers for modules [4] and is given explicitly by Garavaglia [7, Thm. 2].

Theorem 10 *(e.g. [20, 2.18]) Modules M and N are elementarily equivalent iff for every pp-pair ϕ/ψ, the cardinality of $\phi(M)/\psi(M)$ is equal to that of $\phi(N)/\psi(N)$ or both are infinite.*

A somewhat weaker condition is that they are **definably equivalent**, meaning that they generate the same definable subcategory of Mod-R: $\langle M \rangle = \langle N \rangle$. In fact, this is only slightly weaker than M and N being elementarily equivalent. The next result follows by Theorem 2 and, e.g., [20, 2.23].

Corollary 11 *Modules M and N are definably equivalent if and only if $M^{(\aleph_0)}$ and $N^{(\aleph_0)}$ are elementarily equivalent, equivalently if every pp-pair open on M is open on N and vice versa.*

Thus elementary equivalence of M and N adds, to definable equivalence, the requirement that, for each pp-pair ϕ/ψ, the cardinalities of $\phi(M)/\psi(M)$ and $\phi(N)/\psi(N)$ are equal if one of them is finite. However, for most algebraic considerations, the important information is whether the pair is open or closed, not the exact size of that factor group[6].

[6] The condition "$T = T^{\aleph_0}$" which appears in the hypotheses of many results in [20] says that we ignore the exact sizes and look only at which pp-pairs are open and which are closed.

Corollary 12 *If every pp-definable quotient $\phi(M)/\psi(M)$ is infinite if nonzero, then M is elementarily equivalent to each of its powers M^I and copowers $M^{(I)}$.*

If R is an algebra over an infinite field K, then all quotients $\phi(M)/\psi(M)$, being vector spaces over are infinite if nonzero, so we have the following.

Corollary 13 *If R is an algebra over an infinite field, then definable equivalence and elementary equivalence for R-modules coincide.*

Corollary 14 *([32, Cor. 2 to Thm. 4]) For any module M and index set I, $M^{(I)}$ and M^I are elementarily equivalent, and hence definably equivalent.*

Although we state the next theorem for modules it holds true for arbitrary kinds of first-order structures.

Theorem 15 *(Keisler-Shelah Theorem, see, e.g. [10, 9.5.7]) Two R-modules M, N are elementarily equivalent iff they have isomorphic ultrapowers: $M^I/\mathcal{U} \simeq N^J/\mathcal{V}$ for some index sets I, J and ultrafilters $\mathcal{U}$, $\mathcal{V}$ on I, respectively J.*

In fact one may take $I = J$ and $\mathcal{U} = \mathcal{V}$.

Corollary 16 *Two R-modules M, N are elementarily equivalent iff one is an elementary submodule of an ultrapower of the other.*

That follows from Theorem 15 and Remark 9.

Corollary 17 *Two R-modules M, N are definably equivalent iff $M^{(\aleph_0)}$ and $N^{(\aleph_0)}$ have isomorphic ultrapowers.*

Corollary 18 *The definable subcategory $\langle M \rangle$ generated by a module M consists of the pure submodules of ultrapowers of $M^{(\aleph_0)}$, equivalently of ultrapowers of $M^{\aleph_0}$, equivalently of ultraproducts of the various finite powers M^n ($n = 1, 2, \ldots$) of M.*

3 Large and small objects in definable categories

In a module category, every module is the directed colimit of finitely presented objects and every module can be embedded in a direct power of a (fixed) large enough injective module. There are, for definable categories, analogues of these facts.

3.1 Small objects - (strongly) atomic modules

For every definable category $\mathcal{D}$ there is a cardinal κ such that every object is a directed colimit of $< \kappa$-presented objects. To see that, suppose that $\mathcal{D}$ is a definable subcategory of R-Mod and set κ to be the cardinality of R if R is infinite and $\kappa = \aleph_0$

if R is finite. Then, by Corollary 8 every module in $\mathcal{D}$ is a directed colimit of submodules which are also in $\mathcal{D}$. It follows that $\mathcal{D}$ is κ-accessible in the sense of [1], though κ might not be minimal such that $\mathcal{D}$ is κ-accessible, see Example 11 below. So every definable category is an additive accessible category with products.

Example 11

Take $\mathcal{A}$ to be an abelian category consisting of a very large set of non-isomorphic simple objects, all with endomorphism ring some field K, that is, $\mathcal{A}$ is the K-path category of some very large discrete quiver. An (exact) $\mathcal{A}$-module is just a choice of a K-vector space for each object of $\mathcal{A}$; such a module is the directed union of submodules which are small (of cardinality $\leq |K| + \aleph_0$), so $\mathcal{D}$ in this case is $|K| + \aleph_0$-accessible.

Some definable categories $\mathcal{D}$, for example module categories, are finitely accessible, that is, every object is a directed colimit of objects from C^{fp} - the full subcategory of finitely presented objects - which is also required to be skeletally small. Recall that an object A in a category C is **finitely presented** if the hom functor $(A, -)$ commutes with directed colimits in C. But, for instance, in the definable category of divisible abelian groups the zero module is the only finitely presented object, see [22, 18.1.1]. We may ask whether a definable category has a $\varinjlim$-generating set consisting of objects which are more analogous to finitely presented objects than those given by the Downwards Löwenheim-Skolem Theorem. We do, in fact, have the (strictly) atomic objects, which share the following key property with finitely presented objects.

Proposition 19 *([22, 1.2.6, 1.2.7]) If A is a finitely presented R-module, then every pp-type realised in A is finitely generated. Moreover, if $\overline{a}$ is any finite tuple from A, with $\mathrm{pp}^A(\overline{a})$ being generated by, say, ϕ, then, if M is any module and $\overline{b} \in \phi(M)$, then there is a morphism $f : A \to M$ with $f\overline{a} = \overline{b}$.*

These properties don't characterise the finitely presented modules. In fact, the first characterises the Mittag-Leffler R-modules, [31, 2.2] and the stronger property characterises the strictly Mittag-Leffler modules, see [28, 4.1]. But they represent what we do have in definable categories in the possible absence of finitely presented objects - see Corollary 21 below.

Suppose that $\mathcal{D}$ is a definable category. We say that $M \in \mathcal{D}$ is **$\mathcal{D}$-atomic**, if every pp-type realised in M is $\mathcal{D}$-finitely generated, that is, for every finite tuple $\overline{a} = (a_1, \dots, a_n)$ from M, there is a pp formula ϕ (in whichever language for $\mathcal{D}$ that we are using), with free variables $x_1, \dots, x_n$ (say), which **$\mathcal{D}$-generates** $\mathrm{pp}^M(\overline{a})$ in the sense that $\phi \in \mathrm{pp}^M(\overline{a})$ and, for every pp formula $\psi \in \mathrm{pp}^M(\overline{a})$, we have $\phi \leq_{\mathcal{D}} \psi$, meaning that $\phi(D) \leq \psi(D)$ for every $D \in \mathcal{D}$.

The, stronger, **strictly $\mathcal{D}$-atomic** condition on $M \in \mathcal{D}$ is that M is $\mathcal{D}$-atomic and, for every tuple $\overline{a}$ from M, with pp-type $\mathcal{D}$-generated by, say, ϕ, and for every $D \in \mathcal{D}$ and $\overline{b} \in \phi(D)$, there is a morphism $f : M \to D$ with $f\overline{a} = \overline{b}$.

Example 12

Consider the category $\mathcal{DIV}$ of divisible abelian groups as a definable subcategory of **Ab**. The object $\mathbb{Q}$ as an object of $\mathcal{DIV}$ has the propery that the pp-type of every finite tuple in it is generated *modulo (the theory of)* $\mathcal{DIV}$ by some pp formula. For instance, $1 \in \mathbb{Q}$ is, within that category, a free realisation of the single pp formula $x = x$ in the sense that, given any divisible abelian group D and element $d \in D$ realising $x = x$ (that is, given any element $d \in D$), there is a morphism $\mathbb{Q} \to D$ taking 1 to d (see [22, §1.2.2] for free realisations of pp formulas). Similarly it is easy to see that each Prüfer group is strictly $\mathcal{DIV}$-atomic (and clearly they and their finite direct sums form a $\varinjlim$-generating set for $\mathcal{DIV}$). They are, respectively, $\mathcal{DIV}$-(pre)envelopes of $\mathbb{Z}$ and the $\mathbb{Z}_{(p)}$ in the sense defined below.

The theorem and corollary that follow are special cases of a very general theorem of Makkai, for which see [18, 4.3, 4.4]. For this formulation see [28, 4.10].

Theorem 20 *If $\mathcal{D}$ is a definable subcategory of* Mod-R *and A is a finitely presented R-module, then there is a $\mathcal{D}$-preenvelope $A \to D_A$ of A with D_A strictly $\mathcal{D}$-atomic. Every module in $\mathcal{D}$ is a directed colimit of modules of the form D_A with $A \in$* mod-R.

To say that $f : A \to D_A$ is a $\mathcal{D}$-**preenvelope** of A is to say that every morphism from A to a module in $\mathcal{D}$ factors (not necessarily uniquely) through f.

Corollary 21 *If $\mathcal{D}$ is a definable category then $\mathcal{D}$ contains a $\varinjlim$-generating set of strictly $\mathcal{D}$-atomic objects.*

So every definable category has a $\varinjlim$-generating set of objects which are the analogues of strictly Mittag-Leffler modules in a module category and these objects share some key properties with finitely presented modules.

3.2 Large objects - elementary cogenerators

We also have, in any definable category, objects which are somewhat analogous to injective cogenerators in module categories. Recall that an object E in a category C is an **injective cogenerator** for C if E is injective in C and if every object has an embedding into a direct product E^I of copies of E. Every module category Mod-R has an injective cogenerator, indeed many injective cogenerators, a minimal choice being the injective hull $E(\bigoplus_S E(S))$ of the direct sum of one copy of the injective hull $E(S)$ for each simple module S. The corresponding notion in a definable category is that of an elementary cogenerator.

If $\mathcal{D}$ is a definable subcategory of R-Mod, then an **elementary cogenerator for** $\mathcal{D}$ is a pure-injective module $N \in \mathcal{D}$ such that the objects of $\mathcal{D}$ are exactly the R-modules which purely embed in some direct power N^I of N. Since $\mathcal{D}$ is closed in R-Mod under pure submodules we can say this independently of the embedding of $\mathcal{D}$ as a definable subcategory: $N \in \mathcal{D}$ is an elementary cogenerator for $\mathcal{D}$ iff every object of $\mathcal{D}$ purely embeds in a power of N. The R-modules which arise as elementary cogenerators (of some definable subcategory) are characterised in Proposition 23 below.

Contrast this with the fact, stated earlier, that the definable subcategory $\langle M \rangle$ generated by a module M can be obtained, Corollary 18 as the class of pure submodules of ultraproducts of finite powers M^n of M (or, by Corollary 12, just of M if every pp-pair open on M is infinite).

Theorem 22 *(see [22, 5.3.52, 5.3.50]) Every definable category has an elementary cogenerator. A minimal choice is the pure-injective hull of the direct sum of one copy of each neg-isolated pure-injective. So a pure-injective $N \in \mathcal{D}$ is an elementary cogenerator for $\mathcal{D}$ iff it has, as a direct summand, at least one copy of each neg-isolated pure-injective in $\mathcal{D}$.*

Example 13

By Theorem 22 and Example 7 a minimal elementary cogenerator for $\mathcal{DIV}$ is the direct sum of one copy of each Prüfer group. In contrast, by Example 8, a minimal elementary cogenerator for $\mathcal{TF}$ is $H(\bigoplus_p \overline{\mathbb{Z}_{(p)}}) \oplus \mathbb{Q}$ where $H(-)$ denotes pure-injective hull.

We will say that a pure-injective module N is an **elementary cogenerator** if every ultrapower of N purely embeds in a direct power of N.

Proposition 23 *A pure-injective module N is an elementary cogenerator iff it is an elementary cogenerator of the definable category $\langle N \rangle$ that it generates.*

Proof Since every definable subcategory is closed under ultraproducts, we have the direction ($\Leftarrow$).

For the other direction we consider the set of (isomorphism classes of) indecomposable pure-injective modules which are direct summands of modules elementarily equivalent to N (equivalently, Corollary 16, direct summands of ultrapowers of N). This is exactly the closed subset of $\mathrm{Zg}(\mathcal{D})$ corresponding, in the sense of Theorem 4, to the definable subcategory $\langle N \rangle$. In [35] this is denoted $U(N)$ and in [20, p. 87] it is denoted $\mathcal{I}(N)$, but here we will use the perhaps more suggestive notation $\mathrm{supp}(N)$ of [22].

We use the fact, see [22, 5.3.53], that every object in the definable subcategory generated by a module N is a pure submodule of a direct product of modules in $\mathrm{supp}(N)$.

By definition, each of the modules in $\mathrm{supp}(N)$ is a direct summand of a module elementarily equivalent to N; each such module is, Corollary 16, pure in an ultrapower of N and, by assumption on N, each such ultrapower is pure in a direct power of N. Therefore every module in $\langle N \rangle$ is pure in a direct power of N, as required. □

Thus we have a characterisation of the modules which occur as elementary cogenerators of definable subcategories. From Corollary 14 and the fact that $\mathrm{supp}(M) = \mathrm{supp}(M^{(\aleph_0)})$ [20, 4.39], we have the following.

Corollary 24 *N is an elementary cogenerator iff $N^{\aleph_0}$ is an elementary cogenerator iff the pure-injective hull of $N^{(\aleph_0)}$ is an elementary cogenerator. All three of these modules are definably equivalent.*

Note that there is a natural bijection between definable-equivalence classes of R-modules N which are elementary cogenerators and definable subcategories of R-Mod. Furthermore, the relation of inclusion $\mathcal{D}_1 \subseteq \mathcal{D}_2$ between definable subcategories $\mathcal{D}_1$ and $\mathcal{D}_2$ is equivalent to N_1 being a direct summand of some power of N_2, where N_i is an(y) elementary cogenerator of $\mathcal{D}_i$. On the other hand, the lattice operations on the set of definable subcategories of R-Mod are not very well-reflected by elementary cogenerators.

Example 14

If $\mathcal{D}_1, \mathcal{D}_2$ are definable subcategories of R-Mod then so is their intersection but an elementary cogenerator for $\mathcal{D}_1 \cap \mathcal{D}_2$ cannot be manufactured from elementary cogenerators N_i for the $\mathcal{D}_i$. For instance, take $R = \mathbb{Z}$, $\mathcal{D}_1 = \langle \mathbb{Z}_{2^\infty} \rangle$ and $\mathcal{D}_2 = \langle \mathbb{Z}_{3^\infty} \rangle$. Both $\mathbb{Z}_{2^\infty}$ and $\mathbb{Z}_{2^\infty}$ are elementary cogenerators but they have no direct summands in common so do not give an elementary cogenerator for $\mathcal{D}_1 \cap \mathcal{D}_2 = \langle \mathbb{Q} \rangle$. Of course, if the elementary cogenerators N_i have every indecomposable pure-injective in $\mathcal{D}_i$ occurring as a direct summand, then the direct product of their common direct summands will be an elementary cogenerator for $\mathcal{D}_1 \cap \mathcal{D}_2$.

On the other hand, finite join of definable subcategories is better.

Lemma 25 *Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are definable subcategories of R-*Mod *and that N_i is an elementary cogenerator for $\mathcal{D}_i$. Then $N_1 \oplus N_2$ is an elementary cogenerator for the definable subcategory generated by $\mathcal{D}_1 \cup \mathcal{D}_2$.*

Proof By [22, 3.4.9] $\langle \mathcal{D}_1 \cup \mathcal{D}_2 \rangle$ consists of the pure submodules of modules of the form $M_1 \oplus M_2$ with $M_i \in \mathcal{D}_i$, so this follows. □

Example 15

That doesn't work for infinite joins of definable subcategories. Take $R = \mathbb{Z}$ and $\mathcal{D}_n = \langle \mathbb{Z}_{p^n} \rangle$ - the subcategory of direct sums of copies of $\mathbb{Z}_{p^n}$, so each $\mathbb{Z}_{p^n}$ is an elementary cogenerator. The definable subcategory generated by the union of these also contains $\mathbb{Z}_{p^\infty}, \overline{\mathbb{Z}_{(p)}}$ and $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty}$ is neg-isolated for that definable subcategory, so is not a direct summand of (any power of) $\prod_n \mathbb{Z}_{p^n}$.

4 Anti-equivalence of definable categories and abelian categories

Definable categories $\mathcal{D}$ and skeletally small abelian categories $\mathcal{A}$ are linked by $\mathcal{D} \mapsto \mathrm{fun}(\mathcal{D})$ and $\mathcal{A} \mapsto \mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$. This link extends to functors between these types of category and natural transformations between those functors. The framework in which to talk about this link is that of 2-categories.

Because the 2-categories we deal with here are rather concrete we don't need to give the full definition of 2-category or deal with some of the subtleties involved. For that see, for instance, [19, §4.1], [13, B1.1]. Essentially a **2-category** has three layers of structure: 0-arrows (or objects), 1-arrows and 2-arrows (or 2-cells). In all our examples these will be, respectively, categories, functors between those categories, and natural transformations between those functors.

An **equivalence** between 2-categories $\mathbb{A}$, $\mathbb{B}$ is given by a pair $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{B} \to \mathbb{A}$ of 2-functors such that there are natural equivalences $GF \simeq 1_{\mathbb{A}}$ and $FG \simeq 1_{\mathbb{B}}$ (for full details see the references). By an **anti-equivalence** from $\mathbb{A}$ to $\mathbb{B}$ we mean an equivalence between the opposite of $\mathbb{A}$ (which is also a 2-category) and $\mathbb{B}$.

We have already mentioned two of the 2-categories we will consider. Namely the 2-category $\mathbb{ABEX}$ whose objects are the skeletally small abelian categories, whose 1-arrows are the exact functors between these and whose 2-arrows are the natural transformations between exact functors. The other 2-category is $\mathbb{DEF}$, with objects the definable additive categories, 1-arrows the interpretation functors[7] = functors which preserve direct products and directed colimits, and with 2-arrows the natural transformations between these functors.

There is a third 2-category $\mathbb{COH}$ that we can add to the picture. This has, for its objects, the locally coherent Grothendieck categories, for its 1-arrows, the **coherent morphisms**, meaning adjoint pairs (F^*, F_*) of functors, $F_* : \mathcal{G} \to \mathcal{H}$ and $F^* : \mathcal{H} \to \mathcal{G}$, between such categories, with F^* left exact and $F^*\mathcal{H}^{\mathrm{fp}} \subseteq \mathcal{G}^{\mathrm{fp}}$ and, again, natural transformations for the 2-arrows. See [25, §4] for more detail.

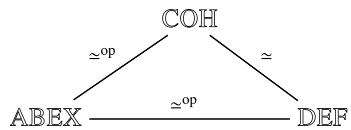
Given a skeletally small abelian category $\mathcal{A}$ its **Ind-completion**, $\mathrm{Ind}(\mathcal{A})$, has, for its objects, the equivalence classes of directed diagrams in $\mathcal{A}$ and the morphisms of $\mathrm{Ind}(\mathcal{A})$ are constructed along the same lines, see e.g. [2, Exp. I, §8] for details. The categories $\mathrm{Ind}(\mathcal{A})$ obtained in this way are precisely the locally coherent Grothendieck categories, [30, Prop. 2]. In the other direction, given a locally coherent Grothendieck category $\mathcal{G}$, the subcategory $\mathcal{G}^{\mathrm{fp}}$ of finitely presented objects is, to natural equivalence, a typical skeletally small abelian category. This construction can be extended to a 2-functor from $\mathbb{ABEX}$ to $\mathbb{COH}$ which is an equivalence of 2-categories, see [25, 4.3].

In fact we have the following commutative diagram of equivalences and anti-equivalences.

Theorem 26 *([29, 2.3 and comments following])*

[7] This terminology refers to their model-theoretic definition which turns out to be equivalent to the stated algebraic preservation properties, see [23, Chpt. 2].

There are equivalences and anti-equivalences between the 2-categories $\mathbb{ABEX}$*,* $\mathbb{COH}$ *and* $\mathbb{DEF}$ *as shown.*



The anti-equivalence from $\mathbb{ABEX}$ to $\mathbb{DEF}$ takes $\mathcal{A}$ to $\mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ and, if $F : \mathcal{A} \to \mathcal{B}$ is exact then we have, by composition, the induced functor $\mathrm{Ex}(\mathcal{B}, \mathbf{Ab}) \to \mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ which commutes with products and directed colimits. In the other direction, starting from a definable category $\mathcal{D}$ we obtain the skeletally small abelian category $\mathrm{fun}(\mathcal{D})$ of functors from $\mathcal{D}$ to $\mathbf{Ab}$ which commute with direct products and directed colimits and, again, the action on functors is just that induced by composition, [23, 13.1]. Further details can be found in [29] or, for a broader account, [25].

We remark that, from the above, $\mathrm{fun}(\mathcal{D})$ can be any small abelian category but those categories arising when $\mathcal{D}$ is the category of all modules over some ring (possibly with many objects) have enough, that is a generating set of, projectives and are of global dimension 0 or 2, [3, p. 205]. Furthermore, [14, 2.3], a definable category $\mathcal{D}$ is finitely accessible iff $\mathrm{fun}(\mathcal{D})$ has enough projectives.

5 Monoidal structure and definable categories

5.1 Monoidal categories

We briefly recall some definitions; for details see, for example, [17, Chpt. 7].

By a **monoidal** structure on an additive category C we mean an additive bifunctor $(- \otimes -) : C \times C \to C$ which is associative - $A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ - and symmetric - $A \otimes B \simeq B \otimes A$ and which has a tensor-unit $\mathbb{1}$ - $A \otimes \mathbb{1} \simeq A \simeq \mathbb{1} \otimes A$. In fact, one has to take care over these isomorphisms and treat them as part of the data of the monoidal structure but we don't need that level of detail in this exposition.

Familiar examples are the category of modules over a commutative ring with the usual tensor structure and the category of K-representations (K a field) of a finite group with $A \otimes B$ defined to be the vector space $A \otimes B$ equipped with the G-action determined by $g(a \otimes b) = ga \otimes gb$.

In each of those examples, there is a right adjoint to the tensor functor, expressed by the natural isomorphisms $(A \otimes B, C) \simeq (A, [B, C])$ where $[B, C]$ is the usual hom in the first case but not in the second. In general, we say that a monoidal structure on a category is **closed** if there is such an adjoint, referred to as **internal hom** and notated $[-, -]$.

We say that a monoidal structure is **rigid** if every object A has a **dual** $A^\vee$, where this duality is a contravariant functor on C and satisfies natural conditions including that $[A^\vee, B] \simeq [\mathbb{1}, A \otimes B]$. Typically (as in vector spaces) one requires that there is such a duality only on the subcategory of “small" objects of C and our typical requirement on a monoidal category C is that the monoidal structure be closed on C and we might add that the monoidal structure should restrict to the category C^{fp} (if C is finitely accessible) or C^{c} (if C is triangulated), and possibly add that this restricted structure be rigid.

A symmetric monoidal structure on a skeletally small additive category C lifts to a closed symmetric monoidal structure on the category $(C, \mathbf{Ab})$ of C-modules. The process, Day convolution [6, 3.3, 3.6], is determined by setting $(A, -) \otimes (B, -) = (A \otimes B, -)$ on representable functors = finitely generated projective modules and then extending to all modules by insisting that $\otimes$ on the functor category be right exact. Clearly this tensor product on C-Mod restricts to C-mod.

5.2 The monoidal anti-equivalence

We now state the monoidal version of the anti-equivalence between $\mathbb{ABEX}$ and $\mathbb{DEF}$. The analogue of the third vertex, $\mathbb{COH}$, appearing in Theorem 26 can be obtained by extending a monoidal structure on an abelian category $\mathcal{A}$ along directed colimits.

The 2-category $\mathbb{ABEX}^\otimes$ has, for its objects, the skeletally small monoidal abelian categories $(\mathcal{A}, \otimes)$ where $\otimes$ is additive, symmetric and *exact*, that is, if $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathcal{A}$ then, for each $D \in \mathcal{A}$, the sequence $0 \to A \otimes D \to B \otimes D \to C \otimes D \to 0$ is exact. The 1-arrows of $\mathbb{ABEX}^\otimes$ are the exact monoidal functors F, where exactness means that, if $F : \mathcal{A} \to \mathcal{B}$ and if $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathcal{A}$, then $0 \to FA \to FB \to FC \to 0$ is an exact sequence in $\mathcal{B}$, and the monoidal condition includes that F takes the tensor-unit of $\mathcal{A}$ to that of $\mathcal{B}$ and that, for each pair A, B of objects of $\mathcal{A}$, there is an isomorphism $\tau_{A,B} : F(A \otimes B) \to FA \otimes FB$ such that, for all morphisms $f : A \to A'$, $g : B \to B'$, we have the commutative diagram shown.

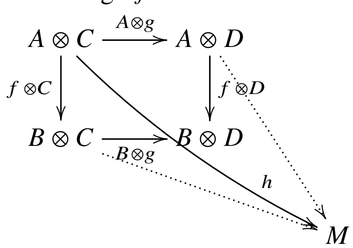
$$\begin{array}{ccc} F(A \otimes B) & \xrightarrow{F(f \otimes g)} & F(A' \otimes B') \\ \downarrow{\scriptstyle \tau_{A,B}} & & \downarrow{\scriptstyle \tau_{A',B'}} \\ FA \otimes FB & \xrightarrow[Ff \otimes Fg]{} & FA' \otimes FB' \end{array}$$

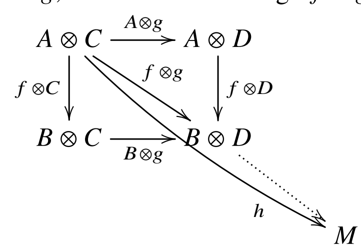
The 2-arrows of $\mathbb{ABEX}^\otimes$ are the natural transformations.

The definition of $\mathbb{DEF}^\otimes$ is more complicated and less “intrinsic". The objects of $\mathbb{DEF}^\otimes$ are triples $(\mathcal{D}, C, \otimes)$ where C is a finitely accessible category with products, $\otimes$ is an additive symmetric *closed* monoidal structure on C such that C^{fp} is a monoidal subcategory (that is C^{fp} is closed in C under $\otimes$ and contains $\mathbb{1}$) and where $\mathcal{D}$ is a definable subcategory of C which is fp-hom-closed and satisfies the exactness criterion - we now define these conditions.

We say that a definable subcategory $\mathcal{D}$ of a monoidal finitely accessible category C with products is **fp-hom-closed** if, for every $A \in C^{\mathrm{fp}}$ and every $M \in \mathcal{D}$, the object $[A, M]$ is in $\mathcal{D}$ - that is, "$\mathcal{D}$ contains its internal hom sorts".

We say that a definable subcategory $\mathcal{D}$ of a monoidal finitely accessible category C with products satisfies the **exactnesss criterion** if, given morphisms $f : A \to B$ and $g : C \to D$ in C^{fp}, and given any $M \in \mathcal{D}$ and morphism $h : A \otimes C \to M$, if h factors through $f \otimes C$ and also factors through $A \otimes g$, then h factors through $f \otimes g$.





The 1-arrows of $\mathbb{DEF}^{\otimes}$ from $(\mathcal{D}, C, \otimes)$ to $(\mathcal{D}', C', \otimes')$ are the morphisms $\mathcal{D} \to \mathcal{D}'$ which commute with direct products and directed colimits and which are such that the exact functor $\mathrm{fun}(\mathcal{D}') \to \mathrm{fun}(\mathcal{D})$ induced by composition, see [23, 13.1], is monoidal.
The 2-arrows of $\mathbb{DEF}^{\otimes}$ are just the natural transformations.

Theorem 27 *([34, 1.3/3.1]) There is an anti-equivalence of 2-categories:*

$$\mathbb{ABEX}^{\otimes} \leftrightarrow^{\mathrm{op}} \mathbb{DEF}^{\otimes}$$

given by

$$\mathcal{A} \mapsto (\mathrm{Ex}(\mathcal{A}, \mathbf{Ab}), \mathcal{A}\text{-}\mathrm{Mod}, \otimes)$$

where the monoidal structure on $\mathcal{A}$-Mod *is induced from that on* $\mathcal{A}$ *by Day convolution and, in the other direction, given by*

$$(\mathcal{D}, C, \otimes) \mapsto (\mathrm{fun}(\mathcal{D}), \otimes)$$

where the monoidal structure on $\mathrm{fun}(\mathcal{D})$ *is induced, by Serre-localisation, from that induced on* $(C^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$ *by Day convolution.*

The last part, about the monoidal structure on $\mathrm{fun}(\mathcal{D})$, needs explanation. We use the fact, see [23, Chpt. 10], that $\mathrm{fun}(\mathcal{D})$ is a Serre quotient of $(C^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}} = \mathrm{fun}(C)$, namely the quotient by the Serre-annihilator $\mathcal{S}_{\mathcal{D}} = \{F \in (C^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}} : \overrightarrow{F}\mathcal{D} = 0\}$ of $\mathcal{D}$, where $\overrightarrow{F}$ is the $\varinjlim$-commuting extension of F to all of C. The fp-hom-closed condition on $\mathcal{D}$ is exactly what is needed to ensure that $\mathcal{S}_{\mathcal{D}}$ is a tensor-ideal of $(C^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$ and hence that the monoidal structure on $(C^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$ induced, via Day convolution, by that on C^{fp} induces a monoidal structure on its Serre-quotient $\mathrm{fun}(\mathcal{D}) = (C^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}/\mathcal{S}_{\mathcal{D}}$.

Theorem 28 *([34, 1.2/3.7]) Suppose that* C *is an additive finitely accessible category with products, with a closed symmetric monoidal structure such that* C^{fp}

is a monoidal subcategory. Equip $(\mathcal{C}^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$ *with the Day-convolution-induced monoidal structure.*

Let $\mathcal{D}$ *be a definable subcategory of* $\mathcal{C}$ *and let* $\mathcal{S}_{\mathcal{D}}$ *denote the Serre-annihilator of* $\mathcal{D}$.

Then $\mathcal{S}_{\mathcal{D}}$ *is a tensor-ideal of* $(\mathcal{C}^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$ *iff* $\mathcal{D}$ *is fp-hom-closed in* $\mathcal{C}$.

The exactness condition on $\mathcal{D}$ is precisely what is needed to ensure that the monoidal structure induced on $\mathrm{fun}(\mathcal{D})$ is exact.

Theorem 29 *([34, 3.10,3.11]) Suppose that* $\mathcal{C}$ *is an additive finitely accessible category with products, with a closed symmetric monoidal structure such that* $\mathcal{C}^{\mathrm{fp}}$ *is a monoidal subcategory. Equip* $(\mathcal{C}^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$ *with the Day-convolution-induced monoidal structure.*

Let $\mathcal{D}$ *be an fp-hom-closed definable subcategory of* $\mathcal{C}$ *and induce the monoidal structure on* $\mathrm{fun}(\mathcal{D})$ *as in Theorem 28.*

Then the monoidal structure on $\mathrm{fun}(\mathcal{D})$ *is exact iff* $\mathcal{D}$ *satisfies the exactness criterion in* $\mathcal{C}$.

Example 16

(see [34, 5.3] Consider the category of modules over $R = K[\epsilon : \epsilon^2 = 0]$ with K any field, equipped with the usual tensor product for commutative rings (so the tensor-unit $\mathbb{1}$ is the unique simple module U which is K with ϵ acting as 0). There are just two nonzero proper definable subcategories of R-Mod, namely $\langle U \rangle$, which consists of direct sums of copies of U, and $\langle R \rangle$, which consists of direct sums of copies of the indecomposable projective module R. We have $(U, U) \simeq U \simeq (R, U)$ so $\langle U \rangle$ is fp-hom-closed in R-Mod, but $(U, R) \simeq U$, so $\langle R \rangle$ is not fp-hom-closed. It is also the case that $\langle U \rangle$ satisfies the exactness condition, by the following result.

Theorem 30 *([34, 5.5]) If* R *is a commutative coherent ring then every fp-hom-closed definable subcategory of* R-Mod *satisfies the exactness condition.*

If the tensor structure restricted to $\mathcal{C}^{\mathrm{fp}}$ is rigid, then we have the following.

Theorem 31 *([34, 4.3]) Suppose that* $\mathcal{C}$ *is an additive finitely accessible category with products, with a closed symmetric monoidal structure such that* $\mathcal{C}^{\mathrm{fp}}$ *is a rigid monoidal subcategory. Equip* $(\mathcal{C}^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$ *with the Day-convolution-induced monoidal structure. Let* $\mathcal{D}$ *be a definable subcategory of* $\mathcal{C}$.

Then $\mathcal{S}_{\mathcal{D}}$ *is a Serre tensor-ideal of* $(\mathcal{C}^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$ *iff* $\mathcal{D}$ *is a tensor-ideal of* $\mathcal{C}$.

Example 17

Suppose that G is a finite group and K is a field. Then the monoidal structure on KG-modules, restricted to finitely presented KG-modules, is rigid. So, by Theorem

31, the definable tensor-ideals of KG-Mod are in natural bijection with the Serre tensor-ideals of $(C^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$.

Example 18

(see [34, 5.14] Consider again the category of modules over $R = K[\epsilon : \epsilon^2 = 0]$ but now with K a field of characteristic 2, so that we may consider this as the category of K-representations of the cyclic group of order 2 equipped with the representations-of-groups tensor product. So now the tensor-unit is the module R.

One can compute that, in this case (cf. Example 16), $\langle R \rangle$ is fp-hom-closed (and a tensor-ideal of R-Mod) but $\langle U \rangle$ is not, so we have $\mathcal{S}_{\langle R \rangle}$ as the unique proper non-trivial Serre tensor-ideal of $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$.

Although we have not discussed elementary duality of definable categories here (for that see [22] for instance), we do note the further result, [34, 4.8], that a definable subcategory $\mathcal{D}$ of $\mathcal{A}$-Mod, where $\mathcal{A}$ has a rigid monoidal structure, is fp-hom-closed iff the dual definable subcategory $\mathcal{D}^{\mathrm{d}}$ of Mod-$\mathcal{A}$ is a tensor-ideal.

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